On the existence of solutions to the two-fluids systems

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Part I – Two-phase viscous fluids

The model of the two-phase flow of compressible fluids

$$\alpha^+ + \alpha^- = 1,$$
$$\partial_t(\alpha^\pm \rho^\pm) + \mathrm{div}(\alpha^\pm \rho^\pm \mathbf{u}^\pm) = 0,$$
$$\partial_t(\alpha^\pm \rho^\pm \mathbf{u}^\pm) + \mathrm{div}(\alpha^\pm \rho^\pm \mathbf{u}^\pm \otimes \mathbf{u}^\pm) + \nabla(\alpha^\pm p^\pm(\rho^\pm)) = D_{\mathbf{u}}(\mathbf{u}_\mp - \mathbf{u}\pm) + p_{int} \nabla \alpha^\pm.$$

In the viscous version we add: $-\mu\Delta(\alpha^\pm \mathbf{u}^\pm) - (\lambda + \mu)\nabla \,\mathrm{div}(\alpha^\pm \mathbf{u}^\pm)$.

- The unknowns: α^+, α^- – volume fractions, ρ^+, ρ^- – species' densities, $\mathbf{u}^+$, $\mathbf{u}^-$ – species velocities.
- The interfacial velocity pressure p_{int} has to be specified, for example

$$p_{int} = \alpha^+ p^+ + \alpha^- p^-.$$

- Closure assumptions:
  -M. Ishii '75, M. Ishii, T. Hibiki '06 (saturation condition $p^+ = p^-$).
  -M.R. Baer, J W. Nunziato '86 ($\partial_t \alpha^+ + \mathbf{u}_{int} \cdot \nabla \alpha^+ = D_p(p^+ - p^-)$).
  -Bresch, Desjardin, Ghidaglia, Grenier, Hillairet '18.

Reduction of the model

- barotropic pressure law $p^+ = (\rho^+)^{\gamma^+}$, $p^- = (\rho^-)^{\gamma^-}$,
- saturation condition $p^+ = p^-$,
- the relaxation parameter $D_{\mathbf{u}} \to \infty$, i.e. $\mathbf{u}^+ = \mathbf{u}^- = \mathbf{u}$,
- the semi-stationary Stokes flow $\partial_t(\alpha^\pm \rho^\pm \mathbf{u}^\pm) + \mathrm{div}(\alpha^\pm \rho^\pm \mathbf{u}^\pm \otimes \mathbf{u}^\pm) = 0$,
- the periodic boundary conditions.

So, the system reduces to

$$\begin{aligned}
&\partial_t(\alpha^\pm \rho^\pm) + \mathrm{div}(\alpha^\pm \rho^\pm \mathbf{u}) = 0,\\
&-\mu \Delta \mathbf{u} - (\lambda + \mu)\nabla \operatorname{div} \mathbf{u} + \nabla p = \mathbf{0},\\
&\alpha^+ + \alpha^- = 1,\\
&p := (\rho^+)^{\gamma^+} = (\rho^-)^{\gamma^-},
\end{aligned}$$

under the constraint

$$\int_{\mathbb{T}^d} \mathbf{u}(t,x)\,\mathrm{d}x = 0 \quad \text{for} \quad t \in (0,T).$$

Reformulation of the system

Let us denote

$$R = \rho^+ \alpha^+, \qquad Q = \rho^- \alpha^-, \qquad Z = \rho^+, \qquad \gamma = \frac{\gamma^+}{\gamma^-}.$$

Then the pressure can be expressed as

$$p = p(R, Q) = Z^{\gamma_+},$$

for $Z = Z(R, Q)$ such that

$$Q = \left(1 - \frac{R}{Z}\right) Z^{\gamma},$$

and

$$R \leq Z.$$

Reformulation of the system

We consider

$$\begin{aligned}
&\partial_t R + \mathrm{div}(R\mathbf{u}) = 0,\\
&\partial_t Q + \mathrm{div}(Q\mathbf{u}) = 0,\\
&-(\lambda + 2\mu)\,\mathrm{div}\,\mathbf{u} + \left(Z^{\gamma^+} - \{Z^{\gamma^+}\}\right) = 0,\\
&\mathrm{rot}\,\mathbf{u} = 0, \qquad \int_{\mathbb{T}^d} \mathbf{u}(t,x)\,\mathrm{d}x = 0,
\end{aligned}$$

where $Z = Z(R,Q)$. The initial conditions are

$$R|_{t=0} = R_0, \quad Q|_{t=0} = Q_0, \qquad R_0 \geq 0, \quad Q_0 \geq 0,$$

with the compatibility condition for $Z|_{t=0} = Z_0$

$$Q_0 = \left(1 - \frac{R_0}{Z_0}\right) Z_0^{\gamma}, \qquad \text{with} \qquad R_0 \leq Z_0.$$

