Stability of the Ginzburg-Landau vortex

Philippe Gravejat
CY Cergy Paris University

Joint work with **Eliot Pacherie** (*New York University Abu Dhabi*) and **Didier Smets** (*Sorbonne University*)

Introduction

1. The Gross-Pitaevskii equation

The Gross-Pitaevskii equation writes as

$$i\partial_t \Psi + \Delta \Psi + \Psi(1 - |\Psi|^2) = 0, \tag{GP}$$

for a function $\Psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$.

The equation is hamiltonian. Its Hamiltonian is the Ginzburg-Landau energy

$$E(\Psi) \equiv \int_{\mathbb{R}^N} e(\Psi) := \int_{\mathbb{R}^N} \Big(\frac{1}{2}|\nabla \Psi|^2 + \frac{1}{4}(1 - |\Psi|^2)^2\Big).$$

2. The stationary background

a. The stationary solutions

Stationary solutions solve the Ginzburg-Landau equation

$$\Delta\psi + \psi(1 - |\psi|^2) = 0. \qquad \text{(GL)}$$

Bethuel and Saut [99] proved that there are no non constant stationary solutions with finite energy in any dimension $N \geq 2$ (see also Brezis, Merle and Rivière [94] for $N = 2$) .

b. The vortex solutions

In dimension $N = 2$, there exist stationary solutions with infinite energy : the vortex solutions of degree $d \in \mathbb{Z}^*$.

These special solutions write under the equivariant form

$$V_d(x) = \rho_d(r)\, e^{id\theta},$$

for $x = (r\cos(\theta), r\sin(\theta))$. The profile ρ_d is the unique solution of the ordinary differential equation

$$\rho_d''(r) + \frac{1}{r}\rho_d'(r) - \frac{d^2}{r^2}\rho_d(r) + \rho_d(r)\,(1 - \rho_d(r))^2 = 0,$$

with $\rho_d(0) = 0$ and $\rho_d(r) \to 1$ as $r \to \infty$ (see [Chen, Elliott and Qi, 94] and [Hervé and Hervé, 94]).

Consider a smooth stationary solution u such that

$$\int_{\mathbb{R}^2} \left(1 - |u|^2\right)^2 < \infty.$$

Brezis, Merle and Rivière [94] established the existence of a number $d \in \mathbb{Z}$ such that

$$\int_{\mathbb{R}^2} \left(1 - |u|^2\right)^2 = 2\pi d^2.$$

The integer d is the degree at infinity of the solution u.

Question : *Is the solution u equal to the vortex solution V_d (up to the geometric invariances) ?*

Theorem [Mironescu, 96]. *Consider a smooth stationary solution u such that*

$$\int_{\mathbb{R}^2} \left(1 - |u|^2\right)^2 = 2\pi d^2.$$

If $d = \pm 1$, then there exist a point $a \in \mathbb{R}^2$ and a number $\varphi \in \mathbb{R}$ such that

$$u(x) = e^{i\varphi} V_d(x - a).$$

(See also [Brezis, Merle and Rivière, 94] for $d = 0$)

c. The minimizing nature of the vortex solutions $V_{\pm 1}$

Let B_R be the ball of $\mathbb{R}^2$ with center 0 and radius R. A stationary solution u is called locally minimizing if and only if

$$\int_{B_R} e(u + \varepsilon) \geq \int_{B_R} e(u),$$

for any number $R > 0$ and any function $\varepsilon \in H_0^1(B_R, \mathbb{C})$.

Corollary [Mironescu, 96]. *The vortex solutions $V_{\pm 1}$ are the unique non-constant locally minimizing stationary solutions, up to the translations and constant phase shifts.*

d. Towards a quantitative stability estimate

Given a function $\varepsilon \in H^1_0(B_R, \mathbb{C})$, we can write

$$\int_{B_R} e(V_1 + \varepsilon) = \int_{B_R} e(V_1) + \frac{1}{2} Q(\varepsilon) + \mathcal{O}\Big(\|\varepsilon\|^3_{H^1(B_R)} \Big),$$

where the quadratic form Q is given by

$$Q(\varepsilon) := \int_{B_R} \Big(|\nabla \varepsilon|^2 - (1 - |V_1|^2)|\varepsilon|^2 + 2\langle V_1, \varepsilon \rangle^2_{\mathbb{C}} \Big).$$

