Soliton resolution for the energy critical wave equation in six dimensions in the radial case

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Part 1: What is soliton resolution?

The soliton resolution conjecture

A "self-similar" solution is a solution that is invariant under the action of a $1D$ group of symmetry:

- A stationary state, invariant by time translation,
- A traveling wave, invariant by simultaneous time and space translation,
- A backward or forward self-similar solution, invariant by scaling.

The soliton resolution conjecture is the belief that any solution of an evolution PDE evolves asymptotically into a sum, as $t \to T$, of "self-similar" solutions, plus a lower order term (typically a linear solution):

$$u(t,x) \approx \sum \lambda_i^{\alpha_i}(t) W_i\left(\frac{x - x_i(t)}{\lambda_i(t)}\right) + v_L(t,x).$$

Hypotheses to ensure such relaxation have to be made, e.g. non-confining domain for global solutions.

The soliton resolution conjecture

Origin of the conjecture: numerical experiments [Fermi-Pasta-Ulam 1955], [Zabusky-Kruskal 1965] and completely integrable equations.

For completely integrable systems, the soliton resolution can be proved with the method of inverse scattering [Eckauss-Schuur 1983] for KdV, [Schuur 1986] for mKdV, [Pocovnicu 2011] for Szegö etc...

For typical completely integrable equations, a multisoliton in the future is also a multisoliton in the past, with the same parameters (thus it is a *pure multisoliton*). The collision between solitons is elastic. This is the case for KdV, mKdV, and also the Szegö equation.

This property is believed to be specific to completely integrable equations. See [Martel-Merle '11, '15] for collisions for non-integrable gKdV.

In this talk we consider the energy critical semilinear wave equation (NLW) in $6D$. We show any solution that remains bounded in the energy space is asymptotically the sum, as $t \to T$, of decoupled stationary states and a linear solution.

Inelastic collision conjecture: there is no pure multisoliton for (NLW).

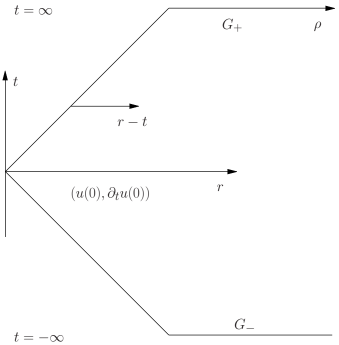
Part 2: Asymptotics for the linear wave equation.

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ (u(0), \partial_t u(0)) \in \dot{H}^1 \times L^2(\mathbb{R}^N). \end{cases} \tag{LW}$$

Proposition (Asymptotic self-similarity) — [Friedlander 60's, Duyckaerts-Kenig-Merle '16]

For $N \geq 3$, u radial solution on $\mathbb{R}^{1+N}$ of (LW) there exist $G_\pm \in L^2(\mathbb{R})$ radiation profiles:

$$\lim_{t\to\pm\infty} \int_0^\infty \left| r^{\frac{N-1}{2}} (\partial_r u, \partial_t u)(t,r) - (G_\pm, \pm G_\pm)(r - |t|) \right|^2 dr = 0.$$



Recent works [Li-Shen-Wei '21, '22, Delort '21, Cote-Laurent '21].

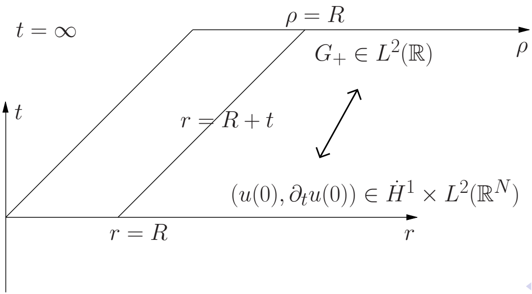
Repartition of energy

The map $(u(0), \partial_t u(0)) \mapsto G_\pm$ is an isometry:

$$\int_{\mathbb{R}^N} |\nabla u(0)|^2 + |\partial_t u(0)|^2 \, dx = \int_{\mathbb{R}} |G_\pm|^2 d\rho$$