Mono-fluid system

Putting $\alpha^+ = 1$ we get the usual semi-stationary compressible Stokes system

$$\begin{aligned}&\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0,\\ &-\mu \Delta \mathbf{u} - (\lambda + \mu)\nabla \operatorname{div} \mathbf{u} + \nabla p(\rho) = \mathbf{0},\end{aligned}$$

that has been studied, for example by P.-L. Lions with a monotone pressure law $p(\rho) = \rho^\gamma$ with $\gamma > 1$, and more recently by D. Bresch and P.-E. Jabin for non-monotone locally Lipschitz, pressure law $p(\rho)$, such that $p(0) = 0$ and

$$C^{-1}\rho^\gamma - C \leq p(\rho) \leq C\rho^\gamma + C, \qquad |p'(s)| \leq \overline{p} s^{\gamma - 1},$$

for some constants $C > 0$, $\overline{p} > 0$ and $\gamma > 1$.

Compressible systems with two continuity equations

Existence of weak solutions to

$$\partial_t n + \mathrm{div}(n\mathbf{u}) = 0$$
$$\partial_t \rho + \mathrm{div}(\rho\mathbf{u}) = 0$$
$$\partial_t([\rho + n]\mathbf{u}) + \mathrm{div}([\rho + n]\mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho, n) - \mu\Delta\mathbf{u} - (\lambda + \mu)\nabla\,\mathrm{div}\,\mathbf{u} = \mathbf{0}$$

for $p(\rho, n) = \rho^\gamma + n^\alpha$ with $\gamma > 9/5$ and $\alpha \geq 1$, A. Vasseur, H. Wen, C. Yu '17.

One-dimensional case with linear pressure law, S. Evje et al. '08, '15.

Navier-Stokes with potential temperature transport

$$\partial_t \rho + \mathrm{div}(\rho\mathbf{u}) = 0$$
$$\partial_t s + \mathbf{u} \cdot \nabla s = 0$$
$$\partial_t(\rho\mathbf{u}) + \mathrm{div}(\rho\mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho, s) - \mu\Delta\mathbf{u} - (\lambda + \mu)\nabla\,\mathrm{div}\,\mathbf{u} = \mathbf{0},$$

D. Maltese, M. Michalek, P.B. Mucha, A. Novotný, M. Pokorný, E. Z. '16.

Sketch of the proof of existence of solutions

Part 1 A-priori estimates
Due to a non-standard form of the pressure, it is actually not clear what sort of uniform estimates for R and Q can be expected.

Part 2 Weak sequential stability of solutions
This means that the hypothetical sequence of sufficiently smooth solutions $\{R_n, Q_n, Z_n, \mathbf{u}_n\}_{n=1}^{\infty}$ satisfying the energy and extra integrability estimates uniformly w.r.t. n, has a limit when $n \to \infty$, that is a weak solution to the same system.

Part 3 Construction of the approximate solution
Involves a) Lagrangian reformulation of the system, b) cut–off of the pressure, and c) application of stability of the flow result by G. Crippa, C. DeLellis '08 to recover the solution in the Eulerian coordinates.
Remark: $\varepsilon\Delta\rho$ is good as well.

Theorem (Existence of weak solutions)

Let $\gamma^{\pm} > 1$, $\lambda + 2\mu > 0$, *and let the initial data satisfy:*

$$\int_{\mathbb{T}^d} (R_0^{\gamma^+} + Q_0^{\gamma^-}) \, \mathrm{d}x < \infty, \qquad 0 < \int_{\mathbb{T}^d} R_0 \, \mathrm{d}x < \infty, \qquad 0 < \int_{\mathbb{T}^d} Q_0 \, \mathrm{d}x < \infty.$$

Then there exists a global-in-time weak solution $(R, Q, \mathbf{u})$ *s.t. such that*

$$R \in L^\infty(0,T; L^{\gamma^+}(\mathbb{T}^d)) \cap L^{2\gamma^+}((0,T)\times\mathbb{T}^d) \cap \mathcal{C}([0,T]; L^{\gamma^+}(\mathbb{T}^d)),$$

$$Q \in L^\infty(0,T; L^{\gamma^-}(\mathbb{T}^d)) \cap L^{2\gamma^-}((0,T)\times\mathbb{T}^d) \cap \mathcal{C}([0,T]; L^{\gamma^-}(\mathbb{T}^d))$$

$$\left[Z \in L^\infty(0,T; L^{\gamma^+}(\mathbb{T}^d)) \cap L^{2\gamma^+}((0,T)\times\mathbb{T}^d) \cap \mathcal{C}([0,T]; L^{\gamma^+}(\mathbb{T}^d)),\right]$$