Let H_Q be the Hilbert space of functions $\varepsilon \in H^1_{\text{loc}}(\mathbb{R}^2, \mathbb{C})$ such that

$$\|\varepsilon\|^2_{H_Q} := \int_{\mathbb{R}^2} \Big(|\nabla \varepsilon|^2 + (1 - |V_1|^2)|\varepsilon|^2 + 2\langle V_1, \varepsilon \rangle^2_{\mathbb{C}} \Big) < \infty.$$

Theorem [del Pino, Felmer and Kowalczyk, 04]. *The quadratic form Q is positive semi-definite on the space H_Q. Moreover, its kernel is spanned by the derivatives $\partial_{x_1} V_1$ and $\partial_{x_2} V_1$.*

3. The dynamical background

a. The Cauchy problem

Theorem [Bethuel and Smets, 07]. *Let*

$$\mathcal{V} := \Big\{ v \in L^\infty(\mathbb{R}^2, \mathbb{C}) \text{ s.t. } \Delta v \in H^\infty, \nabla |v| \in L^2 \text{ and } 1 - |v|^2 \in L^2 \Big\}.$$

Given a function $\Psi_0 = v_0 + \psi_0 \in \mathcal{V} + H^1(\mathbb{R}^2, \mathbb{C})$, *there exists a unique solution* $\Psi = v_0 + \psi$ *of* (GP), *with initial datum* Ψ_0, *and such that* $\psi \in \mathcal{C}^0(\mathbb{R}, H^1(\mathbb{R}^2))$. *Moreover, the renormalized energy*

$$\mathcal{E}_{v_0}(\Psi) = \int_{\mathbb{R}^2} \Big(\frac{1}{2} |\nabla \psi|^2 - \langle \Delta v_0, \psi \rangle_{\mathbb{C}} + \frac{1}{4} (1 - |v_0 + \psi|^2)^2 \Big),$$

is conserved.

(See also [Gérard, 06] and [Gallo, 08])

b. The dynamics of well-prepared vortex configurations

Fix integers $d_1 = \pm 1$, ..., $d_\ell = \pm 1$, with $d_1 + \ldots + d_\ell =: d$. Consider a configuration $\boldsymbol{a}^0 = (a_1^0, \ldots, a_\ell^0)$ of points in $\mathbb{R}^2$ and set

$$V_{\boldsymbol{a}^0}^{\varepsilon}(x) := \prod_{i=1}^{\ell} V_{d_i}\Big(\frac{x - a_i^0}{\varepsilon}\Big),$$

for a small parameter $0 < \varepsilon < 1$.

Theorem [Bethuel, Jerrard and Smets, 08]. *Consider the solution $\boldsymbol{a} = (a_1, \ldots, a_\ell)$ of the point-vortex system*

$$\frac{da_i}{dt}(t) = \sum_{j \neq i} 2 d_j \, \frac{(a_i(t) - a_j(t))^{\perp}}{|a_i(t) - a_j(t)|^2},$$

with $a_i(0) = a_i^0$, and assume that it is well-defined on $[0, T]$ for a number $T > 0$.

If initial conditions $(\Psi^{\varepsilon,0})_{0<\varepsilon<1}$ *are well-prepared for the configuration* $\boldsymbol{a}^0$*, then the corresponding solutions* $(\Psi^{\varepsilon}(t))_{0<\varepsilon<1}$ *are well-prepared for the configuration* $\boldsymbol{a}(t)$ *for any fixed* $t \in [0,T]$*.*

(See also [Colliander and Jerrard, 98], [Lin and Xin, 99] and [Jerrard and Spirn, 07-08])

Question : *Is it possible to extend the previous convergence from finite time intervals to all time ?*

c. The vortex pair solution

For $d_1 = -d_2 = 1$ and $a_1^0 \neq a_2^0$, the point-vortex solution is equal to

$$\begin{cases} a_1(t) = a_1^0 + 2\frac{(a_1^0 - a_2^0)^\perp}{|a_1^0 - a_2^0|^2} t, \\ a_2(t) = a_2^0 + 2\frac{(a_1^0 - a_2^0)^\perp}{|a_1^0 - a_2^0|^2} t, \end{cases}$$

for any $t \in \mathbb{R}$. This is a vortex pair in uniform translation.

