Global well-posedness for the derivative nonlinear Schrödinger equation

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Aim : investigate the global well-posedness for the (DNLS) **equation**

$$\text{(DNLS)}\begin{cases} iu_t + u_{xx} = \pm i\partial_x(|u|^2 u),\ x \in \mathbb{R} \\ u_{|t=0} = u_0 \end{cases}$$

• (DNLS) involves in several physical problems :

- Asymptotic regimes of the propagation of Alfvén waves in polarized plasmas

- MHD equation in the presence of the Hall effect,...

• A considerable literature dealing with the (DNLS) equation since 2 decades :

- Local well-posedness is fully understood

- Global well-posedness is not completely settled

- Study of associated solitary waves : stability, variational characterization,...

Local well-posedness :

- Fully understood in the scale of Sobolev spaces

- Well-posedness for Cauchy data u_0 in $H^s(\mathbb{R})$, $s \geq \frac{1}{2}$ and blow-up criterion

Hayachi-Ozawa (1992) in H^1-setting, Takaoka (1999) for $H^s(\mathbb{R})$, $s \geq \frac{1}{2}$

- Ill-posedness in $H^s(\mathbb{R})$, $s < \frac{1}{2}$:

Biagioni-Linares (2001), Takaoka (2001)

- Main difficulty

- Derivative in the nonlinear term which generates a loss of derivative when investigating directly this nonlinear term

- One can overcome this difficulty by a gauge transformation

- The improvement from H^1 to $H^{\frac{1}{2}}$ is technically very costly

Known results about global well-posedness

• Best results up-to-date

- u_0 in $H^{\frac{1}{2}}(\mathbb{R})$, with small mass $\|u_0\|_{L^2}^2 < 4\pi$: Guo-Wu (2017)

- u_0 in $H^{2,2}(\mathbb{R}) = \left\{f \in H^2 : x^2 f \in L^2\right\}$: Jenkins-Liu-Perry-Sulem (2020)

• Two different approches

- PDE approach

a) First series of results under the assumption $\|u_0\|_{L^2}^2 < 2\pi$: Hayashi-Ozawa (1994), Colliander-Keel-Staffilani-Takaoka-Tao (2002),...

b) Results under the assumptions $\|u_0\|_{L^2}^2 < 4\pi$: Wu (2015), ...

- Inverse scattering approach (integrability structure) :

Pelinovsky-Saalmann-Shimabukuro (2017), Jenkins-Liu-Perry-Sulem (2018), (2020)

Basic properties of the (DNLS) equation

- Symmetry : the change of variable $x \to -x \Longrightarrow \pm \to \mp$

\- In what follows

$$\text{(DNLS)} \begin{cases} iu_t + u_{xx} = -i\partial_x(|u|^2 u) \\ u_{|t=0} = u_0 \in H^{\frac{1}{2}}(\mathbb{R}) \end{cases}$$

- Invariances

\- L^2-critical : $u(t,x) \to u_\mu(t,x) = \sqrt{\mu}\, u(\mu^2 t, \mu x), \quad \mu > 0$

\- 1/2 derivative gap in the H^s-scale : studies in $\widehat{H}^r_{1/2}$ / $\|u\|_{\widehat{H}^r_{1/2}} = \|\langle\cdot\rangle^{\frac{1}{2}}\hat{u}\|_{L^{r'}}$

Grünrock (2005) : local well-posedness for $u_0 \in \widehat{H}^r_{\frac{1}{2}}$, $1 < r \leq 2$

- (DNLS) is completely integrable

Infinite number of conservation laws, a Lax pair, explicit solitary waves,...

There are two philosophies concerning the study of global well-posedness for the (DNLS) equation :

- PDEs methods

behind the results with smallness condition on the mass

- Inverse scattering methods

behind the results in weighted Sobolev spaces

- In this work, we combine the two approaches to improve the known global well-posedness results

We prove the global well-posedness of (DNLS) for general initial data in $H^{\frac{1}{2}}$:

For any $u_0 \in H^{\frac{1}{2}}(\mathbb{R})$, the Cauchy problem associated with (DNLS) is globally well-posed, and the corresponding solution u satisfies

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^{\frac{1}{2}}(\mathbb{R})} < +\infty$$

- Our result closes the discussion in the setting of the Sobolev spaces H^s

- If $u_0 \in H^s(\mathbb{R})$, $s \geq 1/2$, then no turbulence occurs

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^s(\mathbb{R})} < +\infty$$

Keys tools in the known global well-posedness previous results : two strikingly different strategies

- PDEs arguments : to show that $\|u(t,\cdot)\|_{\dot{H}^s}$ is bounded

- Conservation laws : in particular, in H^1 framework

$$
\begin{aligned}
M(u) &= \int_{\mathbb{R}} |u(t,x)|^2 dx \\
P(u) &= \mathrm{Im} \int_{\mathbb{R}} \overline{u(t,x)} u_x(t,x) dx + \frac{1}{2} \int_{\mathbb{R}} |u(t,x)|^4 dx \\
E(u) &= \int_{\mathbb{R}} \left(|u_x(t,x)|^2 - \frac{3}{2} \mathrm{Im} |u(t,x)|^2 u(t,x) \overline{u_x(t,x)} + \frac{1}{2} |u(t,x)|^6 \right) dx
\end{aligned}
$$

- Gauge transformation $\mathcal{G}_a$

$$v(t,x) = \mathcal{G}_a u(t,x) = e^{ia \int_{-\infty}^{x} |u(t,y)|^2 dy} u(t,x)$$

Idea of proof of Hayashi-Ozawa global result : $u_0 \in H^1$ with $\|u_0\|_{L^2}^2 < 2\pi$

