Large time behaviour in nonlocal reaction-diffusion equations of the Fisher-KPP type

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Abstract. The basic question is the evolution of the solutions to equations of the Fisher-KPP type, in which the diffusion is given by an integral operator. The level sets will organize themselves into an invasion front that is asymptotically linear in time, corrected by a logarithmic term. For a special class of nonlinearities, this fact can be deduced from the study of an underlying branching random walk (Aïdekon, 2013). This extends a famous result by Bramson (1983), where the diffusion is given by the Laplacian, and the underlying random walk is the Branching Brownian Motion. Motivated by models arising from epidemiology and ecology, that go beyond the equations arising from the branching random walks, this work explains how this logarithmic correction can be derived by working directly on the integro-differential equation.

1 Introduction

1.1 Model and question

We investigate the large time behaviour of the solution $u(t,x)$ to the equation

$$\partial_t u + \int_{\mathbb{R}} K(x-y)\big(u(t,x)-u(t,y)\big)dy = f(u), \quad t>0,\ x\in\mathbb{R}. \tag{1}$$

The initial datum $u(0,x)=u_0(x)$ will be nonnegative, compactly supported, and the kernel K will be nonnegative, smooth, supported in $[-1,1]$ and with unit mass. To alleviate the notations we will often denote the integral diffusion by

$$\mathcal{J}u(x) = \int_{\mathbb{R}} K(x-y)\big(u(t,x)-u(t,y)\big). \tag{2}$$

The function f will be positive on $(0,1)$ with $f(0)=f(1)=0$. A major assumption will be

$$f(u) \le f'(0)u \quad \text{for all } u\in(0,1). \tag{3}$$

The deviation of f to its initial slope will be denoted $g(u)$, that is :

$$f(u) = f'(0)u - g(u). \tag{4}$$

It will be smooth, nonnegative, not identically 0 with

$$g(0)=g'(0)=0, \quad \lim_{u\to+\infty}\frac{g(u)}{u}=+\infty, \quad g'(u)\ge\frac{g(u)}{u} \text{ for } u\ge 0. \tag{5}$$

Property (5) implies that $g(u) = O_{u\to 0}(u^2)$. It also implies the existence of a least $u_+ > 0$ such that $f(u_+) = 0$, we will without loss of generality that $u_+ = 1$. It also implies that the function $u \mapsto \dfrac{g(u)}{u}$ is nondecreasing.

When is is convenient to us, we will also assume $f'(1) < 0$. All these assumptions will be quite useful at many places, they are probably not, however, the optimal ones. Nonlinearities satisfying (5) are the convex gs such that $\lim\limits_{u\to+\infty} g'(u) = +\infty$, such as $g(u) = u^2$. However, Assumptions (5) are more general.

The issue is to follow, in the most precise fashion as possible, the level sets of u. More precisely, for a given $\gamma > 0$, we look for an asymptotic expansion, as $t \to +\infty$, of the quantity

$$X_\gamma(t) = \sup\{x \in \mathbb{R} : \ u(t,x) = \gamma\}. \tag{6}$$

The main goal of this paper is to prove the following asymptotic expansion for X_γ.

Theorem 1.1 *There exist two constants $c_* > 0$ and $\lambda_* > 0$, and a smooth function $x_\infty(\gamma)$ such that*

$$X_\gamma(t) = c_* t - \frac{3}{2\lambda_*}\ln t + x_\infty(\gamma) + o_{t\to+\infty}(1). \tag{7}$$

1.2 Motivations and known results

The link with second order diffusion

If the kernel K is an approximation of the identity, that is $K(x) = \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$, we derive the familiar Fisher-KPP reaction-diffusion equation (the acronym KPP stands for the authors Kolmogorov, Petrovskii and Piskunov of the seminal work [15]) as a limiting case of (1). Rescale he time and the nonlinear term as

$$\tau = \varepsilon^2 t, \quad f(u) = \varepsilon^2 F(u),$$

where F would bear the same properties as f. Dropping the terms of higher order in ε we obtain

$$\begin{aligned} u_\tau - du_{xx} = &\ \ F(u) \ \ (t > 0, \ x \in \mathbb{R}) \\ u(0,x) = &\ \ u_0(x), \end{aligned} \tag{8}$$

and the same question for (8) as (1). Defining $X_\gamma(\tau)$ as in (6), the main results in the asymptotic behaviour $\tau \to +\infty$ of the solution $u(\tau, x)$ are

Theorem 1.2 *(Kolmogorov, Petrovskii, Piskunov [15]) Define $c_* = 2\sqrt{df'(0)}$, and asume $u_0(x) = 1 - H(x)$, H being the Heaviside function. Then*

$$X_\gamma(\tau) = c_*\tau + o_{\tau\to+\infty}(\tau). \tag{9}$$

This fundamental result is refined in the following theorem.

Theorem 1.3 *(Bramson [4], [5]) Assume*

$$f(u) = f'(0)u - u^2. \tag{10}$$

Define $\lambda_* = \sqrt{\dfrac{f'(0)}{d}}$. Then

$$X_\gamma(\tau) = c_*\tau - \frac{3}{2\lambda_*}\ln \tau + x_\infty + o_{\tau\to+\infty}(1). \tag{11}$$

Bramson's approach is based on the detailed study of the position $X(\tau)$ of the rightmost particle in the Branching Brownian Motion in one space dimension, the bridge between $u(\tau,.x)$ being given by the McKean identity

$$u(\tau,x)=\mathbb{P}\big(\{X(\tau)\geq x\}\big). \tag{12}$$

His results are retrieved by Nolen, Ryzhik and the author in [16], with purely PDE arguments.

Branching random walks

The very introductory material in this section is borrowed from Section 2 of an interesting work of Graham [12], who proves an expansion of $X_\gamma(t)$ up to $O_{t\to+\infty}$ with a mixture of probabilistic and analytic arguments. A branching random walk consists of a system of particles that jump and divide at random times, the law of the length of the jumps being given by K. A striking fact is striking fact is that there is a McKean formula of the type (12), still for f of the form (10). And, for this sort of nonlinearity, Theorem 1.1 is a consequence of the work of Aïdekon [1], in the context of discrete time walks. It is to be noted that the Branching Brownian Motion, at the background of Theorem 1.3, is noting else than a limiting random walk where the particles have continuous trajectories.

The spatial spread of an epidemic

An important application of Theorem 1.1 is that it allows for the complete study of simple models arising from epidemiology. The most basic model is the SIR model with nonlocal interactions : if $x\in\mathbb{R}$, let $S(t,x,)$ denote the fraction of susceptible individuals, assumed not to diffuse. Let $I(t,x)$ the fraction of infected individuals. The number of susceptibles evolves not only from immediate contacts with infected individuals, but also from more remote interactions modelled by a convolution kernel K. The model thus reads

$$\left\{\begin{array}{rl}\partial_t I+\alpha I= & \beta SK*I\quad (t>0,\ x\in\mathbb{R})\\ \partial_t S= & -\beta SK*I\end{array}\right. \tag{13}$$

We assume a uniform initial susceptible density : $S(0,x,y)\equiv S_0>0$, and that $I(0,x)=I_0(x)$ compactly supported. The question is thus how the density of infected will evolve in time and space, and whether the epidemic front will travel at a definite speed. The cumulative numbers of infected individuals $u(t,x)$ solves

$$\partial_t u+\alpha u=S_0(1-e^{-K*u(t,.)})+I_0(x),\quad u(0,x)\equiv 0.. \tag{14}$$

Note that, if $K=\beta\delta_{x=0}$, and the initial density of infected is a constant number I_0 then (14) reduces to the ODE

$$\dot u=S_0(1-e^{-\beta u})-\alpha u+I_0:=f(u)+I+0, \tag{15}$$

and one easily shows the existence of $u^\infty(I_0)$ such that $\lim\limits_{t\to+\infty}u(t)=u^\infty(I_0)$. We have $f'(0)=S_0\beta-\alpha$, so that, defining

$$R_0=\frac{S_0\beta}{\alpha}, \tag{16}$$

we have $f'(0)=\alpha(R_0-1)$. Thus, if $R_0\leq 1$ we have $\lim_{I_0\to 0}u^\infty(I_0)=0$, while this limit is positive if $R_0>1$. This entails the following easy, but important result :

$$\lim_{t\to+\infty}S(t)=S_0e^{-\beta u^\infty(I_0)},$$

so that the density of susceptibles diminishes by a sizeable amount for infinite times, provided $R_0 > 1$.
Coming back to (15), the study is of course not finished once we know the behaviour of $u(t,x)$: we still need to know that its derivative has the expected behaviour. We note that (15) is an integral equation based on the model of (1), as its linearised version around $u = 0$ is simply

$$v_t + S_0\beta\mathcal{J}v = \alpha(R_0 - 1)v. \tag{17}$$

When $R_0 > 1$, spreading occurs, the first to figure that out in a completely rigorous fashion being Aronson [2], where a result analogous to Theorem 1.2 is proved. Parts of this result were understood by Kendall in [13].
Equation (14) can further be reduced to an integral equation on $v(t,x) := K * u(t,x)$:

$$v(t,x) = f(t,x) + \int_0^t \int_{\mathbb{R}} e^{-\alpha(t-s)} K(x-y)(1 - e^{-v(s,y)})dy, \tag{18}$$

where $f(t,x)$ depends on the data. This is an important remark, as models including more effects may not always be amenable to a reduction of the form (14), but are amenable to a reduction to (18). For instance, there is the subsequent study of Diekmann [10], who puts into the model (still with $d = 0$) the fact that an infected individual does not have the same infecting power during the course of the infection. He reduces it to an integral equation of the type (18), where the exponential $e^{-\alpha s}$ is replaced by a more general integrable function of time. These results were proved independently and at the same time by Thieme [18], on the formulation (18). This has generated an important line of research on such integral equations, that is too abundant to be cited exhaustively here.
For $R_0 > 1$, the consequence of Theorem 1.1 on the solution $u(t,x)$ of (14) is that one may, for every $\gamma \in (0, u^\infty(0)$, define $X_\gamma(t)$ by formula (6), and hat it satisfies an expansion of the form (7).

1.3 Organisation of the paper

There are four main sections following this introduction. Section 2 recalls some material which is relatively standard, but that, we feel, deserves to be stated here. The main sections are Section 4 and 5. The logarithmic delay is figured out in Section 4, and we show in 5 that the deviation from the logarithmic delay is asymptotically constant. This is preceded by the computation, in Section 3, of a heat kernel that we deem of independent interest.

2 Preliminary material

2.1 The Cauchy Problem and steady states

The initial value problem

As far as Equation (1) is concerned, the Cauchy Problem really resorts to nothing else than the Cauchy-Lipschitz theorem in the space, say, of all C^k functions of $\mathbb{R}$, the integer k being chosen so that any differentiation that we will need is automatically justified. In particular, if we assume u_0 to be a C^k function, a C^k solution will exist for all time, all derivatives being bounded by exponentials, due to the Gronwall Lemma.
Of course a better L^∞ bound can be proved through comparison. At this stage, it may be useful to state a few facts about sub and super-solutions, that are very much inspired from the notions

pertaining second order elliptic or parabolic equations. While they are elementary, they will turn out to be helpful in the proofs of estimates that go beyond the standard ones. For $c \in \mathbb{R}$, a super-solution (resp. sub-solution) to the equation

$$\partial_t u + \mathcal{J} u - c\partial_x u = f(u) \tag{19}$$

is a locally Lipschitz function $\bar{u}(t,x)$ (resp. $\underline{u}(t,x)$) that satisfies (19) with ≥ 0 (resp. ≤ 0) instead of $= 0$. A similar definition applies for a super-solution or sub-solution of the steady version of (19), that is, with $\partial_t = 0$. The definition easily extends to sub- or super-solutions of (19) on parts of the real line, with the caveat that one should know the function a little outside. For instance, a sub-(resp. super-) solution of (23) on $[0,T] \times (a,b)$ should be defined on $[0,T] \times (a-1,b+1)$. If the support of the integral kernel was not compact (e.g. $\mathbb{R}$), this would impose the function to be defined on $[0,T] \times \mathbb{R}$.
The following proposition is a typical example of a useful result, whose proof is obvious. The reader may be interested in writing down its analogue for elliptic or parabolic equations.

Proposition 2.1 *Let a be a real number, and let us consider a function $\underline{u}^+(t,x)$ defined on $[0,T] \times [a-1,+\infty)$ that is a sub-solution to (19) on $[0,T] \times (a,+\infty)$. Let $\underline{u}^-(t,x)$ be a function defined on $[0,T] \times [a-1,a]$ such that*

$$\underline{u}^-(t,x) \leq \underline{u}^+(t,x) \quad \text{for } t \in [0,T] \text{ and } a-1 \leq x \leq a.$$

Then, $\underline{u}(t,x)$ defined by

$$\underline{u}(t,x) = \begin{cases} \underline{u}^-(t,x) \text{ if } a-1 \leq x \leq a \\ \underline{u}^+(t,x) \text{ if } x \geq a \end{cases}$$

is a sub-solution to (23) on $[0,T] \times [a,+\infty)$. A similar statement would hold for a super-solution.

It is also true that the successive derivatives of $u(t,x)$ are uniformly bounded in time. This, however, is not a triviality anymore. It will be obtained as a corollary of much more precise estimates on $u(t,x)$ for large times, that will also stem from comparison.

Proposition 2.2 *Assume $c \neq 0$.*
Let $\underline{u}(x)$ be a nonnegative Lipschitz continuous sub-solution to (19), that is in addition ≤ 1. Let $v(t,x)$ be the solution of (19) starting from $\underline{u}$. Then $\partial_t v(t,x) \geq 0$ and $v(t,.)$ converges, locally uniformly, to a smooth solution $u_\infty(x)$ of (19) that is above $\underline{u}$.
Let $\overline{u}(x)$ be a nonnegative Lipschitz continuous super-solution to (19). Let $v(t,x)$ be the solution of (19) starting from $\overline{u}$. Then $\partial_t v(t,x) \geq 0$ and $v(t,.)$ converges, locally uniformly, to a smooth solution $u_\infty(x)$ of (19) that is below $\overline{u}$.

Steady states

A steady state to Equation (1) is a nonnegative bounded function $u(x)$ such that

$$\mathcal{J} u = f(u). \tag{20}$$

We note that $u(x)$ is not necessarily continuous. As steady states are expected to be the final state of the solution $u(t,x)$ of (1) after the passage of the front, it is useful to classify them. Obvious solutions are $u \equiv 0$ and $u \equiv 1$. And in fact, they are the only ones.