Theorem [Bethuel and Saut, 99] and [Chiron and Pacherie, 21]. *There exists a number* $\varepsilon_0 > 0$ *such that, given any number* $0 < \varepsilon < \varepsilon_0$, *there exists a travelling wave solution*

$$\Psi_\varepsilon(x_1, x_2, t) = U_\varepsilon(x_1 - \varepsilon t, x_2),$$

of (GP), *whose profile* U_ε *satisfies*

$$U_\varepsilon(x_1, x_2) = V_1(x_1, x_2 + d_\varepsilon)V_{-1}(x_1, x_2 - d_\varepsilon) + R_\varepsilon(x_1, x_2),$$

with $\varepsilon\, d_\varepsilon \to 1$ *and* $\|R_\varepsilon\|_{W^{1,\infty}} \to 0$ *as* $\varepsilon \to 0$.

(See also [Liu and Wei, 20])

I. Main results

1. The functional framework

We introduce the complex Hilbert space H corresponding to the norm

$$\|\Psi\|_H^2 := \int_{\mathbb{R}^2} \Big(|\nabla(\Psi\,\bar{V}_1)|^2 + (1 - |V_1|^2)|\nabla\Psi|^2 \Big),$$

and its subset

$$E := \Big\{ \Psi \in H \text{ s.t. } 1 - |\Psi|^2 \in L^2(\mathbb{R}^2) \Big\},$$

which is a complete metric space for the distance

$$d_E(\Psi_1, \Psi_2) := \|\Psi_1 - \Psi_2\|_H + \Big\| |\Psi_1|^2 - |\Psi_2|^2 \Big\|_{L^2}.$$

2. The minimizing nature of the vortex solution V_1

Theorem 1 [G., Pacherie and Smets, 21]. (i) *The renormalized Ginzburg-Landau energy*

$$\mathcal{E}(\Psi) := \lim_{R \to +\infty} \int_{B_R} \Big(e(\Psi) - e(V_1) \Big),$$

is well-defined on E, and invariant by translations and constant phase shifts.

(ii) *The vortex V_1 is the unique global minimizer of the energy $\mathcal{E}$ on E, up to translations and constant phase shifts.*

3. The quantitative stability estimate

Given a function $\Psi \in E$, define its orbit as

$$\text{Orb}(\Psi) := \left\{ e^{-i\varphi} \, \Psi(\cdot + a) \text{ for } \varphi \in \mathbb{R} \text{ and } a \in \mathbb{R}^2 \right\}.$$

Theorem 2 [G., Pacherie and Smets, 21]. *There exist two numbers $\kappa > 0$ and $\rho > 0$ such that*

$$\mathcal{E}(\Psi) \geq \kappa \, d_E\Big(V_1, Orb(\Psi)\Big)^2,$$

for any function $\Psi \in E$ such that

$$d_E(V_1, \text{Orb}(\Psi)) < \rho.$$

4. The orbital stability of the vortex solution V_1

Theorem 3 [G., Pacherie and Smets, 21]. *Let* $\Psi^0 \in E$. *There exist two numbers* $\tau > 0$ *and* $C > 0$ *such that, if*

$$d_E(V_1, \Psi_0) \leq \tau,$$

then there exist two functions $a \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^2)$ *and* $\varphi \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ *such that the solution* Ψ *of* (GP) *with initial datum* Ψ^0 *satisfies*

$$d_E\Big(V_1, e^{-i\varphi(t)}\,\Psi(\cdot + a(t), t)\Big) \leq C\, d_E(V_1, \Psi_0),$$

for any $t \in \mathbb{R}$. *Moreover, the position* a *and phase shift* φ *are controlled by*

$$\Big|a'(t)\Big| + \Big|\varphi'(t)\Big| \leq C\, d_E(V_1, \Psi_0).$$

II. Sketch of the proofs

1. Proof of Theorem 1

Defining properly the renormalized energy relies on the identity

$$
\begin{aligned}
|\nabla \Psi|^2 - |\nabla V_1|^2 = & |\nabla (\Psi \, \bar{V}_1)|^2 + (1 - |V_1|^2) |\nabla \Psi|^2 \\
& - (1 - |\Psi|^2) |\nabla V_1|^2 - 2 \langle \nabla (\Psi \, \bar{V}_1), \Psi \, \nabla \bar{V}_1 \rangle_{\mathbb{C}},
\end{aligned}
\tag{1}
$$

and on the asymptotics

$$
\nabla \bar{V}_1(x) \sim -i \frac{x^\perp}{|x|^2} \bar{V}_1(x),
$$

in the limit $|x| \to \infty$.