By finite speed of propagation, on exterior wave cones away from origin:

$$\int_{|x| \geq R} |\nabla u(0)|^2 + |\partial_t u(0)|^2 \, dx \gtrsim \int_{\rho \geq R} |G_\pm|^2 d\rho$$



Part 3: Main results

The focusing critical wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = u^2, \qquad \text{or} \qquad \partial_t^2 u - \Delta u = |u|u \\ (u(0), \partial_t u(0)) \in \dot{H}^1 \times L^2(\mathbb{R}^6). \end{cases}$$

where $u : [0, T[\times \mathbb{R}^N \to \mathbb{R}$ is radial. The general equation:

$$\begin{cases} \partial_t^2 u - \Delta u = |u|^{\frac{4}{N-2}} u \\ (u(0), \partial_t u(0)) \in \dot{H}^1 \times L^2(\mathbb{R}^N). \end{cases} \tag{NLW}$$

where $N \geq 3$. (NLW) is locally well-posed in $\dot{H}^1 \times L^2(\mathbb{R}^N)$ [Ginibre-Soffer-Velo 92].
Conserved energy:

$$E = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u(t)|^2 + |\partial_t u(t)|^2 dx - \frac{N-2}{2N} \int_{\mathbb{R}^N} |u(t)|^{\frac{2N}{N-2}} dx.$$

Invariances: If u is a solution and $\lambda > 0$, so is

$$u_{(\lambda),\pm}(t,x) = \pm \frac{1}{\lambda^{N/2-1}} u\left(\frac{\pm t}{\lambda}, \frac{x}{\lambda}\right).$$

(NLW) is energy critical: $E(u_{(\lambda),\pm}) = E(u)$.

Stationary state

$$-\Delta W = |W|^{\frac{4}{N-2}} W, \quad W \in \dot{H}^1(\mathbb{R}^N). \tag{Ell}$$

$\exists$ solutions of (Ell) with arbitrarily large energy [Ding 1986].

Unique nonzero radial solution of (Ell) up to scaling and sign change (ground state):

$$W(x) = \frac{1}{\left(1 + \frac{|x|^2}{N(N-2)}\right)^{\frac{N-2}{2}}}.$$

Remark: W decays faster than the scaling invariant solution $|W(x)| \ll |x|^{\frac{N-2}{2}}$ as $x \to \infty$, which is false for supercritical case.

For $N = 6$:

$$W(x) = \left(1 + \frac{|x|^2}{24}\right)^{-2} \approx \langle x \rangle^{-4}.$$

Main results

Theorem 1 — Soliton resolution $N = 6$ radial [C.-Duyckaerts-Kenig-Merle '22]

Let u be a radial solution of

$$\partial_t^2 u - \Delta u = u^2 \qquad \text{or} \qquad \partial_t^2 u - \Delta u = |u|u.$$

Assume it is global for positive times and remains bounded in energy norm

$$\sup_{t\geq 0} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} < \infty.$$

Then $\exists J \in \mathbb{N}$, $\iota_j \in \{\pm 1\}$ ($\iota_j = 1$ for u^2 nonlinearity),

$$0 < \lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_J(t) \ll t,$$

and v_L a free wave $\partial_t^2 v_L - \Delta v_L = 0$, such that, as $t \to \infty$:

$$(u(t), \partial_t u(t)) = \sum_{j=1}^{J} \left(\frac{\iota_j}{\lambda_j^2(t)} W\left(\frac{x}{\lambda_j(t)}\right), 0 \right) + (v_L(t), \partial_t v_L(t)) + o(1), \text{ in } \dot{H}^1 \times L^2.$$

Main results

Theorem 2 — Nonexistence of pure multisoliton radial [C.-Duyckaerts-Kenig-Merle '22]

Let u be a radial solution of

$$\partial_t^2 - \Delta u = u^2 \qquad \text{or} \qquad \partial_t^2 - \Delta u = |u|u.$$