Theorem 2.3 *Assume the function $u \mapsto \dfrac{f(u)}{u}$ to be nonincreasing. Then any bounded, nonnegative solution of (20) is either $u \equiv 0$ or $u \equiv 1$.*

2.2 Linear and travelling waves

Linear waves

Linear waves are special solutions of (1), linearised around the steady state $u \equiv 0$, that is

$$\partial_t v + \mathcal{J} v = f'(0)v, \quad t \in \mathbb{R}, \ x \in \mathbb{R}. \tag{21}$$

They have the form $v(t,x) = \phi_\lambda(x - ct)$, with

$$\phi_\lambda(x) = e^{-\lambda x.} \tag{22}$$

The function ϕ_λ solves the steady problem

$$\mathcal{J}\phi - c\phi' = f'(0)\phi, \quad x \in \mathbb{R}. \tag{23}$$

When $c > 0$ the wave is said to propagate rightwards, in the opposite case it is said to propagate leftwards. The situation being perfectly symmetric we will look for rightwards propagating waves. Let us define the function

$$D_c(\lambda) = 2\int_0^1 \big(\cosh(\lambda x) - 1\big) K(x)dx - c\lambda - f'(0). \tag{24}$$

We will have many encounters with this function. Plugging (22) into (21) yields the equation

$$D_c(\lambda) = 0. \tag{25}$$

We have

$$D_c'(\lambda) = 2\int_0^1 xK(x)\sinh(\lambda x)dx - c, \quad D_c''(\lambda) = 2\int_0^1 x^2 K(x)(\cosh(\lambda x)dx. \tag{26}$$

From the uniform strict convexity of $u \mapsto \cosh u$, there exists a critical $c_* > 0$ such that (25) has two positive solutions $\lambda_-(c) < \lambda_+(c)$ if $c > c_*$, no solution if $c < c_*$, and exactly one, that we call λ_* if $c = c_*$. If $c > c_*$ we have

$$D_c'(\lambda_-(c)) < 0 < D_c'(\lambda_+(c)). \tag{27}$$

If $c = c_*$ we have

$$D_{c_*}'(\lambda_*) = 0, \quad D_{c_*}''(\lambda_*) > 0. \tag{28}$$

We also notice that $xe^{-\lambda_* x}$ also solves equation (23).
It will be useful to look for complex solutions of (25) for c slightly below c_*. In this range, we write (25) under the form

$$\frac{D_{c_*}(\lambda_*)}{2}(\lambda - \lambda_*)^2 = -\lambda(c_* - c) + O(\lambda - \lambda_*)^3,$$

that is,

$$\lambda - \lambda_* = \pm i\sqrt{\frac{2(c_* - c)}{D_{c_*}''(\lambda_*)}} + O(c_* - c)^{3/4}.$$

We write, therefore

$$\lambda = \lambda_*(c) \pm i\omega_*(c), \quad \lambda_*(c) \in \mathbb{R}, \ \omega_*(c) \in \mathbb{R}, \tag{29}$$

with

$$\lambda_*(c) = \lambda_* + O(c_* - c)^{3/4}, \quad \omega_*(c) = \sqrt{\frac{2(c_* - c)}{D_{c_*}''(\lambda_*)}} + O(c_* - c)^{3/4}. \tag{30}$$

The corresponding linear wave ϕ_λ may be taken as

$$\phi_\lambda(x) = e^{-\lambda_*(c)x}\cos\omega_*(c)x. \tag{31}$$

Travelling waves

Travelling waves are special propagating solutions of the full nonlinear equation (1). They play the same role as the linear waves in equation (21). They have the form

$$u(t,x) = \varphi(x - ct); \tag{32}$$

with the same notion of leftwards or rightwards propagation as for the linear waves. As they are supposed to be transition profiles between the two steady states of the equation, one requires them to connect 0 at one infinity to 1 at the other infinity. So, for the rightwards propagating waves, the problem is

$$\begin{cases} \mathcal{J}\varphi - c\varphi' = & f(\varphi) \\ \varphi(-\infty) = 1, \quad \varphi(+\infty) = & 0. \end{cases} \tag{33}$$

Existence, uniqueness and asymptotic behaviour of solutions is summarised in the following statement. Let c_* be the least velocity of linear waves defined in Section 2.

Theorem 2.4 *For a given $c > 0$, equation (33) has a nonnegative solution $\varphi_c(x)$ if and only if $c \geq c_*$. Moreover, ϕ is a C^∞ function, with $\varphi' < 0$. It is also unique modulo translations.*
Denote by $\varphi_c(x)$ the solution to (33) such that $\varphi(0) = 1/2$. If $c > c_*$, there exists $k_c > 0$ and $\delta > 0$ such that

$$\varphi_c(x) = k_c e^{-\lambda_-(c)x} + O(e^{-(\lambda_-(c)+\delta)x}) \quad \text{as } x \to +\infty. \tag{34}$$

If $c = c_*$, there is $k_* \in \mathbb{R}$ such that

$$\varphi_{c_*}(x) = (x + k_*)e^{-\lambda_* x} + O(e^{-(\lambda_*+\delta)x}) \quad \text{as } x \to +\infty. \tag{35}$$

Existence is due to (Diekman [10], Coville [7], Coville [8]. Uniqueness in certain cases can be found in [10], the full uniqueness result and the asymptotic behaviour being due to Carr-Chmaj [6]).

2.3 Some notations for the sequel

As we will need to observe the solutions of (1) in reference frames moving with various speeds, we set, for any $c \geq c_*$:

$$\mathcal{N}_c u = \mathcal{J}u - c\partial_x u - f(u), \tag{36}$$

and we denote by $\mathcal{L}_c$ the linear operator

$$\mathcal{L}_c\phi = \mathcal{J}\phi - c\partial_x\phi - f'(0)\phi. \tag{37}$$

We denote it by $\mathcal{L}_*$ if $c = c_*$.

$$\partial_t u + \mathcal{N}_c u = 0, \tag{38}$$

or, equivalently,

$$\partial_t u + \mathcal{L}_c u + g(u) = 0. \tag{39}$$

In a similar fashion as above, we denote by $\mathcal{N}_*$ the nonlinear operator with $c = c_*$.
The driving idea in the sequel that 0 is the most unstable value of the range of u, up to the fact that the aspect of $u(t,x)$ in the range where it is almost 0 will have to be scrutinised a little more precisely. In any case, the first move to make is to run at the correct speed, that is c_*. We obtain the equivalent equations (38) or (39), with $c = c_*$. In order to uncover the small values of u, we

remove the exponential at which it is expected to decay, namely the one with exponent λ_*. Let us introduce some notations specific to $c = c_*$; we set, for short :

$$K_*(x) = e^{-\lambda_* x} K(x), \quad \mathcal{J}_* = \int_{\mathbb{R}} \mathcal{J}_*. \tag{40}$$

Note that, even is K is symmetric, K_* is not symmetric anymore. The next step is to set

$$u(t,x) = e^{-\lambda_*} v(t,x), \tag{41}$$

so that the equation for v becomes

$$v_t + \mathcal{J}_* v - c_* v_x + e^{\lambda_* x} g(e^{-\lambda_* x} v), \tag{42}$$

We set

$$h(x,v) = \frac{e^{\lambda_* x} g(e^{-\lambda_* x} v)}{v}, \tag{43}$$

we have, for $x \geq 0$ and v in a bounded set :

$$h(x,v) \geq 0, \qquad h(x,v) \lesssim e^{-\lambda_* x} v. \tag{44}$$

We need to exploit a little more that c_* is the critical speed, for that we recall that c_* is the least c such that there is a solution λ to the equation

$$c\lambda + f'(0) = \int_{\mathbb{R}} e^{\lambda(x-y)} K((x-y)) dy,$$

so that

$$c_* = \int_{\mathbb{R}} (x-y) K_*((x-y)) dy.$$

Let $\mathcal{I}_*$ be the integro-differential operator

$$\mathcal{I}_* v(x) = \int_{\mathbb{R}} K_*(x-y) \big(v(x) + (y-x) v_x(x) - v(y) \big) dy, \tag{45}$$

the equation for v we will work with is thus

$$\partial_t v + \mathcal{I}_* v + h(x,v) v = 0, \tag{46}$$

with the initial datum

$$v_0(x) = e^{\lambda_* x} u_0(x). \tag{47}$$

3 Asymptotics for the heat kernel

3.1 The Dirichlet heat equation

We gather here some exceedingly standard, it is nevertheless useful to the crucial part of the analysis of $v(t,x)$, namely in the area $x \sim t^\gamma$ with $\gamma \in (0, \frac{1}{2})$. So, we decide to describe, in a few lines, what is going to be useful to us. Consider the solution $w(t,x)$ of the Dirichlet heat equation on the half line

$$\begin{aligned} \partial_t w - d_* \partial_{xx} w = &\ 0 \ \ (t > 0,\ x > 0) \\ w(t,0) = &\ 0 \\ w(0,x) = &\ v_0(x) \text{ compactly supported.} \end{aligned} \tag{48}$$

Let v_0^* be the odd extension of v_0 to $\mathbb{R}$, that is, the odd function coinciding with v_0 on $\mathbb{R}_+$; then $w(t,x) = e^{t\partial_{xx}} v_0^*(x)$, that is, the heat kernel of the whole line applied to v_0^*. The expression of $w(t,x)$ is therefore

$$w(t,x) = \frac{1}{\sqrt{4\pi d_* t}} \int_0^{+\infty} \left(e^{-\frac{(x-y)^2}{4d_* t}} - e^{-\frac{(x+y)^2}{4d_* t}} \right) v_0(y) dy. \tag{49}$$

Inspection of this integral yields, setting $\eta = \dfrac{x}{\sqrt{d_* t}}$:

$$w(t,x) \sim_{t\to+\infty} \frac{\eta}{2\sqrt{\pi} d_* t} e^{-\frac{\eta^2}{4}}. \tag{50}$$

The asymptotic slope of $w(t,x)$ is therefore the first moment of v_0 ; in more precise terms we have, for every $\gamma \in (0, \dfrac{1}{2})$:

$$\lim_{t\to+\infty} t^{3/2-\gamma} w(t,t^\gamma) = \frac{1}{2\sqrt{\pi} d_*^{3/2} t^{3/2}} \int_0^{+\infty} y v_0(y) dy. \tag{51}$$

All this is readily seen on formula (49), by the change of variables $z = \dfrac{y}{\sqrt{t}}$, and the fact that v_0 is compactly supported. This simple formula will be our holy grail for the solution $v(t,x)$ of the full Fisher-KPP problem, possibly at the expense of restricting γ a little. When the initial datum is well spread, that is, $v_0(y) = w_0(\dfrac{y}{\sqrt{s}})$, then $e^{td_*\partial_{xx}} v_0$ depends on how the relative values of t and s, as is seen on the formula

$$w(t,x) = \sqrt{\frac{s}{4\pi d_* t}} \int_0^{+\infty} \left(e^{-\frac{s}{4d_* t}(\frac{x}{\sqrt{s}} - \zeta)^2} - e^{-\frac{s}{4d_* t}(\frac{x}{\sqrt{s}} + \zeta)^2} \right) w_0(\zeta) d\zeta. \tag{52}$$

One may for instance see the following properties. If $s \gg t$, and if $\dfrac{x}{\sqrt{s}} \sim s^\gamma$, we have

$$w(t,x) = e^{\frac{s}{t} d_* \partial_{xx}} v_0(x) \big(1 + O(e^{-\frac{s^\gamma}{t}})\big). \tag{53}$$

In particular, it may be seen as a regularisation of v_0 at the scale $\sqrt{s}$. When t becomes significantly larger than s, $w(t,x)$ behaves as the solution of (48) and we have

$$w(t,x) \sim_{\substack{t\to+\infty \\ x=O(\sqrt{t})}} \frac{sx}{2\sqrt{\pi} d_*^{3/2} t^{3/2}} \int_0^{+\infty} \zeta w_0(\zeta) d\zeta. \tag{54}$$

If $w_0(\zeta)$ is proportional to $\dfrac{1}{s}$, formula (54) is compatible with (50).

3.2 The main estimate

Our goal is to analyse, for t large, the exponential of $\mathcal{I}_*$, namely $e^{-t\mathcal{I}_*} v_0$.

Theorem 3.1 *Set*

$$d_* = \frac{1}{2} \int_{\mathbb{R}} z^2 e^{-\lambda_* z} K(z) dz. \tag{55}$$

There is a nonnegative kernel $G_*(t,z)$, defined for $t\geq 1$ and $z\in\mathbb{R}$ and bounded on its domain of definition, such that we have, for all $\gamma\in(0,1/2)$, for all $t\geq 1$ and $x\in\mathbb{R}$:

$$\|e^{-t\mathcal{I}_*}v_0 - G_*(t,.)*v_0\|_{L^\infty(\mathbb{R})} \lesssim e^{-t^{1-2\gamma}}\|v_0\|_{L^2(\mathbb{R})}. \tag{56}$$

Moreover, for every $\delta\in(0,1/2-\gamma)$ we have the following estimates.