The last term in (1) may be written as

$$\langle \nabla(\Psi\,\bar{V}_1), \Psi\,\nabla\bar{V}_1\rangle_{\mathbb{C}} = \Big\langle \nabla(\Psi\,\bar{V}_1), \Big(\nabla\bar{V}_1 + i\,(1-\chi_R)^2\,\frac{x^\perp}{|x|^2}\bar{V}_1\Big)\Psi\Big\rangle_{\mathbb{C}}$$
$$-(1-\chi_R)^2\,\frac{x^\perp}{|x|^2}\,\langle\nabla(\Psi\,\bar{V}_1), i\Psi\,\bar{V}_1\rangle_{\mathbb{C}}, \tag{2}$$

where $\chi_R \in \mathcal{C}_c^\infty(\mathbb{R}^2,[0,1])$ is a radial cut-off function, with $\chi_R \equiv 1$ on a fixed ball B_R.

Observing that the function $\Psi\,\bar{V}_1$ is of finite Ginzburg-Landau energy, it may be decomposed as

$$\Psi\,\bar{V}_1 = e^{i\varphi} + w,$$

with $w \in H^1(\mathbb{R}^2)$ and $\nabla\varphi \in L^2(\mathbb{R}^2)$ (see [Gérard, 08]).

The last term in (2) may be written as

$$\begin{aligned}(1-\chi_R)^2\,\frac{x^\perp}{|x|^2}\,\langle i\Psi\,\bar{V}_1, \nabla(\Psi\,\bar{V}_1)\rangle_{\mathbb{C}} = (1-\chi_R)^2\,\frac{x^\perp}{|x|^2}\times \\ \times\Big(\nabla \mathrm{Re}(\varphi) + \langle ie^{i\varphi}, \nabla w\rangle_{\mathbb{C}} + \langle \nabla\varphi\, e^{i\varphi}, w\rangle_{\mathbb{C}} + \langle iw, \nabla w\rangle_{\mathbb{C}}\Big).\end{aligned} \tag{3}$$

The conclusion follows from integrating by parts the first two terms in the right-hand side of (3).

2. Proof of Theorem 2

Given a function $\Psi = V_1 + \varepsilon \in E$, set $\eta_\varepsilon := -2\langle V_1, \varepsilon\rangle_{\mathbb{C}} - |\varepsilon|^2$ and write $\Psi\,\bar{V}_1 = |V_1|^2\,\big(e^{i\varphi_\varepsilon} + w_\varepsilon\big)$. We can expand the renormalized energy $\mathcal{E}(V_1 + \varepsilon)$ as

$$\mathcal{E}(V_1 + \varepsilon) = \frac{1}{2}\Big(\mathcal{P}_R(\varepsilon) + \mathcal{Q}_R(\varepsilon) + \frac{1}{2}\int_{\mathbb{R}^2} \eta_\varepsilon^2\Big),$$

with

$$\mathcal{P}_R(\varepsilon) := 2\int_{\mathbb{R}^2} (1-\chi_R)^2\,|V_1|^4\,\frac{x^\perp}{|x|^2}\cdot\langle i w_\varepsilon, \nabla w_\varepsilon + 2i\,\nabla\varphi_\varepsilon\, e^{i\varphi_\varepsilon}\rangle_{\mathbb{C}},$$

and

$$\mathcal{Q}_R(\varepsilon) := \|\varepsilon\|_H^2 - \int_{\mathbb{R}^2} (1 - |V_1|^2 - |\nabla V_1|^2)|\varepsilon|^2$$
$$-2\int_{\mathbb{R}^2} \Big\langle \nabla(\varepsilon\,\bar{V}_1), \Big(\nabla\bar{V}_1 + i\,\frac{x^\perp}{|x|^2}\,(1-\chi_R)^2\bar{V}_1\Big)\varepsilon\Big\rangle_{\mathbb{C}}.$$