$$\mathbf{u} \in L^2(0,T; H^1(\mathbb{T}^d)),$$

where equations for R *and* Q *are satisfied in the renormalized sense.*

Energy and pressure estimates

Lemma

Let $(R, Q, \mathbf{u})$ *be sufficiently smooth solution, then for any* $T > 0$ *we have:*

$$\sup_{t<T} \int_{\mathbb{T}^d} (Z^{\gamma^+} + R^{\gamma^+} + Q^{\gamma^-}) \, \mathrm{d}x + \int_0^T \int_{\mathbb{T}^d} |\nabla \mathbf{u}|^2 \, \mathrm{d}x \, \mathrm{d}t \le C,$$

$$\int_0^T \int_{\mathbb{T}^d} (Z^{2\gamma^+} + R^{2\gamma^+} + Q^{2\gamma^-}) \, \mathrm{d}x \, \mathrm{d}t \le C(1+T).$$

The idea of the proof: using formula for $Z(R, Q)$ we get:

$$\nabla P = \nabla Z^{\gamma^+} = \frac{\gamma^+}{\gamma^+ - 1} R \nabla \left(\frac{R}{\alpha}\right)^{\gamma^+ - 1} + \frac{\gamma^-}{\gamma^- - 1} Q \nabla \left(\frac{Q}{1-\alpha}\right)^{\gamma^- - 1}$$

and thus the energy estimate:

$$\sup_{t\in(0,T)} \int_{\mathbb{T}} \left(\frac{1}{\gamma^+ - 1} \left(\frac{R}{\alpha}\right)^{\gamma^+} \alpha + \frac{1}{\gamma^- - 1} \left(\frac{Q}{1-\alpha}\right)^{\gamma^-} (1-\alpha) \right) \mathrm{d}x$$

$$+ \int_0^T \int_{\mathbb{T}^d} |\operatorname{div} \mathbf{u}|^2 \, \mathrm{d}x \, \mathrm{d}t \le E(0).$$

Derivation of the equation for Z

Lemma

Let $\mathbf{u} \in L^2(0,T;W^{1,2}(\mathbb{T}^d))$, $R \in L^{2\gamma^+}((0,T)\times\mathbb{T}^d) \cap L^\infty(0,T;L^{\gamma^+}(\mathbb{T}^d))$, $Q \in L^{2\gamma^-}((0,T)\times\mathbb{T}^d) \cap L^\infty(0,T;L^{\gamma^-}(\mathbb{T}^d))$, *and let* $(R,Q,\mathbf{u})$ *be a weak solution. Then* Z *defined by*

$$Q = \left(1 - \frac{R}{Z}\right) Z^\gamma, \quad \textit{and} \quad R \le Z.$$

belongs to $L^{2\gamma^+}((0,T)\times\mathbb{T}^d) \cap L^\infty(0,T;L^{\gamma^+}(\mathbb{T}^d))$ *and it satisfies*

$$\partial_t Z + \operatorname{div}(Z\mathbf{u}) + \frac{(1-\gamma)(Z-R)Z}{\gamma(Z-R)+R} \operatorname{div}\mathbf{u} = 0,$$

in the sense of distributions.

Remark 1: If $(R,Z,\mathbf{u})$ solves the corresponding system in $\mathcal{D}'$, then $Q = Q(T,Z)$ satisfies the continuity equation in $\mathcal{D}'$.

Remark 2: (Z,u) is a solution to the renormalized equation. Indeed, we have

$$\left| \frac{(1-\gamma)(Z-R)Z}{\gamma(Z-R)+R} \operatorname{div}\mathbf{u} \right| \le CZ|\operatorname{div}\mathbf{u}|.$$

Weak sequential stability of solutions

Theorem

Let $T > 0$*. Assume that for any* n *the quadruple* $(R_n, Q_n, Z_n, \mathbf{u}_n)$ *is a weak solution corresponding to the initial conditions*

$$R_n|_{t=0} = R_{0,n}, \quad Q_n|_{t=0} = Q_{0,n}, \quad R_{0,n} \geq 0, \quad Q_{0,n} \geq 0,$$

with $Z_n|_{t=0} = Z_{0,n}(Q_{0,n}, R_{0,n})$*, and s.t.*

$$R_{n,0} \to R_0 \quad \textit{strongly in } L^1(\mathbb{T}^d),$$
$$Z_{n,0} \to Z_0 \quad \textit{strongly in } L^1(\mathbb{T}^d).$$