Assume it is global for both time directions with

$$\sup_{t\in\mathbb{R}} \|(u(t), \partial_t u(t))\|_{\dot{H}^1\times L^2} < \infty,$$

and that it is non-radiative for $r \geq R + |t|$ for any $R \in \mathbb{R}$:

$$\lim_{t\to\pm\infty} \int_{|x|\geq R+|t|} |\nabla u(t)|^2 + |\partial_t u(t)|^2 \, dx = 0.$$

Then

$$u = 0, \qquad \text{or} \qquad u = W_{(\lambda)} \qquad \text{or} \qquad u = -W_{(\lambda)}.$$

Part 4: Related results

Examples of global solutions

$N = 6$.
[Ginibre-Soffer-Velo '92] $\exists$ scattering solutions

$$(u(t), \partial_t u(t)) = (v_L(t), \partial_t v_L(t)) + o(1), \qquad \text{in } \dot{H}^1 \times L^2.$$

[Krieger-Nakanishi-Schlag '12-'15] $\exists$ manifold of solutions scattering to a ground state

$$(u(t), \partial_t u(t)) = W + (v_L(t), \partial_t v_L(t)) + o(1),$$

[Jendrej '17] $\exists$ a 2-multisoliton

$$(u(t), \partial_t u(t)) = W + \frac{1}{\lambda(t)^2} W\left(\frac{x}{\lambda(t)}\right) + o(1), \qquad \lambda(t) = \kappa^{-1} e^{-\kappa t}, \qquad \kappa = \frac{\sqrt{5}}{2}.$$

General N.
Infinite time concentrating or spreading radial soliton [Krieger-Schlag '07, Krieger Donninger '13]. Non-radial multi-solitons [Martel-Merle '16, Yuan '20 '21]. Radial 2-multisoliton [Jendrej '17].

Proofs of soliton resolution conjecture

Soliton resolution and inelastic collision:

- Analogue results for (NLW) for $N = 3$ [Duyckaerts-Kenig-Merle '13], $N \geq 5$ [Duyckaerts-Kenig-Merle '19], $N = 4$ [Duyckaerts-Kenig-Martel-Merle '19], $N \geq 4$ [Jendrej-Lawrie '22].
- Co-rotational wave maps [Duyckaerts-Kenig-Martel-Merle '21].
- Radial defocusing 3D wave with a potential [Jia-Liu-Schlag '15].
- Nonlinear waves in 3D outside an obstacle [Duyckearts-Yang '19].

Soliton resolution:

- (NLW) without symmetry assumption for (NLW), for a sequence of times [Duyckaerts-Jia-Kenig-Merle '16].
- Partial result for wave maps [Grinis '17], [Duyckaerts-Jia-Kenig-Merle '16].
- Critical Wave Maps in all equivariant classes [Cote-Kenig-Lawrie-Schlag '15], [Cote '15], [Jendrej-Lawrie '18, '20, '21].
- Damped Klein-Gordon [Burq-Raugel-Schlag '17-18].

Inelastic collision:

- For a nonradial multisoliton of (NLW) in 5D [Martel-Merle '19].
- For a two-bubble wave maps [Jendrej-Lawrie '18].

Part 5: Explanation of one aspect of the proof, energy repartition

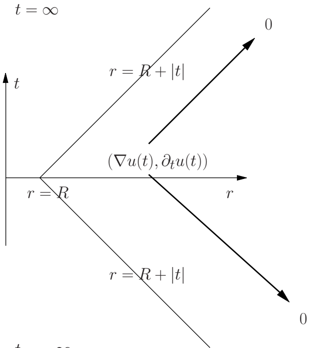
A: classification of solutions that radiate, or do not radiate, energy

Non-radiative solutions — Definition

Definition

A solution is non-radiative for $r \geq R + |t|$ if

$$\lim_{t\to\pm\infty} \int_{|x|\geq R+|t|} |\nabla u(t)|^2 + |\partial_t u(t)|^2 \, dx = 0 \quad \Leftrightarrow \quad G_\pm(\rho) = 0 \quad \forall \rho \geq R.$$



