Monotone solutions for the mean field games master equation

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Introduction: Mean Field Games

Main characteristics of a Mean Field Game (MFG)
- A differential game
- Non atomic players (infinite number of players)

We are interested in Nash equilibria
- Non uniqueness of equilibria in general
- In general uniqueness follows from growth or geometric assumptions
- Equilibria can be characterized in terms of PDE
A mathematical formulation of a Nash equilibrium

\[\downarrow\]

An optimality criterion for a single player (under anticipations), for instance the dynamic programming principle

\[\downarrow\]

An equation posed on the set of probabilities, an Hilbert space, on the measure.... This equation is called the *master equation*
Formally, the state variables of the model (the decision variables) are

\[
\left( \begin{array}{ccc}
t & \mathbf{x} & \mathbf{m} \\
\text{time} & \text{state of the player} & \text{measure of the other players}
\end{array} \right)
\]

\[
\Rightarrow "F(U, \partial_t U, D_x U, D_m U) = 0"
\]

Without noise, or when players are affected by i.i.d. noises, the following simplification holds:

\[
\left( \begin{array}{ccc}
t & \mathbf{x} & \mathbf{m}_0 \\
\text{time} & \text{state of the player} & \text{initial measure of the other players}
\end{array} \right)
\]

\[
\Rightarrow \begin{cases} 
H(u, \partial_t u, D_x u, m) = 0 \\
K(m, \partial_t m, D_x m, D_x u, u) = 0
\end{cases}
\]
For master equations in finite state space: Lions lectures at Collège de France, Bayraktar, Cecchin, Cohen, Delarue, Pelino, ...

For master equations in continuous state space: Lions lectures at Collège de France, Cardaliaguet, Delarue, Lasry and Lions, Porretta, Mou, Zhang, Gangbo, ...

For monotone solutions: B. 20
The master equation in finite state space
Notations

- There are $d$ states
- A time variable $t$ indicates the time remaining in the game.
- $U^i(t, x)$ denotes the value in the state $i \in \{1, ..., d\}$ when the repartition of the other players is $x \in \mathbb{R}^d$ and the time remaining is $t$
- Monotonicity in $\mathbb{R}^d$:
  $$\forall x, y \in \mathbb{R}^d, \langle A(x) - A(y), x - y \rangle \geq 0$$
The form of the master equation in a MFG

- In this context, without common noise, the typical form of the master equation is

\[ \partial_t U^i + (F(x, U) \cdot \nabla_x) U^i = G^i(x, U) \text{ in } (0, \infty) \times \mathbb{R}^d \text{ for } 1 \leq i \leq d; \]

\[ U^i(0, x) = U_0(x) \text{ in } \mathbb{R}^d \text{ for } 1 \leq i \leq d. \]

- The characteristics (analogue of the usual forward-backward system) are in this context \((V(t), x(t) \in \mathbb{R}^d)\)

\[
\begin{align*}
\frac{d}{dt} V(t) &= G(x(t), V(t)); \\
\frac{d}{dt} x(t) &= F(x(t), V(t)); \\
x(t_0) &= x_0, V(0) = U_0(x(0)).
\end{align*}
\]

The following holds

\[ U(t, x(t)) = V(t). \]
Boundary conditions

- Even though the variable $x \in \mathbb{R}^d$ could be describing several mean field terms, we are going to focus on the case

$$x^i = \text{quantity of players in the state } i$$

- Thus the master equation only holds on

$\{x^1 \geq 0\} \cap \ldots \cap \{x^d \geq 0\} =: O_d$.

- We shall make for the rest of this presentation the assumptions

$$\forall x \in O_d, p \in \mathbb{R}^d, x^i = 0 \Rightarrow F^i(x, p) \geq 0$$

When $\|x\|$ is large enough, $\forall p \in \mathbb{R}^d, \sum_{i=1}^{d} F^i(x, p) \leq 0$.