— *If $|z|\leq t^{1/2+\delta}$,*

$$G_*(t,z) = \big(1+O(\frac{1}{t^{1-\delta}})\big)\frac{e^{-z^2/4d_*t}}{\sqrt{4\pi d_* t}}. \tag{57}$$

— *If $|z|\geq t^{1/2+\delta}$, then*

$$G_*(t,z)\lesssim e^{-t^\delta}. \tag{58}$$

Proof. As Problem (55) is homogeneous in x, we gladly resort to Fourier transforms once again. Let $\hat{P}_*(t,\xi)$ the Fourier transform in x of $e^{-t\mathcal{I}_*}v_0$. We have, with the notation (**??**) :

$$\begin{aligned}\hat{P}_*(t,\xi) =& \quad \hat{v}_0(\xi)\mathrm{exp}\bigg(-t\big(\int_{\mathbb{R}}K_*(z)dz - i\xi\int_{\mathbb{R}}zK_*(z)dz - \hat{K}_*(\xi)\big)\bigg)\\ =& \quad \hat{v}_0(\xi)\mathrm{exp}\bigg(-t\big(\hat{K}_*(0)+\xi\hat{K}_*'(0)-\hat{K}_*(\xi)\big)\bigg).\end{aligned} \tag{59}$$

Let us examine the phase in the exponential away from $\xi=0$. As K_* is compactly supported, $\hat{K}_*$ belongs to the Schwartz class, so that there is $M_0>0$ such that , for $|\xi|\geq M_0$ we have :

$$|\hat{K}_*(\xi)|\leq\frac{1}{2}\int_{\mathbb{R}}K_* = \frac{\hat{K}_*(0)}{2}. \tag{60}$$

For $|\xi|\leq M_0$, we start from

$$\mathrm{Re}\ \big(\hat{K}_*(0)-\hat{K}_*(\xi)\big) = \int_{-1}^1\big(1-\cos(z\xi)\big)K_*(z)dz.$$

Pick $\gamma\in(0,1/2)$, and

$$t_\gamma = \big(\frac{\pi}{4}\big)^{-1/\gamma}.$$

For $t\geq t_\gamma$ and $|\xi|\in(t^{-\gamma},M]$, we use $\cos u\lesssim -u^2$ for u close to $t^{-\gamma}$, together with the fact that $u\mapsto\cos u$ is less than 1, while not identically equal to 1. We obtain :

$$\int_{-1}^1\big(1-\cos(z\xi)\big)K_*(z)dz\gtrsim t^{-2\gamma}. \tag{61}$$

Putting everything together we obtain, for $|\xi|\geq t^{-\gamma}$:

$$\mathrm{Re}\ \big(\hat{K}_*(0)-\hat{K}_*(\xi)\big)\gtrsim t^{-2\gamma}. \tag{62}$$

This calls for the decomposition

$$\hat{P}_*(t,\xi) = \big(\mathbf{1}_{|\xi|\leq t^{-\gamma}}+\mathbf{1}_{|\xi|\geq t^{-\gamma}}\big)P_*(t,\xi) := \hat{Q}_*(t,\xi)+\hat{R}_*(t,\xi). \tag{63}$$

From (62) we have

$$|\hat{R}(t,\xi)|\lesssim e^{-t^{1-2\gamma}}.$$

Therefore, if $\tilde{R}_*(t)$ denotes, according to the notation (**??**), the distributional conjugate Fourier transform of $\hat{R}_*(t,.)$ (that is, a linear continuous operator in $L^2(\mathbb{R}_x)$), we have, by Plancherel's equality :

$$\|\tilde{R}_*(t)v_0\|_{L^2(\mathbb{R}_x)} = \|\hat{R}_*(t,.)\|_{L^2(\mathbb{R}_\xi)} \lesssim e^{-t^{1-2\gamma}}\|\hat{v}_0\|_{L^2_\xi} = e^{-t^{1-2\gamma}}\|v_0\|_{L^2_x}.$$

It remains to study $Q_*(t,x)$, the inverse Fourier transform of $\hat{Q}_*(t,\xi)$. Let us write

$$\hat{K}_*(0) + \xi\hat{K}_*'(0) - \hat{K}_*(\xi) = -d_*\xi^2 - \frac{\hat{K}_*'''(0)\xi^3}{6} - \frac{\xi^4}{6}\int_0^1 (1-\sigma)^3\hat{K}_*^{(4)}(\sigma\xi)d\sigma,$$

so that we have, setting $\zeta = \sqrt{t}\xi$:

$$Q_*(t,x) = \frac{1}{2\pi\sqrt{t}}\int_{-t^{1/2-\gamma}}^{t^{1/2-\gamma}}\int_{\mathbb{R}} \exp\Big(\frac{i\zeta(x-y)}{\sqrt{t}} - d_*\zeta^2 - \frac{\hat{K}_*'''(0)\zeta^3}{6\sqrt{t}} - \frac{\zeta^4}{6t}\int_0^1 (1-\sigma)^3\hat{K}_*^{(4)}(\frac{\sigma\zeta}{\sqrt{t}})d\sigma\Big)v_0(y)dy\zeta.$$

By Fubini's Theorem, we gave $Q_*(t,.) = G_*(t,.) * v_0$, with

$$G_*(t,z) = \frac{1}{2\pi\sqrt{t}}\int_{-t^{1/2-\gamma}}^{t^{1/2-\gamma}} e^{\Phi_*(t,z,\zeta)}d\zeta, \tag{64}$$

with

$$\Phi_*(t,z,\zeta) = i\frac{z\zeta}{\sqrt{t}} - d_*\zeta^2 - \frac{\hat{K}_*'''(0)\zeta^3}{6\sqrt{t}} - \frac{\zeta^4}{6t}\int_0^1 (1-\sigma)^3\hat{K}_*^{(4)}(\frac{\sigma\zeta}{\sqrt{t}})d\sigma. \tag{65}$$

Inspired by the computation of the Fourier transform of the Gaussian, we want to change the line of integration. Define the line

$$\Gamma_* = \{\zeta = \eta + \frac{iz}{2d_*\sqrt{t}}, \quad -t^\gamma \le \eta \le t^\gamma\}, \tag{66}$$

If $\hat{K}_*$ was a quadratic polynomial, as in the Gaussian case, the effect of this shift would be to make Φ_* real, equal to $-d_*\eta^2 - \dfrac{z^2}{4d_*t}$ and this would finish the computation. However there is a slight glitch, that is, $\hat{K}_*$ is not a quadratic polynomial. And so, if we wish to mimick the classical computation, we have to make sure that the non-quadratic part of Φ_* is actually negligible. As a consequence, we will have to worry about the size of z, and this will also constrain γ. The constraints are therefore :

$$\frac{1}{\sqrt{t}}(\eta + \frac{iz}{2d_*\sqrt{t}})^3 \ll 1, \ \frac{1}{t}(\eta + \frac{iz}{2d_*\sqrt{t}})^4 \ll 1, \quad \text{with } |\eta| \le t^{1/2-\gamma}.$$

It is enough to impose

$$4(\frac{1}{2} - \gamma) < \frac{1}{2}, \quad \frac{|z|^4}{t^3} \ll 1. \tag{67}$$

So, we choose $\delta > 0$ small enough so that (67) is fulfilled with

$$\gamma < \frac{1}{2} - \delta, \quad \frac{|z|}{\sqrt{t}} \le t^\delta. \tag{68}$$

Let us deal at once with z outside the range (67). If $z \ge 0$, define the vertical segments

$$\Gamma_\pm = \{\zeta = \pm t^\gamma + i\eta, \quad 0 \le \eta \le 1\}.$$

and the horizontal line

$$\Gamma_* = \{\zeta = \eta + i, \quad -t^\gamma \le \eta \le t^\gamma\}.$$

If $z \le 0$ we replace $\eta + i$ by $\eta - i$, and $\pm t^\gamma + i\eta$ by $\pm t^\gamma - i\eta$. We have

$$\int_{-t^{1/2-\gamma}}^{t^{1/2-\gamma}} e^{\Phi_*(t,z,\zeta)} d\zeta = \int_{\gamma_- \cup \Gamma_* \cup \gamma_+} e^{\Phi_*(t,z,\zeta)} d\zeta. \tag{69}$$

For $z \ge 0$ (the same would of course also work for $z \le 0$), the following holds.

— On $\Gamma_\pm$, we have

$$\begin{aligned} \Phi_*(t,z,\zeta) = & \pm izt^{-\gamma} - \frac{\eta z}{\sqrt{t}} - d_*(\pm t^{1/2-\gamma} + i\eta)^2 + O(t^{\gamma-1/2}), \text{ so that} \\ \text{Re } \Phi_*(t,z,\zeta) \le & -d_* t^{1-2\gamma}(1 + O(1)). \end{aligned}$$

— On Γ_* we have

$$\begin{aligned} \Phi_*(t,z,\zeta) = & -\frac{z}{\sqrt{t}} + \frac{i\eta z}{\sqrt{t}} - d_*(\eta + i)^2 + O(t^{\gamma-1/2}), \text{ so that} \\ \text{Re } \Phi_*(t,z,\zeta) \le & -\frac{z}{\sqrt{t}} + O(1) \\ \le & -t^\delta + O(1). \end{aligned}$$

Consequently we have

$$\left|\int_{-t^{1/2-\gamma}}^{t^{1/2-\gamma}} e^{\Phi_*(t,z,\zeta)} d\zeta\right| \lesssim e^{-t^\delta},$$

and this proves (58).

Finally, let us deal with z in the range (67). This time, if $z \ge 0$, define the vertical segments

$$\Gamma_\pm = \{\pm t^\gamma + i\eta, \quad 0 \le \eta \le \frac{z}{2d_*\sqrt{t}}\}.$$

while the horizontal line Γ_* is defined by (66). This time, the definition does not change if $z \le 0$. On $\Gamma_\pm$ we have

$$\begin{aligned} \text{Re } \Phi_*(t,z,\zeta) = & -\frac{z\eta}{\sqrt{t}} - d_*(t^{1-2\gamma} - t^{2\delta}) + O(t^{\gamma-1/2}) \\ \le & -d_* t^{1-2\gamma}(1 + o_{t\to+\infty}(1)), \text{ as } \delta < 1/2 - \gamma. \end{aligned}$$

As formula (69) is valid with the new $\Gamma_\pm$ and the new Γ_*, the above computation accounts for the contribution of $\int_{\Gamma_- \cup \Gamma_+} e^{\Phi_*(t,z,\zeta)} dz$ in the area $|z| \ge t^{1/2+\delta}$. And so, what is left to us now is the Gaussian-like part. Recalling that we have this time $\zeta = \eta + \frac{iz}{2d_*\sqrt{t}}$ we decompose $\Phi_*(t,z,\zeta)$ into

$$\Phi_*(t,z,\zeta) = -d_*\eta^2 - \frac{z^2}{4d_* t} - \psi_*(t,z,\eta).$$

From (65), and remembering that $|z| \le t^{1/2+\delta}$, we estimate ψ_* as

$$\begin{aligned} \psi_*(t,z,\eta) = & \frac{\hat{K}_*'''(0)\zeta^3}{6\sqrt{t}} + \frac{\zeta^4}{6t}\int_0^1 (1-\sigma)^3 \hat{K}_*^{(4)}(\frac{\sigma\zeta}{\sqrt{t}}) d\sigma \\ = & \frac{\hat{K}_*'''(0)}{6\sqrt{t}}(\eta + \frac{iz}{2d_*\sqrt{t}})^3 + O(t^{1-4\gamma}) \\ = & \frac{\hat{K}_*'''(0)\eta^3}{6\sqrt{t}} + O(t^{1-4\gamma}), \end{aligned}$$

where we have taken the worst estimate for z and η, according to the constraint (65) and the fact that $|\eta| \leq t^{1/2-\gamma}$. Also notice that we have $\psi_*(t,z,\eta)$ is an $o_{t+\infty}(1)$ uniformly in $z \in (-t^{1/2+\delta}, t^{1/2}+\delta)$ and $\eta \in (t^{1/2-\gamma}, t^{1/2+\gamma})$, as it decays at most like $1/t^{-\gamma}$. With this final reduction, we may compute G_* in the zone $|z| \leq t^{1/2+\delta}$. We remember that, in the original integral expanding the exponential as :

$$\begin{aligned}
G_*(t,z) = & \frac{1}{2\pi\sqrt{t}} \int_{-t^{1/2-\gamma}}^{t^{1/2-\gamma}} \exp\Big(-d_*\eta^2 - \frac{z^2}{4d_*\sqrt{t}} - \psi_*(t,z,\eta)\Big) d\eta \\
= & \frac{1}{2\pi\sqrt{t}} \int_{-t^{1/2-\gamma}}^{t^{1/2-\gamma}} \Big(1 + \frac{\hat{K}_*'''(0)\eta^3}{6\sqrt{t}} + O(t^{1-4\gamma}) + O(\psi_*(t,z,\eta))^2\Big) e^{-d_*\eta^2 - z^2/4d_*t} d\eta \\
= & \frac{e^{-z^2/4d_*t}}{2\pi\sqrt{t}} \int_{-t^{1/2-\gamma}}^{t^{1/2-\gamma}} \Big(1 + \frac{\hat{K}_*'''(0)\eta^3}{6\sqrt{t}} + O(t^{1-4\gamma}) + O(\psi_*(t,z,\eta))^2\Big) e^{-d_*\eta^2} d\eta \\
= & \frac{e^{-z^2/4d_*t}}{2\pi\sqrt{t}} \int_{-t^{1/2-\gamma}}^{t^{1/2-\gamma}} \Big(1 + O(t^{1-4\gamma}) + O(\psi_*(t,z,\eta))^2\Big) e^{-d_*\eta^2} d\eta \quad \text{by the oddness of } \eta \mapsto \eta^3 \\
= & \big(1 + O(\frac{1}{t^{4\gamma-1}}) + O(\frac{1}{t^{-2\gamma}})\big) \frac{e^{-z^2/4d_*t}}{\sqrt{4\pi d_* t}}.
\end{aligned}$$

As γ is as close to $1/2$ as we wish, and $\delta < 1/2 - \gamma$, this finishes the proof of the estimate of G_*. □

Remark 3.2 *An easy consequence of Theorem 3.1 is the following estimate, similar to that of the classical heat equation :*

$$\|e^{-t\mathcal{I}_*} v_0\|_{L^2(\mathbb{R})} \lesssim \frac{1}{t^{1/4}} \|v_0\|_{L^2(\mathbb{R})}. \tag{70}$$

Remark 3.3 *One may wonder why such a universal behaviour arises for the heat kernel. In fact, Theorem 3.1 has nothing surprising : in spirit, its proof is the same as that of the classical Central Limit Theorem, which essentially assumes, in the landscape, a characteristic function that one can expand near 0. It is all the less surprising that the full model (1) has a probabilistic interpretation, at least for some very special functions f.*

3.3 The solution emanating from a well-spread datum

The objective of this section is to devise specific estimates for $e^{-\tau\mathcal{I}_*} v_s$, where v_s is spread over the characteristic length $\sqrt{s} \gg 1$, for times τ of the form $0 \leq \tau \leq s^\varepsilon$, with $\varepsilon \ll 1$. The reason is that Theorem 3.1 will be sufficient to devise bounds on the solution $v(t,x)$ of the full problem (47) at the door of the diffusive zone, that will be of the correct order of magnitude in time. And so, that will be enough to locate the level sets of v with $O_{t\to+\infty}(1)$ precision. However, in order to reach the next order $o_{t\to+\infty}(1)$, we will need to update our observations at various large times, and Theorem 3.1 will not be able to handle properly these updates on times ranges slightly larger than s, say, a very small power of s. This gap is made up for by Theorem 3.4 below.