Classification of non-radiative linear waves

Proposition [Kenig-Lawrie-Liu-Schlag '15, Duyckaerts-Kenig-Martel-Merle '21, Li-Shen-Wei '21]

A radial solution u of the linear wave equation is non-radiative for $r \geq R + |t|$ iff

$$u(t,r) = a_L[\vec{c}] = \sum_{m=0}^{m_0} \frac{c_m}{r^{N-2-m}} p_m\left(\frac{r}{t}\right), \qquad \forall r > R + |t|,$$

with p_m explicit polynomial of degree m, equivalently

$$u(0,r) \in \operatorname{Span}\left\{\frac{1}{r^{N-2-2l}}\right\}_{0\leq l\leq l_0}, \quad \partial_t u(0,r) \in \operatorname{Span}\left\{\frac{1}{r^{N-2-2l}}\right\}_{0\leq l\leq l_1}, \qquad \forall r > R$$

$m_0 = \lfloor \frac{N-3}{2} \rfloor$, $l_0 = \lfloor \frac{N-3}{4} \rfloor$, $l_1 = \lfloor \frac{N-5}{4} \rfloor$.

Classification of non-radiative nonlinear waves

Proposition [C.-Duyckaerts-Kenig-Merle '22]

Existence. For various energy critical non-linear wave equations there exists a $m_0 + 1$-parameter family of solutions

$$a_{NL}[\vec{c}] = a_L[\vec{c}] + \tilde{a}[\vec{c}] = \sum_{m=0}^{m_0} \frac{c_m}{r^{N-2-m}} p_m\left(\frac{r}{t}\right) + h.o.t.$$

that are non-radiative for $r \geq R_0[\vec{c}] + |t|$.

Uniqueness. If u is a non-radiative solution for $r > R + |t|$, then there exists $\vec{c} \in \mathbb{R}^{m_0+1}$ such that

$$u(t,r) = a_{NL}[\vec{c}](t,r) \qquad \forall r \geq R + |t|.$$

One key difficulty of the proof: in even dimensions, the resonance $\frac{1}{r^{\frac{N}{2}-1}} p_{\frac{N}{2}-1}\left(\frac{r}{t}\right)$ barely fails to belong to the energy space. It does not generate nonlinear solutions thanks to cancellations involving Chebychev polynomials.

B: Quantification of energy radiation for linearised equations

Channels of energy estimates — N Odd dimensions

Proposition — [Duyckaerts-Kenig-Merle '12]

N odd. If $R = 0$ then:

$$\int_{\mathbb{R}^N} |\nabla u(0)|^2 + |\partial_t u(0)|^2 \, dx \approx \sum_{\pm} \int_{\rho \geq 0} |G_\pm|^2 d\rho.$$

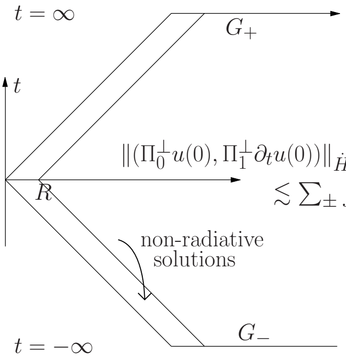
If $R > 0$ and

$$u(0,r) \in \mathrm{Span}\left\{\frac{1}{r^{N-2-2l}}\right\}_{0 \leq l \leq l_0}^{\perp_{\dot{H}^1_{r \geq R}}}, \quad \partial_t u(0,r) \in \mathrm{Span}\left\{\frac{1}{r^{N-2-2l}}\right\}_{0 \leq l \leq l_1}^{\perp_{L^2_{r \geq R}}},$$

then

$$\int_{|x| \geq R} |\nabla u(0)|^2 + |\partial_t u(0)|^2 \, dx \approx \sum_{\pm} \int_{\rho \geq R} |G_\pm|^2 d\rho.$$