- Hence we impose no boundary conditions...
Common noise in discrete state space

- We have to choose a certain type of noise, other noises are possible (see also Bayraktar, Cecchin, Cohen and Delarue)
- We look at the case in which the master equation is of the form
  \[ \partial_t U + (F(x, U) \cdot \nabla_x) U + \lambda(U - T^* U(Tx)) = G(x, U) \text{ in } \mathbb{R}_+ \times \mathbb{R}^d; \]
  where \( T \in \mathcal{L}(\mathbb{R}^d), \lambda > 0. \)
- At random times given by a Poisson process of intensity \( \lambda, \) all the players are affected by the map \( T (x \mapsto T(x)). \)
- We ask that \( T \) is non expansive for the \( \| \cdot \|_1 \) norm and leave \( \mathbb{O}_d \) invariant.
- Fairly general type of noise if we consider limits of this class (see BLL19 for a discussion on this)
Existing results for those master equations

- "Good" class of monotonicity: \((G, F): \mathbb{O}_d \times \mathbb{R}^d \to \mathbb{R}^{2d}\) monotone
- Uniqueness of smooth solutions in the monotone regime
- A priori estimates on \(\|D_x U\|_\infty\) (which helps for existence) in the monotone regime \((+\epsilon)\) if \(F\) and \(G\) are Lipschitz (+ monotone condition)
- Existence theory is difficult and quite unnatural, especially in the continuous state space (quite often very strong assumptions are required)
Theorem

Assume that \((G, F)\) is monotone from \(\mathbb{O}_d \times \mathbb{R}^d\) into \(\mathbb{R}^{2d}\) and \(U_0\) is monotone, then there exists at most one smooth solution \(U\) of

\[
\partial_t U + (F(x, U) \cdot \nabla_x) U + \lambda(U - T^* U(T x)) = G(x, U) \text{ in } (0, \infty) \times \mathbb{O}_d,
\]

with initial condition \(U_0\). Moreover, if \(U\) exists it is monotone.
Proof of the uniqueness result

We consider two such solutions $U$ and $V$ and we introduce

$$W(t, x, y) = \langle U(t, x) - V(t, y), x - y \rangle.$$

The function $W$ is a solution of

$$\partial_t W + (F(x, U) \cdot \nabla_x) W + (F(y, V) \cdot \nabla_y) W + \lambda (W - W(t, Tx, Ty)) =$$

$$= \langle G(x, U) - G(y, V), x - y \rangle + \langle F(x, U) - F(y, V), U - V \rangle.$$

It follows from

$$\nabla_x W = U(x) - V(y) + D_x U \cdot (x - y).$$
Proof of the uniqueness result II

We argue by contradiction to prove that $W \geq 0$ and we consider a point $(t_0, x_0, y_0)$ such that

$$
\begin{cases}
\partial_t W(t_0, x_0, y_0) < 0, \\
W(t_0, x_0, y_0) = 0 = \inf_{x,y} W(t_0, x, y), \\
\nabla_x W(t_0, x_0, y_0) = \nabla_y W(t_0, x_0, y_0) = 0 \text{ (or } \geq 0 \text{ on } \partial \mathbb{O}_d) 
\end{cases}
$$

We deduce that

$$
\partial_t W(t_0, x_0, y_0) \geq \langle G(x_0, U(t_0, x_0)) - G(y_0, V(t_0, y_0)), x_0 - y_0 \rangle 
+ \langle F(x_0, U(t_0, x_0)) - F(y_0, V(t_0, y_0)), U(t_0, x_0) - V(t_0, y_0) \rangle 
\geq 0.
$$

Which is a contradiction, thus for any $t, x, y$

$$
\langle U(t, x) - V(t, y), x - y \rangle \geq 0.
$$

Hence, $U = V$. 

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Monotone solutions of the master equation
A starter: What I needed in the previous proof?

- If we forgot the common noise, the only knowledge needed on $U$ was that at $(t_0, x_0, y_0)$

  $$\partial_t \langle U(t_0, x_0), x_0 - y_0 \rangle \geq G(x_0, U(x_0)), x_0 - y_0 \rangle + \langle F(x_0, U(x_0)), U(x_0) - V(y_0) \rangle.$$

- From the point of view of $U$, the links between $(t_0, x_0), y_0$ and $V(y_0)$ is invisible.