Let the previous estimates hold uniformly with respect to n*. Then, up to the subsequence*

$$R_n \to R \quad \textit{strongly in } L^{2\gamma^+ - \varepsilon}((0,T) \times \mathbb{T}^d),$$
$$Q_n \to Q \quad \textit{strongly in } L^{2\gamma^- - \varepsilon}((0,T) \times \mathbb{T}^d),$$
$$Z_n \to Z \quad \textit{strongly in } L^{2\gamma^+ - \varepsilon}((0,T) \times \mathbb{T}^d),$$
$$\mathbf{u}_n \to \mathbf{u} \quad \textit{weakly in } L^2(0,T; H^1(\mathbb{T}^d)),$$

for any $\varepsilon > 0$*. Moreover,* $(R, Q, Z, \mathbf{u})$ *is another weak solution.*

The compactness criterion

Lemma

Let $\{X_n\}_{n=1}^{\infty}$ *be a sequence of functions uniformly bounded in* $L^p((0,T)\times\mathbb{T}^d)$ *with* $1 \le p < +\infty$. *Assume that* $\mathcal{K}_h$ *is a sequence of positive, bounded functions s.t.*

i) $\forall \eta > 0, \ \ \sup_h \int_{\mathbb{T}^d} \mathcal{K}_h(x) \mathbf{1}_{\{x:\, |x \geq \eta|\}} \,\mathrm{d}x < \infty,$

ii) $\|\mathcal{K}_h\|_{L^1(\mathbb{T}^d)} \to +\infty$ *as* $h \to 0$.

If $\{\partial_t X_n\}_{n=1}^{\infty}$ *is uniformly bounded in* $L^r([0,T], W^{-1,r}(\mathbb{T}^d))$ *with* $r \geq 1$ *and*

$$\limsup_n \left(\frac{1}{\|\mathcal{K}_h\|_{L^1}} \int_0^T \int_{\mathbb{T}^{2d}} \mathcal{K}_h(x-y) |X_n(t,x) - X_n(t,y)|^p \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \right) \to 0,$$

as $h \to 0$, *then,* $\{X_n\}_{n=1}^{\infty}$ *is compact in* $L^p([0,T]\times\mathbb{T}^d)$.
Conversely, if $\{X_n\}_{n=1}^{\infty}$ *is compact in* $L^p([0,T]\times\mathbb{T}^d)$, *then the above* lim sup *converges to* 0 *as* $h \to 0$.

Why does it imply compactness?

In space: denote $\overline{\mathcal{K}_h}(x) = \frac{\mathcal{K}_h(x)}{\|\mathcal{K}_h\|_{L^1(\mathbb{T}^d)}}$, then

$$\|X_n - \overline{\mathcal{K}_h} * X_n\|^p_{L^p(\mathbb{T}^d)} \leq \frac{1}{\|\mathcal{K}_h\|^p_{L^1(\mathbb{T}^d)}} \int_{\mathbb{T}^d} \left(\int_{\mathbb{T}^d} \mathcal{K}_h |X_n(x) - X_n(y)| \, \mathrm{d}x \right)^p \mathrm{d}y$$
$$\leq \frac{1}{\|\mathcal{K}_h\|_{L^1(\mathbb{T}^d)}} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \mathcal{K}_h |X_n(x) - X_n(y)|^p \, \mathrm{d}x \, \mathrm{d}y,$$

which converges to zero as $h \to 0$ uniformly in n.

In time: we apply the standard Aubin-Lions Lemma.

We take:

$$K_h(x) \sim \frac{1}{(|x| + h)^a}, \qquad \overline{K_h}(x) = \frac{K_h(x)}{\|K_h\|_{L^1(\mathbb{T}^d)}}, \qquad \mathcal{K}_{h_0}(x) = \int_{h_0}^1 \overline{K_h}(x) \frac{dh}{h}.$$

We also use the property:

$$\|\mathcal{K}_{h_0}\|_{L^1(\mathbb{T}^d)} \sim |\log h_0|.$$

Introduction of the weights

One does not try to propagate directly

$$\int_{\mathbb{T}^{2d}} K_h(x-y)|R_n(x)-R_n(y)|\,\mathrm{d}x\,\mathrm{d}y, \quad \int_{\mathbb{T}^{2d}} K_h(x-y)|Q_n(x)-Q_n(y)|\,\mathrm{d}x\,\mathrm{d}y$$

Instead, we use

$$S(t) = \int_{\mathbb{T}^{2d}} K_h(x-y)\Big\{|R_n(x)-R_n(y)|+|Z_n(x)-Z_n(y)|\Big\}(w_x+w_y)\,\mathrm{d}x\,\mathrm{d}y,$$

where the weights satisfy a dual equation

$$\begin{cases} \partial_t w + \mathbf{u}\cdot\nabla w + \lambda \mathcal{D} w = 0, \\ w(0,x) = 1, \end{cases}$$

where λ is sufficiently large constant, and $\mathcal{D}$ depends on $M|\nabla \mathbf{u}|$ and Z.