Theorem 3.4 *Consider a function v_0 belonging to $H^m(\mathbb{R})$, for all integer m. Consider $s > 0$, a real number having the possibility of being very large. Set*

$$v_s(y) = v_0(\frac{y}{\sqrt{s}}). \tag{71}$$

Pick $\omega \in (0, 10^{-10})$. Then, there is a (possibly very large) integer m_ω such that, for $\tau \in (0, s^{1/2-\omega})$ we have :

$$\big|e^{-\tau\mathcal{I}_*} v_s(x) - e^{\tau d_* \partial_{xx}} v_s(x)\big| \lesssim \frac{\tau}{s^{3/2-4\omega}} \big(\|v_0\|_{L^1} + \|v_0\|_{H^{m_\omega}}\big). \tag{72}$$

Proof. The strategy for proving the theorem is essentially the same as that for Theorem 3.1 : a Fourier integral will be subject to various changes of variables that will allow an asymptotic expansion of the phase, followed by a passage in the complex plane. There will, however, be an important difference : the passage to complex variables will have, in order to keep the computation meaningful, to occur at a later stage of the computations, thus leading to a lesser precision than that provided by Theorem 3.1.
In any case, the proof starts in the same way as for Theorem 3.1. We set

$$e^{-\tau\mathcal{I}_*}v_s(x) = Q_*(s,\tau;x) + R_*(s,\tau;x),$$

with

$$\begin{aligned} Q_*(s,\tau;x) = & \frac{1}{2\pi}\int_{|\xi|\leq s^{\omega-1/2}} e^{ix\xi-\tau\big(\hat K_*(0)+\xi\hat K_*(0)-\hat K(\xi)\big)}\hat v_s(\xi)d\xi \\ = & \frac{1}{2\pi}\int_{|\xi|\leq s^{\omega-1/2}} e^{i(x-y)\xi-\tau\big(\hat K_*(0)+\xi\hat K_*(0)-\hat K(\xi)\big)}v_s(y)d\xi dy \\ = & \frac{s1}{2\pi}\int_{|\zeta|\leq s^{\omega}} e^{i(\frac{x}{\sqrt{s}}-z)\zeta-\tau\big(\hat K_*(0)+\frac{\zeta}{\sqrt{s}}\hat K_*(0)-\hat K(\frac{\zeta}{\sqrt{s}})\big)}v_0(z)dzd\zeta. \end{aligned} \tag{73}$$

We have not written explicitely the dependence in v_0 or v_s. The integral R_* is readily estimated, using the classical identity $\hat v_s(\xi) = \sqrt{s}\hat v_0(\sqrt{s}\xi)$, the fact that v_0 is in every H^m, and that the real part of $\hat K_*(0) - \hat K_*(\xi)$ is nonnegative :

$$\begin{aligned} R_*(s,\tau;x) \leq & \frac{\sqrt{s}}{2\pi}\int_{|\xi|\geq s^{\omega-1/2}} \hat v_0(\sqrt{s}\xi)d\xi \\ = & \frac{s1}{2\pi}\int_{|\zeta|\geq s^{\omega}}\hat v_0(\zeta)d\zeta \\ \leq & \frac{s1}{2\pi}\Big(\int_{|\zeta|\geq s^{\omega}}\frac{d\zeta}{1+\zeta^{2m}}\Big)^{1/2}\Big(\int_{\mathbb{R}}(1+\zeta^{2m})|\hat v_0(\zeta)|^2d\zeta\Big)^{1/2} \\ \lesssim & s^{-\omega m}\|v_0\|_{H^m}. \end{aligned} \tag{74}$$

Choosing $m = \dfrac{2}{\omega}$ yields estimate (72) for R_*. The integral Q_* is a little more involved, which should be no surprise in view of Theorem 3.1. Let us set

$$Q_*(s,\tau;x) = \frac{1}{2\pi}\int_{\mathbb{R}} G_*(s,\tau;\frac{x}{\sqrt{s}}-y)v_0(y)dy, \tag{75}$$

with, due to the change of variable $\zeta = \sqrt{s}\xi$:

$$G_*(s,\tau;z) = \int_{-s^{\omega}}^{\omega}\exp\Big(iz\zeta - \tau\big(\hat K_*(0) + \frac{\zeta}{\sqrt{s}}\hat K_*(0) - \hat K_*(\frac{\zeta}{\sqrt{s}})\big)\Big)d\zeta. \tag{76}$$

Let $\Phi_*(s,\tau;z,\zeta)$ be the phase in the exponential ; as in Theorem 3.1 we write

$$\Phi_*(s,\tau;x,\zeta) = iz\zeta - \frac{\tau d_*\zeta^2}{s} - \psi_*(s,\tau;z\zeta), \tag{77}$$

with, this time :

$$\psi_*(s,\tau;z,\zeta) = \frac{\tau\hat K_*'''(0)\zeta^3}{6s^{3/2}} + \frac{\tau\zeta^4}{6s^2}\int_0^1(1-\sigma)^3\hat K_*^{(4)}(\frac{\sigma\zeta}{\sqrt{s}})d\sigma. \tag{78}$$

This entails, as ζ is in the range $(-s^\omega, s^\omega)$:

$$\begin{aligned} e^{\Phi_*(s,\tau;z,\zeta)} &= \Big(1-\psi(s,\tau;z,\zeta)+O\big(\psi(s,\tau;z,\zeta)\big)^2\Big)\exp\big(iz\zeta-\frac{\tau d_*\zeta^2}{s}\big) \\ &= \Big(1-\frac{\tau\hat{K}_*'''(0)\zeta^3}{6s^{3/2}}\Big)\exp\big(iz\zeta-\frac{\tau d_*\zeta^2}{s}\big)+O(\frac{\tau}{s^{2-4\omega}}). \end{aligned}$$

This is where we give up the possibility of moving the integration path, at the expense of a less precise error bound. The reward is that it enables an easy computation of G_*. Notice, that, in the second line below, that the difference between the integral over $(-s^\omega, s^\omega)$ is exponentially small in s^ω, so that it is absorbed in the $O(\frac{\tau}{s^{2-4\omega}})$. Once we have realised this, we may shift the integration path from $\mathbb{R}$ to $\mathbb{R}+\frac{isz}{2d_*\tau}$ with our mind at peace, and write the following equality :

$$G_*(s,\tau,z)=\int_{-s^\omega}^{\omega}\Big(1-\frac{\tau\hat{K}_*'''(0)\zeta^3}{6s^{3/2}}\Big)\exp\big(iz\zeta-\frac{\tau d_*\zeta^2}{s}\big)d\zeta+O(\frac{\tau}{s^{2-4\omega}}).$$

We simply estimate the ζ^3 integral as :

$$\Big|\frac{s1}{s^{3/2}}\int_{-s^\omega}^{\omega}\zeta^3\exp\big(iz\zeta-\frac{\tau d_*\zeta^2}{s}\big)d\zeta\Big|\lesssim\frac{\tau}{s^{3/2-4\omega}}.$$

Therefore we have

$$\begin{aligned} G_*(s,\tau,z) = &\int_{-s^\omega}^{\omega}\exp\big(iz\zeta-\frac{\tau d_*\zeta^2}{s}\big)d\zeta+O(\frac{\tau}{s^{3/2-4\omega}})=\int_{\mathbb{R}}\exp\big(iz\zeta-\frac{\tau d_*\zeta^2}{s}\big)d\zeta+O(\frac{\tau}{s^{3/2-4\omega}}) \\ = &\sqrt{\frac{\pi s}{d_*\tau}}e^{-\frac{sz^2}{4d_*\tau}}+O(\frac{\tau}{s^{3/2-4\omega}}). \end{aligned}$$

Convolution of $G_*(s,\tau,.)$ with v_s, and changes of variables that are standard at this stage, yield the result. □

3.4 Additional information

As is usually the case when it comes to computing a heat kernel, Theorem 3.1 gives precise information about it behaviour slightly beyond the range of similarity, that is, in the present context, for $\frac{x}{\sqrt{t}}\geq t^\delta$, for $\delta>0$ small. In order to use it in comparisons, one needs to know how it behaves for $|x|$ very large. Here, a precise behaviour is not needed, what is requested is an estimate that will beat the algebraic expressions in t.

Proposition 3.5 *If v_0 is compactly supported, then, for all $A>0$, there exists $B>0$ such that, if $\frac{x}{\sqrt{t+1}}\geq B$, we have*

$$\big|e^{-t\mathcal{I}_*}v_0(x)\big|\leq\|v_0\|_\infty e^{-\frac{Ax}{\sqrt{t+1}}}. \tag{79}$$

Proof. We begin by rephrasing Section 2.2, with the following innocent computation. For $k\in\mathbb{R}$, and $v(x)=e^{kx}$ we have

$$\begin{aligned} \mathcal{I}_*v(x) &= \frac{1}{2}\int_{\mathbb{R}}K_*(x-y)(x-y)^2\int_0^1(1-\sigma)v''\big(x+\sigma(y-x)\big)d\sigma dy \\ &= \frac{k}{2}\int_{\mathbb{R}}zK_*(z)\big(e^{kz}-1\big)dz \\ &= d_*k^2\big(1+\omega(k)\big), \end{aligned} \tag{80}$$

where, this time, $\omega(k)$ is a real analytic function such that $\omega(0) = 0$. Setting, for short, $\xi = \dfrac{x}{\sqrt{t+1}}$ we have, from (80) :

$$\big(\partial_t + \mathcal{I}_*)\big)e^{-\frac{Ax}{\sqrt{t+1}}} = \frac{A}{t+1}\Big(\frac{\xi}{2} - Ad_*\omega\big(\frac{A}{\sqrt{t+1}}\big)\Big)e^{-\frac{x}{\sqrt{t+1}}} \geq 0 \ \text{ if } \xi \geq Ad_*\|\omega\|_{L^\infty([0,1])}.$$

As $e^{-t\mathcal{I}_*}v_0(x) \leq \|v_0\|_\infty$, the maximum principle entails the estimate. □

4 The logarithmic delay : a rough estimate and gradient bounds

The goal of this section is to prove the following theorem.

Theorem 4.1 *We have*

$$u(t,x) = \varphi_{c_*}\big(x - c_*t - \frac{3}{2\lambda_*}\ln\, t + O_{t\to+\infty}(1)\big), \tag{81}$$

uniformly in $x \in \mathbb{R}_+$. A similar statement holds on $\mathbb{R}_-$.

Given estimate (44) on $h(x,v)$ and the guideline that we should know, in priority, the behaviour of v far from the origin (that, is, the small values of u) it is to be expected that (46), linearised around 0, will play an important role. We will endeavour to describe it before attempting to tackle the nonlinear equation at large distance. At the expense of spoiling the suspense, this large distance will be $x \sim t^\gamma$, the positive real number γ being chosen $< 1/2$, but not necessarily small. Once these two issues are understood, we will see how to retrieve the information to bounded x, thus proving the theorem.
The consequence of this result will be L^∞ bounds for all the derivatives of v.

Theorem 4.2 *Assume that $u_0 \in C^\infty(\mathbb{R})$. For all $m \in \mathbb{N}^*$, the quantity $\|\partial_x^m u(t,.)\|_{L^\infty(\mathbb{R})}$ is bounded.*

Given estimate (44) on $h(x,v)$ and the guideline that we should know, in priority, the behaviour of v far from the origin (that, is, the small values of u) it is to be expected that (46), linearised around 0, will play an important role. We will endeavour to describe it before attempting to tackle the nonlinear equation at large distance. At the expense of spoiling the suspense, this large distance will be $x \sim t^\delta$, with $\delta > 0$ small and, in any case, $< 1/2$. Once these two issues are understood, we will see how to retrieve the information to bounded x, thus proving the theorem.

4.1 Some computations

The study of $v(t,x)$ at $x \sim t^\gamma$ will involve the construction of sub and super solutions built on the previously constructed fundamental solutions, and will be slightly technical. We will also need to understand the behaviour of the solutions in the areas where they are not small, and it turns out that the same sort of computations will be useful. It may thus be good to gather, in a separate section, a set of computational results that will be readily available for the main part of the study.

4.1.1 Two simple identities

The first one that we wish to mention is the analogue of Kato's inequality for the laplacian : for all $v \in W^{2,p}(\mathbb{R}^N)$ we have

$$-\Delta|v| \leq -\text{sgn}(v)\Delta v. \tag{82}$$

Similary to that we have, for all Lipschitz function v on $\mathbb{R}$:

$$\mathcal{I}_*|v| \leq \text{sgn}(v)\mathcal{I}_* v. \tag{83}$$

We have indeed

$$\begin{array}{rl} \text{sgn}(v)\mathcal{I}_* v = & |v| \displaystyle\int K_*(x-y)dy + \partial_x|v| \int (y-x)K_*(x-y)dy \\ & -\text{sgn}(v) \int K_*(x-y)v(y)dy \\ \geq & |v| \displaystyle\int K_*(x-y)dy + \partial_x|v| \int (y-x)K_*(x-y)dy - \int K_*(x-y)|v(y)|dy \\ = & \mathcal{I}_*|v|. \end{array} \tag{84}$$

If now $\text{sgn}^+(v)$ equals 1 if $v > 0$, and 0 if $v \leq 0$, the identity

$$v^+ = \frac{|v| + v}{2} = \text{sgn}^+(v)v, \tag{85}$$

yields the inequality

$$\mathcal{I}_* v^+ \leq \text{sgn}^+(v)\mathcal{I}_* v. \tag{86}$$

We mention the following result property, valid for two Lipschitz functions $u(x)$ and $v(x)$:

$$\mathcal{I}_*(uv)(x) = u(x)\mathcal{I}_* v(x) + v(x)\mathcal{I}_* u(x) - \int_{\mathbb{R}} K_*(x-y)\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)dy. \tag{87}$$

The proof is trivial and left to the reader, who should convince him/herself that this identity is the exact analogue of that for the Laplacian :

$$(-\Delta u)v = u(-\Delta v) - 2\nabla u.\nabla v + u(-\Delta v), \tag{88}$$

the operator $\mathcal{I}_*$ playing the role of $(-\Delta)$.