We can bound the quadratic term $\mathcal{P}_R(\varepsilon)$ by

$$\big|\mathcal{P}_R(\varepsilon)\big| \leq \frac{C}{R}\big(\|\varepsilon\|_H^2 + \|\eta_\varepsilon\|_H^2\big),$$

an we can expand the integral $\int_{\mathbb{R}^2} \eta_\varepsilon^2$ as

$$\frac{1}{2}\int_{\mathbb{R}^2} \eta_\varepsilon^2 = \int_{\mathbb{R}^2} \Big(2\langle V_1, \varepsilon\rangle_{\mathbb{C}}^2 + 2\langle V_1, \varepsilon\rangle_{\mathbb{C}}|\varepsilon|^2 + \frac{1}{2}|\varepsilon|^4\Big).$$

However, the quadratic term in this expression does not necessarily make sense.

Instead, we decompose this term as

$$\frac{1}{2}\int_{\mathbb{R}^2} \eta_\varepsilon^2 = \mathcal{I}_R(\varepsilon) + N_R(\varepsilon),$$

with

$$\mathcal{I}_R(\varepsilon) := 2\int_{\mathbb{R}^2} \chi_R^2 \langle V_1, \varepsilon\rangle_{\mathbb{C}}^2.$$

We can control the higher order term $N_R(\varepsilon)$ by

$$N_R(\varepsilon) \geq \kappa \int_{\mathbb{R}^2} \Big(\frac{1}{2}\eta_\varepsilon^2 - 2\,\chi_R^2 \langle V_1, \varepsilon\rangle_{\mathbb{C}}^2\Big) - C\,\|\varepsilon\|_H^3,$$

for any number $0 < \kappa < 1$.

We are left with the quadratic form

$$Q_R(\varepsilon) := \mathcal{Q}_R(\varepsilon) + \mathcal{I}_R(\varepsilon),$$

for which we show

Lemma 1. *There exist a number $\kappa_0 > 0$ and an integer $N_0 \geq 1$ such that, given any function $\varepsilon \in H$ satisfying the orthogonality conditions*

$$\int_{\mathbb{R}^2} \chi_1 \langle \varepsilon, \partial_{x_1} V_1 \rangle_{\mathbb{C}} = \int_{\mathbb{R}^2} \chi_1 \langle \varepsilon, \partial_{x_2} V_1 \rangle_{\mathbb{C}} = \int_{\mathbb{R}^2} \chi_1 \langle \varepsilon, i V_1 \rangle_{\mathbb{C}} = 0,$$

and any number $R_0 \geq 1$, there exists a number $R_0 \leq R \leq 2^{N_0} R_0$ such that

$$Q_R(\varepsilon) \geq \kappa_0 \left(\|\varepsilon\|_H^2 + \int_{\mathbb{R}^2} \chi_R^2 \langle \varepsilon, V_1 \rangle_{\mathbb{C}}^2 \right).$$

3. Proof of Lemma 1

We decompose the perturbation $\varepsilon \in H$ into orthogonal Fourier sectors through the formula

$$\varepsilon\Big(r\cos(\theta), r\sin(\theta)\Big) = e^{i\theta} \sum_{j\in\mathbb{Z}} \varepsilon_j(r)\, e^{ij\theta}.$$

Setting $\varepsilon_j := a_j + ib_j$, we can write the quadratic form $\mathcal{Q}_R$ as

$$\mathcal{Q}_R(\varepsilon) = \sum_{j\in\mathbb{Z}} \Big(Q_{R,j}(a_j) + Q_{R,j}(b_j)\Big),$$

with

$$Q_{R,j}(e) = \int_0^{+\infty} \Big(|e'|^2 + |e|^2\Big(\frac{(j+1)^2}{r^2} - \frac{2j}{r^2}(1-\chi_R)^2 |V_1|^2 - 1 + |V_1|^2\Big)\Big) r\, dr.$$

Similarly, the quadratic form $\mathcal{I}_R$ can be expanded as

$$\mathcal{I}_R(\varepsilon) = I_R(a_0) + \frac{1}{2} \sum_{j=1}^{+\infty} \Big(I_R(a_j + a_{-j}) + I_R(b_j - b_{-j}) \Big),$$

with

$$I_R(e) := 2 \int_0^{+\infty} \chi_R^2 \, |V_1|^2 \, |e|^2 \, r \, dr.$$

We observe that the quadratic form I_R is positive semi-definite.