- We are now ready for the oldest trick in the book of mathematicians!
Definition of a monotone solution

A continuous function $U$, $\mathcal{C}^1$ in time, is a monotone solution of

$$\partial_t U + (F(x, U) \cdot \nabla_x) U = G(x, U),$$

if for any $V \in \mathbb{R}^d$, $y \in \mathbb{O}_d$, $t_0 > 0$, $R > 0$, $x_0$ point of strict minimum of $x \rightarrow \langle U(t_0, x) - V, x - y \rangle$ on $\{x \in \mathbb{O}_d, \|x\|_1 \leq R\}$

$$\partial_t \langle U(t_0, x_0), x_0 - y \rangle \geq G(x_0, U(x_0)), x_0 - y \rangle + \langle F(x_0, U(x_0)), U(x_0) - V \rangle.$$

Theorem: There exists at most one such solution given a monotone initial condition!
Remarks on the previous definition of monotone solutions

- The time regularity can clearly be removed (and will be in a few slides)
- We shall also add common noise terms
- Uniqueness : by construction
- The fact that we only ask for information at strict minima helps for stability properties
- Reminiscent of viscosity solutions
Sketch of the uniqueness proof

• For $U$ and $V$ two solutions, define

$$W(t, x, y) = \langle U(t, x) - V(t, y), x - y \rangle.$$
Sketch of the uniqueness proof

- For $U$ and $V$ two solutions, define
  \[ W(t, x, y) = \langle U(t, x) - V(t, y), x - y \rangle. \]

- Assume $W \geq 0$ and consider strict minima $(t_0, x_0, y_0)$ of $W$ on $[0, t_M] \times \{ x \in \mathbb{O}_d, \|x\|_1 \leq R \}^2$. 
Sketch of the uniqueness proof

- For $U$ and $V$ two solutions, define

$$W(t, x, y) = \langle U(t, x) - V(t, y), x - y \rangle.$$ 

- Assume $W \not\geq 0$ and consider strict minima $(t_0, x_0, y_0)$ of $W$ on $[0, t_M] \times \{x \in \mathbb{R}^d, \|x\|_1 \leq R\}^2$

- Use the information of monotone solutions to arrive at a contradiction
For $U$ and $V$ two solutions, define

$$W(t, x, y) = \langle U(t, x) - V(t, y), x - y \rangle.$$ 

Assume $W \not\geq 0$ and consider strict minima $(t_0, x_0, y_0)$ of $W$ on $[0, t_M] \times \{x \in \Omega_d, \|x\|_1 \leq R\}^2$

Use the information of monotone solutions to arrive at a contradiction

Technicalities: same as before for the time variable, for the strict minimum, Stegall’s Lemma solve the problem (we just add $\langle a, x \rangle + \langle b, y \rangle$ in $W$ for $a$ and $b$ small)
Lemma

Assume $f : \Omega \rightarrow \mathbb{R}$ is weakly sequentially lower semi continuous function on the compact set $\Omega \subset X$ where $X$ is a separable Hilbert space. Then there is a dense number of point $c$ in $X'$ such that $x \rightarrow f(x) + \langle c, x \rangle$ has a strict minimum.
Definition

A continuous function $U$, is a monotone solution of

$$\partial_t U + (F(x, U) \cdot \nabla_x) U + \lambda(U - T^* U(Tx)) = G(x, U),$$

if for any $\phi$ smooth function of the time, $V \in \mathbb{R}^d$, $y \in \mathbb{O}_d$, $R > 0$, $(t_0, x_0) \in (0, \infty) \times B_1(0, R)$ a point of strict minimum of

$(t, x) \rightarrow \langle U(t, x) - V, x - y \rangle - \phi(t)$ on $[0, t_0] \times B_1(0, R)$

$$\partial_t \phi(t_0) + \lambda(\langle U(t_0, x_0) - T^* U(t_0, Tx_0), x_0 - y \rangle) \geq G(x_0, U(x_0)), x_0 - y \rangle + \langle F(x_0, U(x_0)), U(x_0) - V \rangle.$$

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a. $B_1(0, R) := \{x \in \mathbb{O}_d, \|x\|_1 \leq R\}$
Definition