4.1.2 Barriers around the origin

We apply Computation (80) to the following situation : pick $A > 1$ and $\alpha \in (0, 1/2)$. In (80), we take $k = i(1+t)^{-\alpha}$; there are two real analytic functions $\omega_j(k)$ satisfying $\omega_j(0) = 0$ such that :

$$\begin{array}{rl} & \big(\partial_t + \mathcal{I}_*\big)\big((1+t)^{-A}\cos(\dfrac{x}{(1+t)^\alpha})\big) \\ = & \text{Re}\left(\big(\partial_t + \mathcal{I}_*\big)\big((1+t)^{-A}e^{ix/(1+t)^\alpha}\big)\right) \\ = & \text{Re}\left[\left(\dfrac{d_*}{(1+t)^{2\alpha}}\big(1+\omega_1((1+t)^{-\alpha})\big) - \dfrac{A}{1+t} - i\big(\dfrac{\alpha x}{(1+t)^{1+\alpha}} - \dfrac{d_*\omega_2((1+t)^{-\alpha})}{(1+t)^{2\alpha}}\big)\right)\dfrac{e^{ix/(1+t)^\alpha}}{(1+t)^A}\right] \\ & = (1+t)^{-A}\left(\dfrac{d_*}{(1+t)^{2\alpha}}\big(1+\omega_1((1+t)^{-\alpha})\big) - \dfrac{A}{1+t}\right)\cos(\dfrac{x}{(1+t)^\alpha}) \\ & +(1+t)^{-A}\big(\dfrac{\alpha x}{(1+t)^{1+\alpha}} - \dfrac{d_*\omega_2((1+t)^{-\alpha})}{(1+t)^{2\alpha}}\big)\sin(\dfrac{x}{(1+t)^\alpha}). \end{array}$$

This computation, in view of the targetted expression (93), tells us how we are going to control $v(t,x)$ in a (large) vicinity of $x = 0$, and where the cosine comes from. We may push the computation a little more, as we are interested in a restriction of $\cos(\dfrac{x}{(1+t)^\alpha})$. Notice indeed that $\cos(\dfrac{x}{(1+t)^\alpha})$

vanishes at $x = \frac{3\pi(1+t)^\alpha}{s2}$, and remains nonnegative from there until $x = \frac{5\pi(1+t)^\alpha}{s2}$, that is, well outside the support of K_*. Thus we may apply Proposition 2.1 to infer that, for $1 \leq x \leq \frac{3\pi(1+t)^\alpha}{2}$, we have

$$\begin{aligned} & (1+t)^A(\partial_t + \mathcal{I}_*)\Big(\mathbf{1}_{(-\infty,\frac{3\pi(1+t)^\alpha}{2}]}\big((1+t)^{-A}\cos(\frac{x}{(1+t)^\alpha})\big)\Big) \\ \geq\ & \Big(\frac{d_*}{(1+t)^{2\alpha}}\big(1+\omega_1((1+t)^{-\alpha})\big) - \frac{A}{1+t}\Big)\cos(\frac{x}{(1+t)^\alpha}) \\ & +\big(\frac{\alpha x}{(1+t)^{1+\alpha}} - \frac{d_*\omega_2((1+t)^{-\alpha})}{(1+t)^{2\alpha}}\big)\sin(\frac{x}{(1+t)^\alpha}). \end{aligned} \tag{89}$$

4.2 Analysis at $x \sim t^\gamma$

Theorem 4.3 *For all $\gamma \in (1/3, 1/2)$ we have*

$$\frac{1}{t^{3/2-\gamma}} \lesssim v(t, t^\gamma) \lesssim \frac{1}{t^{3/2-\gamma}}. \tag{90}$$

Another way to formulate (90) is that, as $t \to +\infty$ and $x \in (t^\gamma, t^{1/2})$, the slope of $v(t,x)$ is of the same order as $\frac{s1}{t^{3/2}}$. The main ingredient in the analysis of $v(t,x)$ for large x is a comparison with the heat kernel computed in the preceding section.
The core of the proof of Theorem 4.3 takes its inspiration from the work [11] of Fife and McLeod [11]. It is hard to overstate the importance of this work, the idea contained there deserve a very careful analysis, as their impact goes far beyond bistable reaction-diffusion equations.
Assume v_0 to be supported in $(1, +\infty)$; we always may assume this at the expense of a translation.

Lemma 4.4 *Fix $\alpha \in (\frac{1}{3}, \frac{1}{2})$. There are two bounded positive functions $\xi_0^\pm(t) > 0$, with ξ_0^+ nondecreasing and ξ_0^- nonincreasing, bounded away from 0, such that the following holds.*

- *For $x \geq \frac{3\pi t^\alpha}{2}$, we have*

$$\xi_0^-(t)e^{-t\mathcal{I}_*}v_0^*(x) \leq v(t,x) \leq \xi_0^+(t)e^{-t\mathcal{I}_*}v_0^*(x). \tag{91}$$

- *Fix $\beta \in (0, \frac{1}{2})$. If $-1 \leq x \leq \frac{3\pi t^\alpha}{2}$, we have*

$$\xi_0^-(t)e^{-t\mathcal{I}_*}v_0^*(x) + O(\frac{1}{t^{3/2-\beta}}) \leq v(t,x) \leq \xi_0^+(t)e^{-t\mathcal{I}_*}v_0^*(x) + O(\frac{1}{t^{3/2-\beta}}). \tag{92}$$

4.2.1 The upper barrier

In an ideal world, the solution $v(t,x)$ of (42) would only be controlled from above by the sole function $e^{-t\mathcal{I}_*}v_0(x)$, which is indeed a super-solution. Alas, we do not very well know what the value of $v(t,0)$ is, and, in particular, if it is actually below $e^{-t\mathcal{I}_*}v_0\Big|_{x\in[0,1]}$. Thus we have to compare it to the barrier slightly to the left, say, at a distance t^δ. There, v is exponentially small, thus controllable by $e^{-t\mathcal{I}_*}v_0$. But then, assuming that we have extended v_0 in an odd fashion, $e^{-t\mathcal{I}_*}v_0$ is negative, thus to be supplemented by an additional ingredient in the barrier. This is the role of the cosine function.

So, consider $\delta \in (0, 1/2)$, that will be as small as needed, we will estimate $v(t,x)$ ahead of $x = -t^\delta$. The idea is to construct a barrier function of the form :

$$\overline{v}(t,x) = \xi_0^+(t)e^{-t\mathcal{I}_*}v_0(x) + \frac{1}{(1+t)^A}\cos(\frac{x}{(1+t)^\alpha})\mathbf{1}_{(-\infty, \frac{3\pi(1+t)^\alpha}{2}]}(x), \tag{93}$$

where ξ_0^+ as well as the positive parameters $\alpha \in (\delta, \frac{1}{2})$ and $A > 0$ are to be adjusted in the course of the investigation. The computation, that will be done for $t > 0$ large enough. We have

$$(\partial_t + \mathcal{I}_*)\overline{v} = \dot{\xi}_0^+ e^{-t\mathcal{I}_*}v_0^*(x) + (\partial_t + \mathcal{I}_*)\Big(\mathbf{1}_{(-\infty, \frac{3\pi(1+t)^\alpha}{2}]}\big((1+t)^{-A}\cos(\frac{x}{(1+t)^\alpha})\big)\Big), \tag{94}$$

and three zones are to be singled out. Define $x_0 > 0$ such that $e^{-t\mathcal{I}_*}v_0^*(x) \geq 0$ for all $t \geq 0$ and $x \geq x_0$.

– *The case* $x_0 \leq x \leq \frac{\pi(1+t)^\alpha}{4}$. We have $\cos(\frac{x}{(1+t)^\alpha}) \geq \frac{\sqrt{2}}{2}$, and, for all $A > 0$ and $\alpha \in (0, \frac{1}{2})$ there is $t_{A,\alpha} > t_0$ such that we have, for $t \geq t_{A,\alpha}$:

$$\frac{d_*}{(1+t)^{2\alpha}}\big(1 + \omega_1((1+t)^{-\alpha})\big) - \frac{A}{1+t} \geq \frac{d_*}{2(1+t)^{2\alpha}},$$

and, by possibly by enlarging $t_{A,\alpha}$ we have, from (89) :

$$(\partial_t + \mathcal{I}_*)\big((1+t)^{-A}\cos(\frac{x}{(1+t)^\alpha})\big) \geq \frac{d_*}{2(1+t)^{2\alpha}}. \tag{95}$$

Indeed we have $\frac{x}{(1+t)^{1+\alpha}} \lesssim \frac{1}{t}$. So, if we now request $\dot{\xi}_0^+(t) \geq 0$, we win.

– *The case* $-t^\delta \leq x \leq x_0$. This time, $e^{-t\mathcal{I}_*}$ can be negative, and we have to rely even more on the cosine perturbation. Theorem 3.1 teaches us that $e^{-t\mathcal{I}_*}v_0^*$ is, at worst, distant from $e^{t\partial_{xx}}v_0^*$ by a quantity that decays exponentially in time. Therefore, estimate (51) teaches us in turn that we have, at worst :

$$|\dot{\xi}_0^+(t)e^{-t\mathcal{I}_*}v_0^*(x)| \gtrsim \frac{\dot{\xi}_0^+(t)}{(1+t)^{3/2-\delta}}, \tag{96}$$

while we still have (95). Therefore we should impose

$$\frac{1}{(1+t)^{2\alpha+A}} \gg \frac{\dot{\xi}_0^+(t)}{(1+t)^{3/2-\delta}}. \tag{97}$$

– *The case* $\frac{\pi(1+t)^\alpha}{4} \leq x \leq \frac{3\pi(1+t)^\alpha}{2}$. In this area, the cosine perturbation is negative. However, we may use the full force of $e^{-t\mathcal{I}_*}v_0^*$, which is like $x/t^{3/2}$, with $x \sim (1+t)^\alpha$. So, we require this time

$$\frac{\dot{\xi}_0^+(t)}{(1+t)^{3/2-\alpha}} \gg \frac{1}{(1+t)^{2\alpha+A}}. \tag{98}$$

–*Summary.* From (97) and (98) we should achieve

$$\frac{1}{(1+t)^{3\alpha+A-3/2}} \ll \dot{\xi}_0^+(t) \ll \frac{1}{(1+t)^{2\alpha+\delta+A-3/2}}, \tag{99}$$

which is always possible as soon as $\delta < \alpha$, something we have assumed. It remains to choose A, we devise it to have : for all $x \in [0,1]$, $\overline{v}(t, -t^\delta + x) \geq v(t, -t^\delta + x)$, the last quantity being of the order e^{-t^δ}. Therefore it is enough to have the cosine perturbation dominate $e^{-t\mathcal{I}_*} v_0^*(-t^\delta)$ by a small algebraic order. This is achieved if we choose $A = \dfrac{3}{2} - \beta$, with $\delta < \beta < 2\beta < \alpha$, and

$$\dot{\xi}_0^+(t) = \frac{1}{(1+t)^{3\alpha - 2\beta}}. \tag{100}$$

If $\delta \in (0, 1/2)$ and $\beta < \alpha$ are small enough, we may choose $\alpha > \dfrac{1}{3}$ so that $3\alpha - 2\beta > 1$ and $\dot{\xi}_0^+$ is integrable.
With this, we not only have $\overline{v}(t,x) \geq 0$ everywhere, but there exists $t_0 \geq 0$ such that $\big(\partial_t + \mathcal{I}_*\big)\overline{v}(x) \geq 0$ for $t_0 > 0$, $x \geq -t^\delta$. The only item that remains to be checked is that $\overline{v}(t_0, x)$ is above $v_0(x)$ for $x \geq -1$. This may not be the case ; however, we may put a large multiple of $\overline{v}(t_0, x)$ above $v_0(x)$. So, we have proved the right handsides of (91) and (92).

4.2.2 The lower barrier

In an ideal world once again, we would just try to control $v(t,x)$ by some multiple of $e^{-t\mathcal{I}_*} v_0^*$ from below. Indeed, we do not care about $e^{-t\mathcal{I}_*} v_0^*$ to be negative, so we could think of repeating the above argument, with this time a boundary control near x_0. There is one case where the world is ideal, namely when the function g, entering in the composition (4), is zero on a small range of parameters $(0, \theta)$. Consider indeed the function $\phi_*(x) = x^+ e^{-\lambda_* x}$, it is a steady linear wave solution, positive on $\mathbb{R}_+$, zero on $\mathbb{R}_-$. Now, if $A > 0$ is large enough, the equation $A\phi_*(x) = 1$ has two roots $0 < x_A^- < x_A^+$, and their distance is larger than 1, even if it means enlarging A. Fix such an A, and define

$$\overline{u}(x) = \begin{cases} A\phi_*(x) \text{ if } x \geq x_A^+, \\ 1 \text{ if } x \leq x_A^+ \end{cases} \tag{101}$$

As $x_A^+ - x_A^- > 1$, Proposition 2.1 applies, and $\overline{u}$ is a super-solution to equations (38) or (39) with $c = c_*$, that can additionally be put above u_0. Consider x_0 such that $\overline{u}(x) \leq \theta$ if $x \geq x_0$, we may, at the expense of a translation, assume $x_0 = -2$. Therefore, as $u(t,x) \leq \overline{u}(x)$, we have $g(u(t,x)) \equiv 0$ if $x \geq -2$. The equation for v is therefore the pure diffusive equation, and we may devise the following control of $v(t,x)$ from below. From Theorem 3.1 and Formula (48) for $e^{td_*\partial_{xx}}$ we have

$$e^{-t\mathcal{I}_*} v_0^*(x) \lesssim -\frac{1}{t^{3/2}}\big(1 + o_{t\to+\infty}(1)\big) \text{ for } -2 \leq x \leq -1. \tag{102}$$

Therefore there is $t_0 > 0$ such that $e^{-t\mathcal{I}_*} v_0^* \leq 0$ on $[-2,-1]$ as soon as $t \geq t_0$. For any $\varepsilon > 0$, define

$$\underline{v}_\varepsilon(0,x) = \varepsilon v_0^*(x),$$

as $v(t,x) > 0$ for all $x \in \mathbb{R}$ and $t > 0$ we may choose $\varepsilon > 0$ small enough so that

$$\underline{v}_\varepsilon(0,x) \leq v(1,x) \text{ for } x \geq -2. \tag{103}$$

From that we set

$$\underline{v}_\varepsilon(t,x) = \varepsilon e^{-t\mathcal{I}_*} v_0^*(x), \tag{104}$$

restricting ε further we may assume that

$$\underline{v}_\varepsilon(t,x) \leq v(t+1,x) \text{ for } 1 \leq t \leq t_0 \text{ and } -2 \leq x \leq -1. \tag{105}$$

From (103), (104), (105) and the maximum principle, we have

$$\underline{v}_\varepsilon(t,x) \le v(t+1,x) \ \text{ for } t \ge 0 \text{ and } x \ge -2\ .$$

Another application of Theorem 3.1 and Formula (48) for $e^{td_*\partial_{xx}}$ proves the left handisdes of (91) and (92).