Some tedious computations...

$$\frac{\mathrm{d}}{\mathrm{dt}} S(t) = \int_{\mathbb{T}^{2d}} \nabla K_h(x-y)(\mathbf{u}_x - \mathbf{u}_y) O_{x-y}(w_x + w_y)\ \mathrm{d}x\,\mathrm{d}y$$

$$- \int_{\mathbb{T}^{2d}} K_h(x-y)(\mathrm{div}_x\, \mathbf{u}_x - \mathrm{div}_y\, \mathbf{u}_y)[R_x s_R + Z_x s_Z] w_x\ \mathrm{d}x\,\mathrm{d}y$$

$$- 2\int_{\mathbb{T}^{2d}} K_h(x-y)\left[\frac{(1-\gamma)(Z_x - R_x)Z_x}{\gamma(Z_x - R_x) + R_x}\,\mathrm{div}_x\, \mathbf{u}_x - \frac{(1-\gamma)(Z_y - R_y)Z_y}{\gamma(Z_y - R_y) + R_y}\,\mathrm{div}_y\, \mathbf{u}_y\right] s_Z w_x\ \mathrm{d}x\,\mathrm{d}y$$

$$+ 2\int_{\mathbb{T}^{2d}} K_h(x-y) O_{x-y}\left(\partial_t w_x + \mathbf{u}_x \cdot \nabla w_x + \mathrm{div}_x\, \mathbf{u}_x w_x\right)\ \mathrm{d}x\,\mathrm{d}y$$

$$= I_1 + I_2 + I_3 + I_4.$$

where $O_{x-y} = |R_x - R_y| + |Z_x - Z_y|$.

We prove that

$$S_{h_0}(t) \le C|\log(h_0)|^{1/2} + S_{h_0}(0),$$

where

$$S_{h_0}(t) = \int_{h_0}^{1} \frac{S(t)}{\|K_h\|_{L^1}}\,\frac{dh}{h}.$$

Removal of the weights

Proposition

Assume that $\mathcal{D}$ is bounded in $L^2(0, T \times \mathbb{T}^d)$. Then, there exists a weight w solving the transport equation. Moreover, we have

(i) *For any $(t, x) \in (0, T) \times T^d$, $0 \leq w(t, x) \leq 1$.*

(ii) *If we assume moreover that the pair $(X, \mathbf{u})$ is a solution to the continuity equation:*

$$\partial_t X + \mathrm{div}(X\mathbf{u}) = 0,$$

and X is bounded in $L^2((0, T) \times \mathbb{T}^d)$, there exists $C \geq 0$, such that

$$\int_{\mathbb{T}^d} X|\log w| \, \mathrm{d}x \leq C\lambda.$$

We apply it to both transported unknowns

$$\int_{\mathbb{T}^d} R_n |\log w_n| \, \mathrm{d}x \leq C\lambda, \quad \text{but also} \quad \int_{\mathbb{T}^d} Z_n |\log w_n| \, \mathrm{d}x \leq C\lambda.$$

Existence of approximate solutions

Let $\mathcal{T}_k : \mathbb{R}_+ \to \mathbb{R}_+$ for $k \in \mathbb{R}_+$ such that

$$\mathcal{T}_k(t) = t \text{ for } t \leq k \quad \text{and} \quad \mathcal{T}_k(t) = k \text{ for } t \geq k.$$

We consider the following approximate system for $(t, x) \in (0, T) \times \mathbb{T}^d$:

$$\begin{aligned}
&\partial_t R + \operatorname{div}(\mathbf{u} R) = 0, \\
&\partial_t Q + \operatorname{div}(\mathbf{u} Q) = 0, \\
&Q = \left(1 - \frac{R}{Z}\right) Z^\gamma, \quad R \leq Z, \\
&\operatorname{div} \mathbf{u} = (\mathcal{T}_k(Z))^{\gamma_+} - \{(\mathcal{T}_k(Z))^{\gamma_+}\}, \\
&\mathrm{rot}\, \mathbf{u} = \mathbf{0}, \quad \int_{\mathbb{T}^d} \mathbf{u}(t, x) \, \mathrm{d}x = 0.
\end{aligned}$$

Theorem

Let k be fixed. There exists a global-in-time weak solution to the truncated system, s.t.

$$R_k^{-1},\ R_k,\ Q_k^{-1},\ Q_k,\ Z_k^{-1},\ Z_k \in L^\infty((0,T)\times\mathbb{T}^d),$$

$$\partial_t R_k + \mathbf{u}_k\cdot\nabla R_k \in L^\infty((0,T)\times\mathbb{T}^d),$$

$$\partial_t Q_k + \mathbf{u}_k\cdot\nabla Q_k \in L^\infty((0,T)\times\mathbb{T}^d),$$

$$\partial_t Z_k + \mathbf{u}_k\cdot\nabla Z_k \in L^\infty((0,T)\times\mathbb{T}^d),$$

$$\nabla_x\mathbf{u} \in L^\infty(0,T;BMO(\mathbb{T}^d)),\quad \operatorname{div}\mathbf{u} \in L^\infty((0,T)\times\mathbb{T}^d),\quad \partial_t\mathbf{u} \in L^\infty(0,T;L^2(\mathbb{T}^d)),$$

and the equations are satisfied in the sense of the distributions.