A continuous function $U$, is a monotone solution of

$$rU + (F(x, U) \cdot \nabla_x)U + \lambda(U - T^*U(Tx)) = G(x, U),$$

if for any $V \in \mathbb{R}^d$, $y \in \mathbb{O}_d$, $R > 0$, $x_0 \in B_1(0, R)$ point of strict minimum of $x \rightarrow \langle U(x) - V, x - y \rangle$ on $B_1(0, R)$

$$r\langle U(x_0), x_0 - y \rangle + \lambda(\langle U(x_0) - T^*U(Tx_0), x_0 - y \rangle)$$

$$\geq G(x_0, U(x_0)), x_0 - y \rangle + \langle F(x_0, U(x_0)), U(x_0) - V \rangle.$$
Properties of monotone solutions
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Properties of monotone solutions

Uniqueness of solutions

Theorem
Assume that \((G, F) : \mathbb{O}_d \times \mathbb{R}^d \to \mathbb{R}^{2d}\) and \(U_0\) are monotone. Then there is at most one monotone solution of

\[
\partial_t U + (F(x, U) \cdot \nabla_x) U + \lambda (U - T^* U(Tx)) = G(x, U) \text{ in } (0, \infty) \times \mathbb{O}_d,
\]

with initial condition \(U_0\).

Theorem
Assume that \((G, F) : \mathbb{O}_d \times \mathbb{R}^d \to \mathbb{R}^{2d}\) is monotone. Then there is at most one monotone solution of

\[
r U + (F(x, U) \cdot \nabla_x) U + \lambda (U - T^* U(Tx)) = G(x, U) \text{ in } \mathbb{O}_d.
\]
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Stability of solutions

Theorem

Assume that $(F_n, G_n)_{n \geq 0}$ converges locally uniformly toward $(F, G)$ and that for all $n \geq 0$, $U_n$ is a monotone solution of the master equation associated to $F_n$ and $G_n$ and that it converges locally uniformly toward a continuous function $U$. Then $U$ is a monotone solution of the master equation associated to $F$ and $G$.

Proof (in the stationary case) : Take $V, y$ and $x_*$ strict minimum of $x \rightarrow \langle U(x) - V, x - y \rangle$. One can build a sequence of $(x_n)_{n \geq 0}$ of (almost) strict minima of $x \rightarrow \langle U_n(x) - V, x - y \rangle$ and show that $x_n \rightarrow x_*$. 
Theorem

Assume that $U \in W^{2,\infty}$ is a monotone solution of

$$rU + (F(x, U) \cdot \nabla_x)U + \lambda(U - T^*U(Tx)) = G(x, U) \text{ in } \mathbb{O}_d.$$ 

Assume furthermore that for all $x \in \mathbb{O}_d$, $D_x U(x) > 0$ in the order of positive definite matrix. Then $U$ satisfies

$$rU^i + (F(x, U)\cdot\nabla_x)U^i + \lambda\left(U^i - (T^*U(t, Tx))^i\right) = G^i(x, U) \text{ in } \{x_i > 0\};$$
$$rU^i + (F(x, U)\cdot\nabla_x)U^i + \lambda\left(U^i - (T^*U(t, Tx))^i\right) \leq G^i(x, U) \text{ in } \{x_i = 0\}.$$ 

Proof: for any $x_0, y \in \mathbb{O}_d$ there exists $V \in \mathbb{R}^d$ such that $x_0$ is a strict minimum of $x \to \langle U(x) - V, x - y \rangle$. The rest of the proof easily follows.
A generalization of this approach

- At the core of monotone solution is the idea that if we manage to show that

\[ \forall x, y, \langle U(x) - V(y), x - y \rangle \geq 0 \]

then we deduce that \( U = V \).

- There are plenty other maps \( \psi : \mathbb{O}_d \to \mathbb{R}^d \) such that if

\[ \forall x, y, \langle U(x) - V(y), \psi(x) - \psi(y) \rangle \geq 0 \]

then \( U = V \).

- One can then adapt the previous technique for other assumptions on \( F \) and \( G \) such as

\[
\langle G(x, U) - G(y, V), \psi(x) - \psi(y) \rangle + \\
+ \langle F(x, U) - F(y, V), U \cdot D_x \psi(x) - V \cdot D_x \psi(y) \rangle \geq 0.
\]
Concluding remarks

- Existence holds under more standard assumptions
- Uniqueness: by construction
- Stability of solutions is true
- Can be extended to: MFG with optimal stopping, impulse control, a priori no obstacle to push up to continuous state space (work in progress)
Thank you!