However, the nonlinear term is usually positive. So, if we wish to have a hope to control $v(t,x)$ by $e^{-t\mathcal{I}_*}v_0^*(x)$, we would be well advised to do it in an area where the nonlinear term is negligible. Hence the idea to estimate $v(t,x)$ ahead of t^δ.

In this area, we use the fact that $v(t,x) \le \overline{v}(t,x)$, where $\overline{v}(t,x)$ is the constructed super-solution given by (93). We use the rough estimate that $\overline{v}(t,x)$ is bounded by a constant ; so, if $h(x,v)$ is given by (43), we have, using (44) :

$$\text{for } x \ge t^\delta, \qquad h(x,v(t,x)) \lesssim e^{-t^\delta} := \bar{h}(t). \tag{106}$$

Therefore, any solution $\underline{v}(t,x)$ of

$$\partial_t \underline{v} + \mathcal{I}_* \underline{v} + \bar{h}(t)\underline{v} \le 0, \tag{107}$$

on a domain of the form $\{t \ge t_0,\ x \ge t^\delta\}$ and that satisfies

$$\underline{v}(t_0,x) \ge 0 \text{ for } x \ge 0, \qquad v(t,x) \ge \underline{v}(t,x) \text{ for } t^\delta \le x \le t^\delta + 1$$

will be a barrier for v. Notice that, as $\bar{h}(t)$ is an integrable function, the change of functions $\underline{w}(t,x) = e^{-\int_0^t \bar{h}(s)ds}\underline{v}(t,x)$ allows the search of a subsolution for the sole integral equation $\partial_t w + \mathcal{I}_* w = 0$, in other words the computations of the preceding section remain valid.

And so, a function of the form

$$\underline{v}(t,x) = \xi_0^-(t)e^{-t\mathcal{I}_*}v_0^*(x) - \frac{1}{(1+t)^{3/2-\beta}}\cos\big(\frac{x}{(1+t)^\alpha}\big)\mathbf{1}_{(-\infty,\frac{3\pi(1+t)^\alpha}{2}]}(x), \tag{108}$$

with this time

$$\dot{\xi}_0^-(t) = -\frac{1}{(1+t)^{3\alpha-2\beta}}. \tag{109}$$

is a good candidate. There are, however, one small catch, which is that $\xi_0^-(t)$ should be positive for large times, something that is not guaranteed by (109). It is, however, sufficient to modify the definition (108) by

$$\underline{v}(t,x) = \varepsilon\xi_0^-(t)e^{-t\mathcal{I}_*}v_0^*(x) - \frac{\varepsilon}{(1+t)^{3/2-\beta}}\cos\big(\frac{x}{(1+t)^\alpha}\big)\mathbf{1}_{(-\infty,\frac{3\pi(1+t)^\alpha}{2}]}(x), \tag{110}$$

$\varepsilon > 0$ small. The function $\xi_0^-(t)$ is characterised by $\xi_0^-(0) = \varepsilon$, and $\dot{\xi}_0^-$ given by (109). These modifications ensure the lower barrier property.

4.3 Retrieving the information to $x = O(1)$

The situation is the following : around $x = 0$, the solution $v(t,x)$ of (42) is essentially of the order $\dfrac{1}{t^{3/2}}$. While this is an interesting information, this does not answer the question we are really interested in, that is, the location of a level set of $u(t,x)$ of the original equation (1). Obviously, trying to fish for a level set of $u(t,x)$ of fixed value at this place is bound to fail, and we have to

look a little back in order to see nontrivial values for $u(t,x)$. This is exactly the rationale of the logarithmic delay.
In order to retrieve the information to the back, we need a vehicle, and we have a beautiful one : the travelling wave solution. Let us recall that ϕ_* is the wave with bottom speed, that is, a steady solution of (39) connecting 1 to the left to 0 to the right. We could normalise it once and for all to $\phi_*(0)=\dfrac{1}{2}$, but there is something smarter to do here. From Theorem 2.4, we may normalise it according to its behaviour at $+\infty$, and enforce equation (35) for ϕ_*. In other words we ask :

$$\phi_*(x)\sim_{x\to+\infty}(x+k_*)e^{-\lambda_*x}. \tag{111}$$

We are sure that it is the only one : for any translate of ϕ_*, the exponential in (111) would find itself multiplied by a constant different from 1. This being set once and for all, we are now going to look for two functions $x_-(t)<x_+(t)$ such that

$$\phi_*(x-x_-(t))\leq u(t,x)\leq\phi_*(x-x_+(t))\ \text{ for } -t^\delta\leq x\leq t^\delta. \tag{112}$$

As $x_\pm$ will be much smaller than t^δ, this will locate the nontrivial level sets of u.
Consider $\delta>0$ small, $\beta\in(0,\delta)$ and $\alpha>\dfrac{1}{3}$, close to $\dfrac{1}{3}$. We start from the following consequence of Theorem 42 :

$$u(t,x)\leq\frac{Cxe^{-\lambda_*x}}{t^{3/2}}\ \text{ for } x\in[t^\delta,t^\delta+1]. \tag{113}$$

Let us devise $x_+(t)$ so that $\phi_*(x-x_+(t))\geq u(t,x)$ for $x\in[t^\delta,t^\delta+1]$. Given the equivalent (112), it is sufficient to impose

$$\big(x-x_+(t)+k_*\big)e^{-\lambda_*(x-x_+(t))}\geq\frac{Cxe^{-\lambda_*x}}{t^{3/2}}\ \text{ for } x\in[t^\delta,t^\delta+1],$$

for a constant C that is possibly different from the one in (113). An elementary computation yields

$$x_+(t)\geq-\frac{3}{2\lambda_*}\ln t+K_+, \tag{114}$$

with $K_+>0$ large enough. Similarly, a sufficient condition for a function $x_-(t)$ to satisfy the left handside of (112) is

$$x_-(t)\geq-\frac{3}{2\lambda_*}\ln t-K_-, \tag{115}$$

with, again, $K_->0$ large enough. The functions $x_\pm(t)$ are now fixed as in (114) and (115), with equalities in the place of inequalities, and this is of course where we are going to look for the level sets of $u(t,x)$. The common point between x_+ and x_- being the $\ln t$ term, we make the change of reference frame

$$y=x+\frac{3}{2\lambda_*}\ln t.$$

We rename x the variable y ; in the new reference frame the equation for u is

$$u_t+\frac{3}{2\lambda_*t}u_x+\mathcal{L}_*u-u+g(u)=0, \tag{116}$$

so that the equation for

$$v(t,x)=e^{\lambda_*x}u(t,x) \tag{117}$$

is

$$v_t + \frac{3}{2\lambda_* t}(v_x - v) + \mathcal{I}_* v + h(x,v) = 0,$$

the nonlinearity h being defined by (43). We also set

$$\psi_*(x) = e^{\lambda_* x}\phi_*(x).$$

We study the differences

$$w_\pm(t,x) = v(t,x) - \psi_*(x - K_\pm). \tag{118}$$

The domain of study would be, strictly speaking : $\{-t^\delta + \frac{3}{2\lambda_*}\ln t \leq x \leq t^\delta + \frac{3}{2\lambda_*}\ln t\}$. As t is supposed to be large, we may keep the domain as $\{-t^\delta \leq x \leq t^\delta\}$.
We are going to prove that

$$\limsup_{\substack{t\to+\infty \\ -t^\delta \leq x \leq t^\delta}} w_+(t,x) \leq 0, \quad \limsup_{\substack{t\to+\infty \\ -t^\delta \leq x \leq t^\delta}} w_-(t,x) \geq 0. \tag{119}$$

The equation for $w_\pm(t,x)$ is

$$\partial_t w_\pm + \frac{3}{2\lambda_* t}(\partial_x w_\pm - w_\pm) + \mathcal{I}_* w_\pm + a(t,x) w_\pm = 0,$$

with as usual,

$$a(t,x) = \frac{e^{\lambda_* x}\Big(g\big(e^{-\lambda_* x}v(t,x)\big) - g\big(e^{-\lambda_* x}\psi_*(x-K_\pm)\big)\Big)}{w_\pm(t,x)}.$$

Note that $a(t,x) \geq 0$ because $g' \geq 0$. Let us, for definiteness, examine the large time behaviour of $w_+(t,x)$. The right handside of (119) is equivalent to

$$\lim_{\substack{t\to+\infty \\ -t^\delta \leq x \leq t^\delta}} w_+^+(t,x) = 0, \tag{120}$$

where $w_+^+(t,x)$ is the positive part of $w_+(t,x)$. Let us prove it, we will in fact see that the above limit is uniform in $x \in [-t^\delta, t^\delta]$. Using the analogue (83) of Kato's inequality, we easily derive an inequation for $|w_+^+(t,x)|$:

$$\begin{aligned} \partial_t w_+^+ + \frac{3}{2\lambda_* t}(\partial_x w_+^+ - w_+^+) + \mathcal{I}_* w_+^+ \leq & \ \frac{3}{2\lambda_* t}(\psi_*' + \psi_*)(x - K_+), \quad (-t^\delta \leq x \leq t^\delta) \\ \lesssim & \ \frac{1}{t^{1-\delta}} \\ w_+^+(t,x) = & \ O(e^{-t^\delta}), \quad (-t^\delta \leq x \leq -t^\delta + 1) \\ w_+^+(t,x) = & \ 0, \quad (-t^\delta \leq x \leq t^\delta). \end{aligned} \tag{121}$$

Note that the last line of (121) is perhaps the most crucial one : the region $\{t^\delta - 1 \leq x \leq t^\delta\}$ is indeed the zone where the long range behaviour of $v(t,x)$ and that where the finite range behaviour of $v(t,x)$ communicate. A barrier for w_+^+ will be devised using, once again, the computation toolbox of Section 4.1 will be useful. We pick again $\alpha > \frac{1}{3}$, close to $\frac{1}{3}$, and we keep in mind that δ is small. For $t \geq 1$ we set

$$\overline{w}(t,x) = \frac{1}{t^A}\cos\left(\frac{x}{t^\alpha}\right);$$

from (89) we have, for $t \geq 1$ and $-t^\delta \leq x \leq t^\delta$:

$$\big(\partial_t + \mathcal{I}_*\big)\overline{w}(t,x) \gtrsim \frac{1}{t^{A+2\alpha}},$$

while

$$\frac{1}{t}\big|\partial_x \overline{w}(t,x) - \overline{w}(t,x)\big| \lesssim \frac{1}{t^{A+\alpha+1}}.$$

Therefore, $\overline{w}(t,x)$ is a barrier as soon as $\dfrac{1}{t^{A+2\alpha}}$ dominates the right handside of (121), that is, $\dfrac{1}{t^{1-\delta}}$. This is achieved as soon as δ is small and α slightly larger than $\dfrac{1}{3}$: in this setting there is a room for finding a suitable A; here, any $A < 1 - 2\alpha - \delta$ will work. The barrier $\overline{w}(t,x)$ tending to 0 as $t \to +\infty$, we may put a large multiple of $\overline{w}(t,x)$ above $w_+^+(t,x)$, and this roves (120). As similar considerations can be made for $w_-^-(t,x)$, we have proved (119). This in turn proves the main result of the whole section.

4.4 Bounds on the derivatives of $u(t,.)$

As already mentionned, there is no smoothing mechanism in (1), so that we have to rely on the conservation of initial smoothness and explore a mechanism that prevents the inflation of its derivatives. The starting point is equation for $u(t,x)$ (116) in the reference frame moving like $c_*t - \dfrac{s3}{2\lambda_*}\ln t$. The derivative $u_x(t,x)$ solves

$$\partial_t u_x + u_x - K * u_x - (c_* - \frac{3}{2\lambda_* t})\partial_x u_x = u_x - g'(u)u_x. \tag{122}$$

We define the function $v(t,x)$ as, this time,

$$u_x(t,x) = e^{-\lambda_* x} v(t,x); \tag{123}$$

the function $|v(t,x)|$ solves the equation

$$\partial_t |v| + \mathcal{I}_* |v| - \frac{3}{2t}(\partial_x |v| - |v|) + g'(u)|v| = 0. \tag{124}$$

We always may arrange the picture to be translated by a suitable constant so that there is $q_* > 0$ so that

$$g'\big(u(t,x)\big) \geq q_* \quad \text{for } x \leq 0. \tag{125}$$

We are going to construct an upper barrier $\overline{v}(t,x)$ for $|v(t,x)|$ for large times; as $|v(t,x)|$ is bonded for finite times by the sole virtue of the Gronwall lemma, this will provide the sought for barrier. The barrier function $\overline{v}(t,x)$ will be the infimum of two super-solutions to (124), that we call $\overline{v}^+(t,x)$ and $\overline{v}^-(t,x)$, some care being taken with the fact that things do not exactly work as with classical diffusion.