Strategy of the proof – Step 1: Lagrangian reformulation

We show that the map

$$\Phi \;:\; L^\infty((0,T)\times\mathbb{T}^d)\times L^\infty((0,T)\times\mathbb{T}^d)\to L^\infty((0,T)\times\mathbb{T}^d)\times L^\infty((0,T)\times\mathbb{T}^d),$$

$$\Phi(r,q)=(\bar r,\bar q),$$

where $\bar r, \bar q$ are the solutions to the following system in $(0,T)\times\mathbb{T}^d$

$$\begin{aligned}
&\partial_t \bar r + r\sigma = 0,\\
&\partial_t \bar q + q\sigma = 0,\\
&\sigma = (\mathcal{T}_k(z))^{\gamma+} - \{(\mathcal{T}_k(z))^{\gamma+}\}_{\mathcal{L}},\\
&q = \left(1-\frac{r}{z}\right) z^\gamma, \quad r \le z,
\end{aligned}$$

is a contraction, at least for short time $t\in(0,T)$.

Notation:

$$\{f\}_{\mathcal{L}} := \frac{1}{|\mathbb{T}^d|}\int_{\mathbb{T}^d} f(t,y)\exp\left(\int_0^t \sigma(s,y)\,\mathrm{d}s\right)\mathrm{d}y.$$

Strategy of the proof – Step 2: Back to Eulerian formulation

We want to “solve” the equation:

$$\mathrm{div}_x\, \mathbf{u}(t,x) = \sigma(t,y).$$

For given σ, we will find $\mathbf{u}$ and $x = x(t,y)$ satisfying the above.
Our candidate $x = x(t,y)$ is a solution to an ODE defining the Lagrangian transformation

$$\frac{dx}{dt} = \mathbf{u}(t,x), \qquad x|_{t=0} = y.$$

Fixed point argument

Wdefine a map $\Phi(\bar{\mathbf{u}}) = \mathbf{u}$

$$\Phi \ : \ \mathcal{C}(0,T;W^{2-\alpha,p}(\mathbb{T}^d)) \to \ \mathcal{C}(0,T;W^{2-\alpha,p}(\mathbb{T}^d)),$$

for $\alpha > 0$ arbitrary small, and $p > 1$, in the following way:

1. For a given $\bar{\mathbf{u}} \in L^p(0,T;W^{2-\alpha,p}(\mathbb{T}^d))$ we use the Cauchy-Lipschitz theorem to find a unique $x = x(t,y)$ such that

$$\frac{dx(t,y)}{dt} = \bar{\mathbf{u}}(t,x(t,y)), \qquad x(t,y)|_{t=0} = y.$$

2. We prove that $H(t,y) = \frac{\partial x}{\partial y}(t,y)$ is invertible, and so we can express y as a function of t and x.

3. For such $y = y(t,x)$ we will look for solution

$$\mathbf{u}(t,x) = \nabla\phi(t,x), \quad \text{where} \quad \Delta_x\phi(t,x) = \sigma_\delta(t,y(t,x)),$$

and where

$$\sigma_\delta(t,y) = \sigma(t,y) * \kappa_\delta(y).$$

Fixed point argument

We solve this problem using the Lerey-Schauder fixed point theorem.

Theorem

Let Φ *be a continuous, compact mapping,* X *a Banach Space. Let for any* $\lambda \in [0,1]$ *the fixed point* $v = \lambda\Phi(v)$, $v \in X$ *be bounded. Then* Φ *possesses at least one fixed point in* X.

Remark

This approach guarantees not only that $\mathrm{div}_x\, \mathbf{u}(t,x) = \sigma_\delta(t, y(t,x))$, *but also, that* $\mathbf{u}$ *has a structure of a gradient flow and will satisfy* $\mathrm{rot}\, \mathbf{u} = \mathbf{0}$ *and* $\int_{\mathbb{T}^d} \mathbf{u}(t,x)\, \mathrm{d}x = 0$.

Remark

The solution $\mathbf{u}$ *constructed above depends on the parameter* δ *and should be denoted* $\mathbf{u}^\delta$.