Channels of energy estimates — Odd dimensions



$$\|(\Pi_0^\perp u(0), \Pi_1^\perp \partial_t u(0))\|_{\dot{H}^1\times L^2(|x|\geq R)} \lesssim \sum_\pm \int_{\rho\geq R} |G_\pm|^2 d\rho$$

Channels of energy estimates — Even dimensions

Proposition — [Cote-Kenig-Schlag '14]

$N = 6$. If $R = 0$ then:

$$\int_{\mathbb{R}^N} |\partial_t u(0)|^2 \, dx \lesssim \sum_{\pm} \int_{\rho \geq 0} |G_\pm|^2 d\rho \quad \text{holds true,}$$

$$\int_{\mathbb{R}^N} |\nabla u(0)|^2 \, dx \not\lesssim \sum_{\pm} \int_{\rho \geq 0} |G_\pm|^2 d\rho \quad \text{fails.}$$

If $R > 0$ and

$$u(0,r) \in \operatorname{Span}\left\{\frac{1}{r^4}\right\}^{\perp_{\dot{H}^1_{r \geq R}}}, \quad \partial_t u(0,r) \in \operatorname{Span}\left\{\frac{1}{r^4}\right\}^{\perp_{L^2_{r \geq R}}},$$

then

$$\int_{|x| \geq R} |\Pi_1^F \partial_t u(0)|^2 \, dx \lesssim \sum_{\pm} \int_{\rho \geq R} |G_\pm|^2 d\rho \quad \text{holds true,}$$

$$\int_{|x| \geq R} |\Pi_0^F \nabla u(0)|^2 \, dx \not\lesssim \sum_{\pm} \int_{\rho \geq R} |G_\pm|^2 d\rho \quad \text{fails.}$$

Channels around a ground state

Ground state positive solution of $-\Delta W = |W|^{\frac{4}{N-2}} W$, potential $V = -\frac{N+2}{N-2} W$.

$$\text{Zero:} \qquad (-\Delta + V)\Lambda W = 0, \qquad \Lambda W = x.\nabla W + \frac{N-2}{2} W.$$

Proposition — [Duyckaerts-Kenig-Merle '19]

$N \geq 3$ odd. $v = \Lambda W$ and $v = t\Lambda W$ are non-radiative solutions of

$$\partial_t^2 v - \Delta v + Vv = 0.$$

For any solution u with $u(0) \in \text{Span}\{\Lambda W\}^{\perp_{\dot{H}^1}}$ and $\partial_t u(0) \in \text{Span}\{\Lambda W\}^{\perp_{L^2}}$:

$$\int_{\mathbb{R}^N} |\partial_t u(0)|^2 + |\partial_t u(0)|^2 \, dx \lesssim \sum_{\pm} \int_{\rho \geq 0} |G_\pm|^2 d\rho.$$

Channels around a ground state $N=6$

Logarithmic weakening of $\|u\|_{\dot{H}^1(\mathbb{R}^6)}$:

$$\|u\|_Z = \sup_{R>0} \frac{1}{\langle \ln R\rangle} \|\nabla u\|_{L^2(R\le r\le 2R)} \qquad \left(\lesssim \|u\|_{\dot{H}^1(\mathbb{R}^6)}\right)$$

Proposition — [C.-Duyckaerts-Kenig-Merle '22]

$N=6,8$. $v=\Lambda W$ and $v=t\Lambda W$ are non-radiative solutions of

$$\partial_t^2 v - \Delta v + Vv = 0.$$

For any solution u with $u(0)\in \mathrm{Span}\{\Lambda W\}^{\perp_{\dot{H}^1}}$ and $\partial_t u(0)\in \mathrm{Span}\{\Lambda W\}^{\perp_{L^2}}$:

$$\|\Pi_0^L u(0)\|_Z^2 + \int_{\mathbb{R}^6} |\Pi_1^\perp \partial_t u(0)|^2\, dx \lesssim \sum_{\pm} \int_{\rho\ge 0} |G_\pm|^2 d\rho. \tag{1}$$

Remark

Extends to higher dimensions. $\log^{1/2}$ should be optimal.

Thank you for your attention!!