4.4.1 Construction of $\overline{v}^+$

It will be devised according to the pattern that has already proved to be useful in Section 4.2, up to the fact that the new definition of $v(t,x)$ means the suppression of the $t^{-3/2}$ factor. Let $\Gamma(\eta)$ be a smooth nonnegative nondecreasing function that is zero on $[0,1]$ and equal to 1 on an interval of the form $[\eta_0,+\infty)$. We set

$$\overline{v}^+(t,x)=\xi(t)\big(t^{3/2}e^{-t\mathcal{I}_*}\overline{v}_0^*\big)+q(t)\Gamma(\eta)e^{-b\eta}. \tag{126}$$

We have denoted $\eta=\dfrac{x}{\sqrt{t}}$, in order to alleviate the notations, and the functions $\overline{v}_0$, $\xi(t)$ and $q(t)$, as well as the constant $b>0$, are to be properly chosen. Without spoiling the suspense too much, we can already say that we will impose $\dot\xi\geq 0$. We have denoted by $\overline{v}_0^*$ the odd extension of $\overline{v}_0$ to the left of $x=0$. In order to alleviate the notations somehow we set

$$S_*(t,x)=t^{3/2}e^{-t\mathcal{I}_*}\overline{v}_0^*(x). \tag{127}$$

We may now compute

$$\begin{array}{rl}&\Big(\partial_t+\mathcal{I}_*+\dfrac{3}{2t}(\partial_x-1)\Big)\overline{v}^+(t,x)=\dot\xi(t)S_*+\dot q\Gamma e^{-b\eta}+q\Gamma(\partial_t+\mathcal{I}_*)e^{-b\eta}\\ + & \dfrac{3}{2t}\xi\partial_xS_*-q\Gamma'\dfrac{x}{2t^{3/2}}e^{-b\eta}+qe^{-b\eta}\mathcal{I}_*\Gamma(\eta)+\dfrac{3q}{2t}\partial_x\big(\Gamma(\eta)e^{-b\eta}\big)-\dfrac{3q\Gamma e^{-b\eta}}{2t}\\ - & q\displaystyle\int_{\mathbb{R}}K_*(x-y)\big(\Gamma(\frac{x}{\sqrt t})-\Gamma(\frac{y}{\sqrt t})\big)\big(e^{-bx/\sqrt t}-e^{-by/\sqrt t}\big)dy.\end{array}$$

This makes a string of nine terms, that we number from T_1 to T_9. In order to reach the super-solution property, we will prove that T_1 to T_3 prevail over the others, that will be treated as perturbations. Fortunately, they are relatively easy to examine. As is now usual, three zones are to be investigated.
1. The area $\eta\geq\eta_0+1$. As $\Gamma\equiv 1$ here, we have, provided η_0 is large enough :

$$\begin{aligned}T_1+T_2+T_3\geq&\ \ \big(\dot q+qe^{b\eta}(\partial_t\mathcal{I}_*)e^{-b\eta}\big)e^{-b\eta}\\ \geq&\ \ (\dot q+\frac{b\eta}{t}q)e^{-b\eta}.\end{aligned}$$

On the other hand, as $\Gamma'\equiv 0$ we only have to worry about T_4, T_6, T_7, T_8. We have

$$|T_4|\lesssim\frac1t|\partial_xS_*(t,x)|\lesssim\frac{x^2}{t}e^{-x^2/4d_*t}\lesssim\eta^2e^{-\eta^2/4d_*}\ll\frac{e^{-b\eta}}{t},$$

as soon as η_0 is large enough. As for the remaining terms, we have

$$|T_6|+|T_7|+|T_8|\lesssim\frac{qe^{-b\eta}}{t}.$$

Therefore this last sum is absorbed by $\dfrac{b\eta}{t}q$, so that the function $q(t)$ should, in the end, satisfy

$$\dot q+b\eta_0tq\geq\frac1t,$$

that is

$$q(t)\simeq\frac{1}{b\eta_0},\quad \dot q\simeq\frac{1}{t^{1+b\eta_0}}. \tag{128}$$

2. The range $\varepsilon_0\leq\eta\leq\eta_0+1$. Now that we have chosen the function $q(t)$, we need to choose $\xi(t)$ and we do it here. this time the term that will help us will be $T_1=\dot\xi S_*(t,x)$. We have indeed, still taking for granted that $\dot\xi\geq 0$:

$$T_1\gtrsim\dot\xi x\gtrsim\sqrt t\dot\xi. \tag{129}$$

On the other hand, we have

$$|T_4| \lesssim \frac{\xi}{2t}, \tag{130}$$

while all he other terms are estimated by $\frac{1}{t}$. Let us, for instance, examine T_9. We have indeed

$$\int_{\mathbb{R}} K_*(x-y)\Big|\big(\Gamma(\frac{x}{\sqrt{t}}) - \Gamma(\frac{y}{\sqrt{t}})\big)\big(e^{-bx\sqrt{t}} - e^{-by/\sqrt{t}}\big)\Big| \lesssim \int_{\mathbb{R}} K_*(x-y)\frac{(x-y)^2}{t}dy \lesssim \frac{1}{t}.$$

From (129), it is sufficient to choose

$$\sqrt{t}\dot{\xi} \geq \frac{\xi}{t} + \frac{1}{t}, \tag{131}$$

that is, $\xi(t) = O(1) + O(\frac{1}{\sqrt{t}})$.

3. The range $0 \leq \eta \leq \varepsilon_0$**.** This time we have $\Gamma = \Gamma' \equiv 0$, so that only T_1 and T_4 matter. In this range, however, we have $\partial_x S_*(t,x) \geq 0$; as $\dot{\xi} \geq 0$ we still have the super-solution property.
Note that our computations may not be, strictly speaking, valid up to $x = 0$. However they certainly hold for x larger than a suitably large constant, which will be sufficient for the sequel. To sum up, if $q(t)$ is given by (128) and $\xi(t)$ is given by (131), then there is $x_0 > 0$ and $t_0 > 0$ such that the function $\overline{v}^+(t,x)$ given by (126) solves

$$\Big(\partial_t + \mathcal{I}_* + \frac{3}{2t}(\partial_x - 1)\Big)\overline{v}^+(t,x) \geq 0 \text{ for } t \geq t_0 \text{ and } x \geq x_0.$$

In other words, $\overline{v}^+(t,x)$ is a super-solution to (124).

4.4.2 Construction of $\overline{v}^-$

As $\overline{v}^+(t,x)$ becomes negative for negative x, we need another ingredient. What we have gained is that we know with $O(1)$ precision where the nontrivial level sets of $u(t,x)$ are, that is, they are at finite distance from 0. In particular, a sign of this fact is equation (125), that we did not know before proving Theorem 4.1, which opens new possibilities. For small $\gamma > 0$, let us repeat the following estimate, which is by now usual :

$$\mathcal{I}_* e^{\gamma x} \gtrsim -\gamma^2.$$

This implies

$$\Big(\partial_t + \mathcal{I}_* + \frac{3}{2t}(\partial_x - 1) + g'(u)\Big)e^{\gamma x} \geq q_* + O(\gamma^2) + O(\frac{1}{t}).$$

So, there is $t_1 > 0$ such that $\overline{v}^-(t,x) := e^{\gamma x}$ is a super-solution to (124) on $\{t \geq t_1,\ x \leq -1\}$.

4.4.3 Construction of the super-solution

As equation (127) is linear, we may work with any multiples of $\overline{v}^+$ or $\overline{v}^-$. One may wonder how one can glue $\overline{v}^+$ and $\overline{v}^-$ together so as to make a global super-solution, the remark that will make everything work is that $\overline{v}^+$ is a super-solution not only of (127) on $[t_0, +\infty) \times [x_0, +\infty)$, but also of the equation without the term $g'(u)v$, in other words a translation invariant equation. Thus, for every $a > 0$, the function $\overline{v}^+(t, x+a)$ is also a super-solution to (127) on $[t_0, +\infty) \times [x_0 - a, +\infty)$. Moreover, we have by construction :

$$\overline{v}^+(t,x) \simeq (x+a)e^{-x^2/4d_*t} + O(1) \text{ for } t \geq t_0 \text{ and } x \geq x_0 - a,$$

the $O(1)$ term being independent of a. There is $\alpha_\infty > 0$ such that

$$\lim_{t\to+\infty} \overline{v}^+(t,x+a) = \alpha_\infty(x+a)$$

uniformly on every interval of the form $[x_0 - a, t^\delta]$ with $\delta \in (0, \frac{1}{2})$. Consider the function $a\alpha_\infty \overline{v}^-(x)$; it is larger than $\alpha_\infty(x+a)$ on $\mathbb{R}_+$ and smaller on an interval $(-x_a, 0]$ with $\lim_{a\to+\infty} x - a = -\infty$. And so, for $t > 0$ large enough, there is a function $\bar{x}(t)$ tending to 0 as $t \to +\infty$ such that $\overline{v}^+(t,x)$ is below $a\alpha_\infty \overline{v}^-(x)$ on $\bar{x}(t) - 1, \bar{x}(t)]$, and above for $x \geq x(t)$. Thus, the function

$$\overline{v}(t,x) = \begin{cases} \overline{v}^+(t,x) \text{ if } x \geq \bar{x}(t) \\ a\alpha_\infty \overline{v}^-(x) \text{ if } x \leq \bar{x}(t). \end{cases} \tag{132}$$

It remains to pick any compactly supported, nonnegative, nonzero function $\overline{v}_0(x)$; a large enough multiple of $\overline{v}(t_0, .)$ will dominate the initial datum v_0.

Proof of Theorem 4.2. All the previous development argument proves the global boundedness of $v(t,x)$, hence the boundedness of $u_x(t,x)$ on every half line containing $\mathbb{R}_+$. As the equation is essentially linear on $\mathbb{R}_+$, it also proves bounds for all spatial derivatives of v, henceforth all spatial derivatives of u on every positive half line.

The function $u_x(t,x)$ is then easily bounded on every negative half line : indeed, from equation (122) we have

$$\Big(\partial_t + \mathcal{J} - c_*\partial_x + q_*\Big)|u_x| \leq 0 \quad \text{for } x \leq 0. \tag{133}$$

Let M be an upper bound for u_x for $t \geq 0$ and $x \geq -1$. Then

$$\overline{u}(t) = M + e^{-q*t}$$

is a super-solution to (133) that dominates $|u_x|$ for $t = 0$ and all $x \leq -1$, as well as for $t \geq 0$ and $x \in [-1, 0]$. Therefore it dominates u for $t \geq 0$ and $x \leq 0$. The successive bounds are proved by induction. □

5 Large time convergence

We now have all the elements for the precise study of the solution $u(t,x)$ of the Cauchy Problem (1), starting from a nonnegative, nontrivial, compactly supported initial datum $u_0(x)$. If φ_{c_*} is the travelling wave solution to (1) with bottom speed c_* to the right, recall that λ_* is the exponent at which it decays at $+\infty$.

Theorem 5.1 *There exists $x_\infty \in \mathbb{R}$ such that the following asymptotic identity holds :*

$$\lim_{t\to+\infty} \Big(u(t,x) - \varphi_{c_*}\big(x - c_*t - \frac{3}{2\lambda_*}\ln t + x_\infty\big)\Big) = 0, \tag{134}$$

uniformly in $x \in \mathbb{R}_+$. A similar statement holds on $\mathbb{R}_-$, with a possibly different x_∞.

5.1 Analysis in the region $x \sim t^\delta$

Given how we have proved the weak version of the logarithmic delay, Theorem 4.1, it is clear that the first objective that we should pursue is to obtain a refinment of Theorem 4.3 on the slope of $v(t,x)$ in the sub-diffusive region. This is the object of the next section, where we prove that we have an actual equivalent of $v(t, t^\gamma)$ for $\gamma \in (1/3, 1/2]$. This is where the main part of the effort is cast.

Theorem 5.2 *There exists $p_\infty > 0$ such that, for all $\gamma \in (1/3, 1/2)$ we have*

$$v(t, t^\gamma) = \frac{p_\infty}{t^{3/2-\gamma}}\big(1 + o_{t\to+\infty}(1)\big). \tag{135}$$

Once it is proved, retrieval of information to $x = O(1)$ will essentially the line of the preceding section 4.3.

The proof of Theorem 5.2 is similar in spirit to that of Theorem 4.3, the difference being that, instead of letting the process run by itself from $t = 0$ onwards, we interrupt it at arbitrary large times in order to estimate it more precisely by solutions of the linear equation. The validity of the barriers that we construct will rely on slope estimates performed at various times, that we detail next. With this in hand, we may proceed to the construction of the barriers, leading to the proof of Theorem 5.2.

The sought for sub and super solutions, that we denote by $v_s(t,x)$ for the moment, will be defined for $s > 1$ and $t \geq s$. We want to look for them under the form

$$v_s^\pm(t,x) = \xi_s^\pm(t) e^{-(t-s)\mathcal{I}_*} v^*(s,.) \pm \frac{1}{t^{3/2-\beta}} \cos(\frac{x}{t^\alpha}) \mathbf{1}_{(-1, \frac{3\pi t^\alpha}{2}]}(x), \tag{136}$$

where $v^*(s,.)$ is the odd extension of $v(s,.)$ around a well-chosen point. The important feature is here that $\xi_s^\pm(s) = 1$, and $\dot\xi_s^+ \geq 0$, $\dot\xi_s^-(t) \leq 0$. In other words, we update the observation at time s by replacing v_0^* by $v^*(s,.)$ but the perturbations ξ_s^+ and q_s^+ keep the memory of the past. In this computation, the time s will be assumed to be as large as needed. Notice, first, that $v_s^+(s,x) \geq v(s,x)$ if $x \geq 0$. So, if we manage to prove

$$\partial_t v_s^+ + \mathcal{I}_* v_s^+ \geq 0, \quad \partial_t v_s^- + \mathcal{I}_* v_s^- \leq 0 \qquad (t > s,\ x > 0), \tag{137}$$

and

$$v(t,x) \leq v_s^+(t,x), \quad v(t,x) \geq v_s^-(t,x) \qquad \text{for } t \geq s \text{ and } x \in [-1, 0], \tag{138}$$

we will have upper and lower bounds for $v(t,x)$ in the range $t \geq s$.

5.1.1 Boundary and slope estimates

Let $v^*(s,.)$ be the odd extension of $v(s,.)$ on $\mathbb{R}_-$. As the main building brick of the candidates $v_s^\pm$ for the super and sub-solution property will be a suitable translation of $e^{-(t-s)\mathcal{I}_*}v^*(s,.)$, the goal of this section is to gather the main ingredients that will enable this property. We will essentially need to prove that $v^\pm$ dominates v at the boundary of our domain, in other words, that $e^{(t-s)\mathcal{I}_*}$ is not too big in comparison to the perturbation function at that place. The other property is, from the experience of the proof of Lemma 4.4, is an estimate of the slope of $e^{-(t-s)\mathcal{I}_*}v^*(s,.)$ from below, which will in turn allow us to choose the modulation function $\xi_s^\pm$.

At time $t = s$, the typical length scale is $x \sim \sqrt{s}$. It is therefore logical to try to understand the behaviour of $e^{-(t-s)\mathcal{I}_*}v^*(s,.)$ in an $\varepsilon\sqrt{s}$-neighbourhood of $x = 0$, this is the object of the next lemma. In what follows, we make the slight abuse of notations consisting in denoting by $e^{-(t-s)\mathcal{I}_*}(s,x)$ the number $\big(e^{-(t-s)\mathcal{I}_*}v^*(s,.)\big)(x)$.