Strategy of the proof – Step 3: Removal of δ

To let $\delta \to 0$ in the equation

$$\Delta_x \phi^\delta(t, x^\delta) = \sigma_\delta(t, y(t, x^\delta))$$

we use the uniform estimates

$$\|\nabla_x \mathbf{u}^\delta\|_{L^\infty(0,T;BMO(\mathbb{T}^d))} + \|\operatorname{div}_x \mathbf{u}^\delta\|_{L^\infty((0,T)\times\mathbb{T}^d)} + \|\partial_t \mathbf{u}^\delta\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} \leq C,$$

and the weak formulation

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^d} \Delta_x\phi^\delta(t,x^\delta)\xi(t,x^\delta)\mathrm{d}x^\delta\ \mathrm{d}t &= \int_0^T\int_{\mathbb{T}^d} \sigma * \kappa_\delta(t, y(x^\delta, t))\xi(t,x^\delta)\mathrm{d}x^\delta\ \mathrm{d}t \\ &= \int_0^T\int_{\mathbb{T}^d} \sigma * \kappa_\delta(t,y)\xi(t,x^\delta(t,y))J^\delta(t,y)\mathrm{d}y\ \mathrm{d}t,\end{aligned}$$

for any smooth ξ, where we denoted

$$J^\delta(t,y) = \exp\left(\int_0^t \operatorname{div}_x \mathbf{u}^\delta(t', x^\delta(t',y))\mathrm{d}t'\right) = \exp\left(\int_0^t \sigma * \kappa_\delta(t',y)\mathrm{d}t'\right).$$

Strategy of the proof – Step 3: Removal of δ

1. From the uniform bounds

$$\phi^\delta \to \phi \quad \text{weakly} \quad \text{in } L^\infty(0,T; W^{2,p}(\mathbb{T}^d)),$$

$$\sigma * \kappa_\delta \to \sigma \quad \text{a.e. in } (0,T)\times\mathbb{T}^d.$$

2. Using Crippa-Dellelis '08 stability of the flow result:

$$\sup_{t\in(0,T)} \|x(t,y) - x^\delta(t,y)\|_{L^1(\mathbb{T}^d)} \le C|\ln(\|\mathbf{u} - \mathbf{u}^\delta\|_{L^1((0,T)\times\mathbb{T}^d)})|^{-1}$$

we therefore get $x^\delta(t,y) \to x(t,y)$ in $L^\infty(0,T; L^1(\mathbb{T}^d))$, since $\mathbf{u}^\delta$ is compact in $L^1((0,T)\times\mathbb{T}^d)$.

3. Letting $\delta \to 0$ in both sides of weak formulation

$$\int_0^T \int_{\mathbb{T}^d} \Delta_x \phi(t,x) \xi(t,x) \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_{\mathbb{T}^d} \sigma(t,y) \xi(t, x(y,t)) J(t,y) \, \mathrm{d}y \, \mathrm{d}t.$$

4. Substituting $\Delta_x \phi = \mathrm{div}_x \mathbf{u}$, and $\sigma = (\mathcal{T}_k(z))^{\gamma+} - \{(\mathcal{T}_k(z))^{\gamma+}\}_{\mathcal{L}}$ we obtain

$$\int_0^T \int_{\mathbb{T}^d} \mathrm{div}_x \mathbf{u}(t,x) \xi(t,x) \, \mathrm{d}x \, \mathrm{d}t$$
$$= \int_0^T \int_{\mathbb{T}^d} \Big((\mathcal{T}_k(Z(t,x)))^{\gamma+} - \{(\mathcal{T}_k(Z(t,x)))^{\gamma+}\} \Big) \xi(t,x) \, \mathrm{d}x \, \mathrm{d}t,$$

where $Z(t,x)$ is defined by $Z(t, x(t,y)) = z(t,y)$.
Note that we used the change of variables formula

$$\int_0^T \int_{\mathbb{T}^d} \xi(t, x(y,t)) J(t,y) \, \mathrm{d}y \, \mathrm{d}t = \int_0^T \int_{\mathbb{T}^d} \xi(t,x) \, \mathrm{d}x \, \mathrm{d}t,$$

see Colombo-Crippa-Spirito '15.

Conclusion

We proved existence of a weak solution to the system:

$$
\begin{aligned}
&\partial_t R + \operatorname{div}(\mathbf{u} R) = 0,\\
&\partial_t Q + \operatorname{div}(\mathbf{u} Q) = 0,\\
&Q = \left(1 - \frac{R}{Z}\right) Z^{\gamma}, \quad R \leq Z,\\
&\operatorname{div} \mathbf{u} = (\mathcal{T}_k(Z))^{\gamma_+} - \{(\mathcal{T}_k(Z))^{\gamma_+}\},\\
&\mathrm{rot}\, \mathbf{u} = \mathbf{0}, \quad \int_{\mathbb{T}^d} \mathbf{u}(t,x)\, \mathrm{d}x = 0.
\end{aligned}
$$

To remove the truncation parameter k further uniform estimates are needed (difficult), and adjustment of the compactness argument (not difficult: the truncations are monotone functions of Z).