For $j=0$, a direct computation shows that

$$Q_{R,0}(e)=\int_0^{+\infty}|V_1|^2\left|\left(\frac{e}{|V_1|}\right)'\right|^2 r\,dr\geq 0.$$

Hence there exists a number $\kappa_0>0$ such that

$$Q_{R,0}(a_0)+I_R(a_0)\geq\kappa_0\left(\|a_0\|_H^2+I_R(a_0)\right),$$

while

$$Q_{R,0}(b_0)\geq\kappa_0\,\|b_0\|_H^2,$$

as soon as b_0 satisfies the orthogonality condition

$$\left(\int_{\mathbb{R}^2}\chi_1\,\langle\varepsilon,iV_1\rangle_{\mathbb{C}}=2\pi\right)\int_0^{+\infty}\chi_1(r)\,b_0(r)\,|V_1|(r)\,r\,dr=0.$$

When $j \notin \{\pm 1, \pm 2\}$, we check that

$$Q_{R,j}(e) - Q_{r,0}(e) \geq \frac{1}{3} \int_0^{+\infty} \frac{j^2}{r^2} |e|^2 \, r \, dr,$$

so that

$$Q_{R,j}(e) \geq \kappa_0 \, \|e\|_H^2,$$

for a possibly further number $\kappa_0 > 0$.

For $j = \pm 2$, a similar computation gives

$$Q_{R,2}(e_+) + Q_{R,-2}(e_-) + \frac{1}{2} I_R(e_+ \pm e_-) \geq \kappa_0 \Big(\|e_+\|_H^2 + \|e_-\|_H^2 + \frac{1}{2} I_R(e_+ \pm e_-) \Big).$$

For $j = \pm 1$, we write

$$Q_{R,1}(e_+) + Q_{R,-1}(e_-) + \frac{1}{2} I_R(e_+ \pm e_-)$$
$$= Q_\pm\Big(\chi_R e_+, \chi_R e_-\Big) + Q_\infty\Big((1-\chi_R)e_+, (1-\chi_R)e_-\Big) + \mathcal{R}_R\Big(e_+, e_-\Big).$$

The localized quadratic forms $Q_\pm$ are exactly those studied in [del Pino, Felmer and Kowalczyk, 04]. Under the orthogonality conditions

$$\int_0^{+\infty} \chi_1 \left((u_+ \pm u_-)|V_1|' - (u_+ \mp u_-)\frac{|V_1|}{r} \right) r\, dr = 0,$$

they satisfy

$$Q_\pm(u_+, u_-) \geq \kappa_0 \left(\|u_+\|_H^2 + \|u_-\|_H^2 + \frac{1}{2} \int_0^{+\infty} |V_1|^2 \, |u \pm v|^2 \, r\, dr \right).$$

The quadratic form Q_∞ at infinity

$$Q_\infty(u_+,u_-) := \int_0^{+\infty} \Big(|u_+'|^2 + |u_-'|^2 + \Big(\frac{4-2|V_1|^2}{r^2} - 1 + |V_1|^2\Big)\times$$
$$\times |u_+|^2 + \Big(\frac{2|V_1|^2}{r^2} - 1 + |V_1|^2\Big)|u_-|^2\Big)\, r\,dr,$$

is positive definite when the number R is chosen large enough.

The remainder term $\mathcal{R}_R$ is supported in $[R, 2R]$. Given any integer $N_0 \geq 1$ and any number $R_0 > 0$, we can find a number $R_0 \leq R \leq 2^{N_0} R_0$ and a universal constant $C > 0$ such that

$$\mathcal{R}_R(e_+, e_-) \geq -\frac{C}{N_0}\Big(\|e_+\|_H^2 + \|e_-\|_H^2\Big).$$

Gathering the previous estimates, the conclusion follows from the Parseval formula.

Thank you very much!