Lemma 5.3 *Let us consider $s > 0$ and $\varepsilon > 0$. The time s and the parameter ε are respectively suitably large and suitably small, and should be thought of as $\varepsilon \gg \dfrac{1}{s}$. We have, for $t \geq s$:*

$$e^{-(t-s)\mathcal{I}_*} v^*(s, \varepsilon\sqrt{s}) \gtrsim \frac{\varepsilon\sqrt{s}}{t^{3/2}}. \tag{139}$$

Similarly we have, to the left :

$$e^{-(t-s)\mathcal{I}_*}v^*(s,-\varepsilon\sqrt{s}) \lesssim -\frac{\varepsilon\sqrt{s}}{t^{3/2}}. \tag{140}$$

Proof. At this stage, it is relevant to assess the situation at time $t = s$. We claim the existence of a function $w_s(\eta) \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, such that we have, for some $A > 1$ and $\delta > 0$, the last constant being as small as wanted :

$$(\eta - s^{\delta-\frac{1}{2}})e^{-A\eta} \lesssim w_s(\eta) \lesssim (\eta + s^{\delta-\frac{1}{2}})e^{-A\eta}, \tag{141}$$

and such that

$$v(s,x) = \frac{1}{s}w_s(\frac{x}{\sqrt{s}}). \tag{142}$$

The reason for (141) and (142) stems from two points. The first one is Lemma 4.4, which ensures that $v(s,.)$ has the correct order of magnitude $\dfrac{x}{s^{3/2}}$ at a distance from 0 larger than t^α, and less than a large multiple of $\sqrt{s}$. The second is Proposition a refinement of 3.5, which ensures that v decays like an exponential for $x \gg \sqrt{s}$. We have indeed, for $A > 0$ and all $t > 0$:

$$\begin{aligned}(\partial_t + \mathcal{I}_*))\frac{e^{-\frac{Ax}{\sqrt{t+1}}}}{\sqrt{t+1}} &= \frac{A}{t+1}\Big(\frac{\xi}{2} - \frac{1}{A(t+1)} - Ad_*\omega(\frac{A}{\sqrt{t+1}})\Big)e^{-\frac{x}{\sqrt{t+1}}} \\ &\geq 0 \ \text{ if } \xi \geq 1 + Ad_*\|\omega\|_{L^\infty([0,1])}.\end{aligned}$$

Let us see why (139) and (140) are true. In what follows, we will argue according to the relative sizes of $t-s$ and s. As soon as $t - s \geq s^\kappa$, for a small $\kappa > 0$, we use Theorem 3.1 to claim

$$e^{-(t-s)\mathcal{I}_*}v^*(s,x) = e^{(t-s)d_*\partial_{xx}}v^*(s,x) + O\big(e^{-(t-s)^\delta}\big). \tag{143}$$

So, $e^{-(t-s)d_*\partial_{xx}}$ is the dominant term, and we only need to invoke Formula (52) in Section 4.1, with t replaced by $t-s$. We then argue as follows :

- If $t - s \geq \varepsilon s$, that is, $t \geq (1+\varepsilon)s$, we may write, for $x = \varepsilon\sqrt{s}$:

  $$e^{-\frac{(x-y)^2}{4d_*(t-s)}} - e^{-\frac{(x+y)^2}{4d_*(t-s)}} \simeq \frac{\varepsilon\sqrt{s}y}{d_*(t-s)},$$

  and use the asymptotic formula (54) to conclude.
- If $s + s^\delta \leq t \leq (1+\varepsilon)s$, the order of magnitude of the term $e^{-\frac{(x+y)^2}{4d_*(t-s)}}$ becomes negligible in front of that of $e^{-\frac{(x-y)^2}{4d_*(t-s)}}$. So we use, this time, formula (53).
- In the regime $s \leq t \leq s + s^\delta$, we resort to Theorem 3.4 on the well-spread initial data, to the caveat that the smoothness of $v(s,.)$ is an issue : indeed, the theorem involves the H^m norm of w_s, m possibly large, and we do not have an estimate for $\|w_s\|_{H^m}$, apart from the trivial - and insufficient - one $\|w_s\|_{H^m} \lesssim s^{\frac{m}{2}}$. To circumvent this, we use (141) ; in order to prove (139) we invoke the left inequality : let $\underline{w}_s(\eta)$ be equal to a multiple of $(\eta + s^{\delta-\frac{1}{2}})e^{-A\eta}$ on $[\varepsilon^2, +\infty)$, to a multiple of $[-\eta + s^{\delta-\frac{1}{2}})$, and regular on $\mathbb{R}$, so that it is in the end larger than $w_s^*(\eta)$. Then, Theorem 3.4 yields

  $$\begin{aligned}e^{-(t-s)\mathcal{I}_*}v^*(s,x) &\leq \frac{1}{s}e^{-(t-s)\mathcal{I}_*}\underline{w}_s^*(\frac{x}{\sqrt{s}}) \\ &= \frac{1}{s}e^{(t-s)d_*\partial_{xx}}\underline{w}_s^*(\frac{x}{\sqrt{s}}) + O\big(\frac{\|\underline{w}_s^*\|_{L^1} + \|\underline{w}_s^*\|_{H^m}}{s^{\frac{5}{2}-4\omega-\delta}}\big).\end{aligned}$$

Now, in the integral (52), computed at $x = \varepsilon\sqrt{s}$, only the contribution of the interval $[0, \varepsilon\sqrt{s}]$ really matters, the contribution coming from the integration on $\mathbb{R}_-$ being of the order $e^{-\varepsilon^2 s^{1-\delta}}$. The latter, hence, is negligible in comparison of the integral on $\mathbb{R}_+$, which is of the order $\frac{\varepsilon}{s}$: this entails (139). For (140) we repeat the argument with, this time, the left inequality in (141).

The proof of the lemma is therefore complete. □

The next ingredient is, as already advertised, the following slope estimate.

Lemma 5.4 *For all $B > 0$, and all $\gamma \in (\frac{1}{3}, \frac{1}{2})$, there is $q_{\gamma,B} \in (0,1)$ such that, for all $s \geq 1$, all $t > s$, the following estimates hold.*

– *If $x \in (t^\gamma, B\sqrt{t})$, then*

$$\frac{q_{\gamma,B}x}{t^{3/2}} \leq e^{-(t-s)\mathcal{I}_*}v^*(s,.)(x) \leq \frac{x}{q_{\gamma,B}t^{3/2}}. \tag{144}$$

– *If $0 \leq x \leq t^\gamma$, then, for all $\delta > 0$ we have*

$$e^{-(t-s)\mathcal{I}_*}v^*(s,.)(x) \leq \min\left(\frac{x}{q_{\gamma,B}t^{3/2}}, \frac{1}{t^{3/2-\delta}}\right). \tag{145}$$

Proof. It is essentially identical to the proof of Lemma 5.3, as we just have to use Theorem 3.1 for $t - s \geq s^\delta$ and Theorem 3.4 in the range $t - s \leq s^\delta$; only the place of x changes in the former range. The arguments are, however, the same. □

5.1.2 Barrier property and convergence

If now the functions $v_s^\pm(t,x)$, defined for $t \geq s$ by (136), and if the functions $\xi_s^\pm(t)$ satisfy, as in the proof of the rough bounds :

$$\dot{\xi}_s^\pm(t) \approx \frac{1}{(t+1)^{3\alpha-2\beta}}, \tag{146}$$

with $\alpha > \frac{1}{3}$, $\beta < 1$ small enough so that $3\alpha - 2\beta > 1$, Lemma 5.4 indeed allows a repetition of the computations in Lemma 4.4, and lead to the integro-differential inequalities (137).

However, the ordering (138) between v and $v_s^\pm$ for $x \in [0,1]$ is not necessarily true : as it is certainly true for $t \gg s$, given the asymptotics of $e^{-(t-s)\mathcal{I}_*}$, there is still a large time frame, of the order, say, s^δ where this is not so obvious. One could indeed imagine a scenario where the cosine perturbation would not be large enough to counterbalance $e^{-(t-s)\mathcal{I}_*}v^*(s,x)$ for x bounded and $t \leq s + s^\delta$; this is not forbidden by the asymptotics concerning the heat semigroup with well spread data. This is why we need to make an $\varepsilon\sqrt{s}$ translation to $v^*(s,.)$ in order to neutralise possible spurious effects of the semigroup generated by $\mathcal{I}_*$.

This entails a modification of the definition of $v_s^\pm$, that we conduct as follows. Choose $\varepsilon > 0$ small, and $s > 0$ such that $\varepsilon \gg \frac{1}{s}$. We first use Lemma 5.3 to infer :

$$e^{-(t-s)\mathcal{I}_*}v^(s, x - \varepsilon\sqrt{s}) \leq 0 \leq e^{-(t-s)\mathcal{I}_*}v^*(s, x + \varepsilon\sqrt{s}) \quad \text{for } t \geq s \text{ and } x \in [0,1].$$

On the other hand, Lemma 4.4, and more precisely inequality (92), estimates $v(t,x)$ in the vicinity of 0. It means, in particular, that $v(t,x)$ is dominated by $\frac{1}{t^{3/2-\beta}}$, for any small $\beta > 0$. We therefore modify slightly the definition of $\xi_s^\pm$ by setting :

$$v_s^\pm(t,x) = \xi_s^\pm(t)e^{-(t-s)\mathcal{I}_*}v^*(s, . \pm \varepsilon\sqrt{s}) \pm \frac{1}{t^{3/2-2\beta}}\cos(\frac{x}{t^\alpha})\mathbf{1}_{(-1,\frac{3\pi t^\alpha}{2}]}(x), \tag{147}$$

with

$$\dot{\xi}_s^{\pm}(t) \approx \frac{1}{(t+1)^{3(\alpha-\beta)}}, \tag{148}$$

and omit the dependence in ε in order to avoid an inflation of indices. The real number β is small enough so that $3(\alpha-\beta)>1$. We have, by construction : $v_s^{\pm}(t,x-\varepsilon\sqrt{s}) \leq v(t,x) \leq v_s^{\pm}(t,x+\varepsilon\sqrt{s})$ for $t \geq s$ and $x \in [0,1]$. Then, the definition (147) of $v_s^{\pm}$ implies :

$$v_s^-(t,x-\varepsilon\sqrt{s}) \leq v(t,x) \leq v_s^+(t,x+\varepsilon\sqrt{s}) \quad \text{for } t \geq s \text{ and } x \in [0,1],$$

And we finally have :

$$v_s^-(t,x-\varepsilon\sqrt{s}) \leq v(t,x) \leq v_s^+(t,x+\varepsilon\sqrt{s}) \quad \text{for } t \geq s \text{ and } x > 0. \tag{149}$$

It remains to analyse the large time behaviour of $v_s^-(t,x \pm \varepsilon\sqrt{s})$. The behaviour of the cosine perturbation being clear, let us analyse the main term $\xi_s^{\pm}(t)e^{-(t-s)\mathcal{I}_*}v(t,x \pm \varepsilon\sqrt{s})$. Recall that $\xi_s^{\pm}(t)$ is chosen, according to (108) : This entails

$$\xi_s^{\pm}(t) = 1 + O\big(\frac{1}{s^{3(\alpha-\beta)-1}}\big), \tag{150}$$

On the other hand we have already seen that $e^{-(t-s)\mathcal{I}_*}$ can, at no real cost, be replaced by $e^{(t-s)d_*\partial_{xx}}$. Therefore, from equation(54), we have

$$\frac{1+o(1)}{2\sqrt{\pi}d_*^{3/2}}\int_{\mathbb{R}_+}\eta w_s(\eta-\varepsilon)d\eta = \liminf_{\substack{t\to+\infty \\ x/t^\alpha\to+\infty \\ x/\sqrt{t}\to 0}} \frac{t^{3/2}}{x}v_s^-(t,x) \leq \limsup_{\substack{t\to+\infty \\ x/t^\alpha\to+\infty \\ x/\sqrt{t}\to 0}} \frac{t^{3/2}}{x}v_s^+(t,x) = \frac{1+o(1)}{2\sqrt{\pi}d_*^{3/2}}\int_{\mathbb{R}_+}\eta w_s(\eta+\varepsilon)d\eta,$$

the expression $o(1)$ being understood as $s \to 0$. Once this is established, proving Theorem 5.2 becomes an exercise in elementary topology. Denote, for short :

$$p_s = \frac{1}{2\sqrt{\pi}d_*^{3/2}}\int_0^{+\infty}\eta w_s(\eta)d\eta.$$

We have, for all $\gamma \in (\alpha, \frac{1}{2})$:

$$\big(1+o_{\substack{s\to+\infty\\ \varepsilon\to 0}}(1)\big)p_s \leq \liminf t^{3/2-\gamma}v(t,t^\gamma) \leq \limsup_{t\to+\infty} t^{3/2-\gamma}v(t,t^\gamma) \leq \big(1+o_{\substack{s\to+\infty\\ \varepsilon\to 0}}(1)\big)p_s.$$

In other words, all asymptotic values (that is, all limits of subsequences) of $t^{3/2-\gamma}v(t,t^\gamma)$ lie in an interval centred at p_s and of width $1+o_{\substack{s\to+\infty\\ \varepsilon\to 0}}(1)$. As the function $s \mapsto p_s$ is bounded due to Theorem 4.3, we may let $s \to +\infty$ and $\varepsilon \to 0$, with $\dfrac{\varepsilon}{s} \to +\infty$, and discover that all limiting values of $t^{3/2-\gamma}v(t,t^\gamma)$. This ends the proof of the theorem.

5.2 Convergence to the shifted wave

At this stage, there is no additional idea, other than those presented in Section 4.3 for the large time bound of the level sets. Select $\delta \in (0,\frac{1}{2})$; in the reference frame moving like $-\dfrac{3}{2\lambda_*}\ln t$, we still have, from Theorem 5.2, and with the notations (116) and (117) of Section 4.3 :

$$v(t,t^\delta) \sim_{t\to+\infty} \alpha_\infty t^\delta. \tag{151}$$

Note indeed that the new definition of $v(t,x)$ removes the $t^{-3/2}$ factor, and also that adding a logarithmic correction does not fundamentally alter t^δ.
For any $\varepsilon > 0$, define the two translations $\sigma^{\pm}_{\infty,\varepsilon}$ such that

$$\varphi_*\big(t^\delta - \sigma^{\mp}_{\infty,\varepsilon}(t)\big) = (\alpha_\infty \mp \varepsilon)t^\delta e^{-\lambda_* t^\delta}. \tag{152}$$

For $x \in [t^\delta - 1, t^\delta]$ we have

$$\varphi_*\big(t^\delta - \sigma^{-}_{\infty,\varepsilon}(t)\big) \le u(t,x) \le \varphi_*\big(t^\delta - \sigma^{+}_{\infty,\varepsilon}(t)\big).$$

Arguing as in Section 4.3 we obtain

$$\limsup_{t\to+\infty}\Big(u(t,x) - \varphi_*\big(t^\delta + \sigma^{-}_{\infty,\varepsilon}(t)\big)\Big) \ge 0, \quad \liminf_{t\to+\infty}\Big(u(t,x) - \varphi_*\big(t^\delta - \sigma^{+}_{\infty,\varepsilon}(t)\big)\Big) \le 0.$$

As $\varepsilon > 0$ is arbitrarily small, this puts an end to the proof of Theorem 5.1.

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