Part II – One-dimensional system

The starting point is the system

$$
\begin{aligned}
&\partial_t R + \partial_x(Ru) = 0,\\
&\partial_t Q + \partial_x(Qu) = 0,\\
&\partial_t[(R+Q)u] + \partial_x[(R+Q)u^2] + \partial_x Z^{\gamma_+} = \mu\partial_{xx}u,
\end{aligned}
$$

written in the mass Lagrangian coordinates $y := \int_0^x (R+Q)(\xi,t)d\xi$, $s := t$,

$$
\begin{aligned}
&\partial_t \tau = \partial_y u,\\
&\partial_t(Q\tau) = 0,\\
&\partial_t u = \partial_y\left(\mu\frac{\partial_y u}{\tau} - Z(R,Q)^{\gamma_+}\right),
\end{aligned}
$$

where $y \in \Omega$, $t \in (0,T)$ and

$$
\tau := \frac{1}{R+Q}.
$$

Theorem

Let $\gamma_\pm > 1$ *and* R_0, Q_0, u_0 *satisfy*

$$0 < \underline{R_0} \le R_0 \le \overline{R_0} < \infty, \quad 0 < \underline{Q_0} \le Q_0 \le \overline{Q_0} < \infty, \quad u_0 \in L^2.$$

Then there exists a unique global-in-time weak solution.
Moreover, if (R, Q, u) *and* $(\widetilde{R}, \widetilde{Q}, \widetilde{u})$ *are two weak solutions on* Ω_T *corresponding to the initial data* (R_0, Q_0, u_0) *and* $(\widetilde{R_0}, \widetilde{Q_0}, \widetilde{u_0})$, *respectively, then*

$$\left(\|R - \widetilde{R}\|_{L^\infty(\Omega_T)} + \|Q - \widetilde{Q}\|_{L^\infty(\Omega_T)} + \|u - \widetilde{u}\|_{V_2(\Omega_T)}\right)$$
$$\le C\left(\|R_0 - \widetilde{R_0}\|_{L^\infty} + \|Q_0 - \widetilde{Q_0}\|_{L^\infty} + \|u_0 - \widetilde{u_0}\|_{L^2}\right),$$

where C *is a positive constant depending on the lower and upper bounds of* $(R_0, Q_0, \widetilde{R_0}, \widetilde{Q_0})$, *the* L^2*-norms of* $(u_0, \widetilde{u_0})$, μ, $\gamma_\pm$, *and* T.

Remark: We also proved the exponential decay to the unique steady state

$$\|(R - R_\infty, Q - Q_\infty, u - u_\infty)\|_{L^2} \le C_1 \exp(-C_2 t).$$

Part III – Inviscid system

Infinitely many weak solutions

Theorem

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $\gamma_\pm > 1$. Suppose that

$$R_0 > \underline{R} > 0, \;\; Q_0 > \underline{Q} > 0,$$

$$R_0, Q_0 \in C^3(\overline{\Omega}), \; \mathbf{u}_0 \in C^3(\overline{\Omega}; \mathbb{R}^3), \;\; \mathbf{u}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Then there exist infinitely many weak solutions to

$$\begin{cases} \partial_t R + \mathrm{div}_x(R\mathbf{u}) = 0, \\ \partial_t Q + \mathrm{div}_x(Q\mathbf{u}) = 0, \\ \partial_t[(R+Q)\mathbf{u}] + \mathrm{div}_x[(R+Q)\mathbf{u} \otimes \mathbf{u}] + \nabla_x p = \mathbf{0}. \end{cases}$$

Remark: The drawback of these solutions is that they admit initial energy jump due to the convex integration method applied. Therefore the energy inequality is not satisfied.

Non-uniqueness of the suitable weak solution

Theorem

Let Ω and $\gamma_\pm$ be as in the previous theorem. Assume that the initial densities R_0 and Q_0 are piecewise constant and bounded.
Then there exists $\mathbf{u}_0 \in L^\infty(\Omega; \mathbb{R}^3)$ such that the problem admits infinitely many weak solutions emanating from the same initial data $[R_0, Q_0, \mathbf{u}_0]$. Furthermore, these solutions comply with the conservation of total energy.

Question: Is there any way to choose the right solution?

$\rightarrow$ PhD project available $\leftarrow$

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Thank you for your attention!