- Gauge transformation $v(t,x) = \mathcal{G}_{\frac{3}{4}} u(t,x)$

- Conservation laws

$$M(v) = \|u_0\|_{L^2}^2, \; E(v) = \|\partial_x v(t,\cdot)\|_{L^2}^2 - \frac{1}{16}\|v(t,\cdot)\|_{L^6}^6$$

- Gagliardo-Nirenberg inequality

$$\|f\|_{L^6}^6 \leq \frac{4}{\pi^2}\|\partial_x f\|_{L^2}^2 \|f\|_{L^2}^4$$

Then

$$\|\partial_x v(t,\cdot)\|_{L^2}^2 \leq E(v) + \frac{1}{16}\|v(t,\cdot)\|_{L^6}^6 \leq E(v) + \left(\frac{1}{2\pi}\|u_0\|_{L^2}^2\right)^2 \|\partial_x v(t,\cdot)\|_{L^2}^2$$

In $H^{\frac{1}{2}}$ setting, the proof is more involved (I-method of Bourgain)
Colliander-Keel-Staffilani-Takaoka-Tao

- Inverse scattering technics

- They are linked to the integrability structure of the equation

- The integrability structure of the equation imposes a sort of rigidness

- They require some regularity and decay : as for instance the weighted Sobolev spaces

$$H^{2,2}(\mathbb{R}) = \left\{ f \in H^2(\mathbb{R}) : x^2 f \in L^2(\mathbb{R}) \right\}$$

in the article of Jenkins-Liu-Perry-Sulem (2020)

- One can weaken the hypothesis on the spaces taking for instance $H^2 \cap H^{1,1}$ as in the article of Pelinovsky-Saalmann-Shimabukuro, but by imposing some generic conditions on the set of the scattering data

The strategy amounts to solve an inverse problem by recovering u from the scattering data

General strategy of proof of the global well-posedness

- By contradiction assuming that there is (t_n) such that

$$\mu_n = \|u(t_n,\cdot)\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R})} \to +\infty$$

- We rescale $u(t_n,\cdot)$ defining $U_n = \frac{1}{\sqrt{\mu_n}} u(t_n, \frac{\cdot}{\mu_n})$: (U_n) bounded in $H^{\frac{1}{2}}(\mathbb{R})$

- One can then apply the profile decompositions method to (U_n) (bubbles) : Brezis-Coron (1985),..., Gérard (1998), Merle-Vega (1998),..., Kenig-Merle (2008),... Jaffard (1999), Bahouri-Cohen-Koch (2011), Tintarev...

Other approaches : P.-L. Lions, Tartar, Murat-Tartar, Gérard,...

- The result we obtain here has additional properties coming from the integrability structure of the equation

- Finally, we get a contradiction by using scattering transform tools

- The starting point : Kaup-Newell paper (1978)

- (DNLS) is the integrability condition for the overdetermined system :

$$\partial_x \psi = \underbrace{-i\sigma_3\Big(\lambda^2 + i\lambda U\Big)}_{\mathcal{U}(\lambda)}\psi$$

$$\partial_t \psi = \underbrace{\Big(-i(2\lambda^4 - \lambda^2 |u|^2)\sigma_3 + (2\lambda^3 - \lambda |u|^2)\sigma_3 U + i\lambda U_x\Big)}_{\Upsilon(\lambda)}\psi$$

$\lambda \in \mathbb{C}$, $\psi(t,x,\lambda)$ a $\mathbb{C}^2$-valued function, σ_3 the Pauli matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad U(t,x) = \begin{pmatrix} 0 & u(t,x) \\ \overline{u}(t,x) & 0 \end{pmatrix}$$

- u satisfies the DNLS equation if and only if (Lax pair)

$$\frac{\partial \mathcal{U}}{\partial t} - \frac{\partial \Upsilon}{\partial x} + [\mathcal{U}, \Upsilon] = 0$$

- The scattering transform is defined via the first equation

$$L_u(\lambda)\psi = 0, \; L_u(\lambda) = i\sigma_3\partial_x - \lambda^2 - i\lambda U$$

- The heart of the matter relies on the study of the operator $L_u(\lambda)$

• If $u \in \mathcal{S}$, then there are unique solutions ψ_1^-, ψ_2^+ holomorphic on $\Omega_+ = \{\lambda \in \mathbb{C} : \; \mathrm{Im}\, \lambda^2 > 0\}$, C^∞ on $\overline{\Omega}_+$ (Jost solutions)

$$\begin{aligned} \psi_1^-(x,\lambda) &= e^{-i\lambda^2 x}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \circ(1)\right], \quad \text{as} \quad x \to -\infty \\ \psi_2^+(x,\lambda) &= e^{i\lambda^2 x}\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \circ(1)\right], \quad \text{as} \quad x \to +\infty \end{aligned}$$

This issue amounts to study a Volterra operator type : integrability condition on u is needed

• Since U is a traceless matrix, a_u the Wronskian of ψ_1^- and ψ_2^+ is independent of x (transmission coefficient $1/a_u$)

$$a_u(\lambda) = \det(\psi_1^-(x,\lambda), \psi_2^+(x,\lambda))$$

• Other ways to define a_u : a coefficient in the transfer matrix, regularized Fredholm determinant that can be defined for $u \in L^2$

• If $u = u(t)$ is a solution of (DNLS), then (using the second equation)

$$\partial_t a_{u(t)}(\lambda) = 0 \Leftrightarrow a_{u(t)}(\lambda) = a_{u_0}(\lambda)$$

• a_u satisfies several useful properties :

- $a_u(0) = 1$

- Invariances : $a_{u_\mu}(\lambda) = a_u\left(\frac{\lambda}{\sqrt{\mu}}\right),\ a_u = a_u(\cdot - x_0),\ a_u = a_{e^{i\theta}u},\ \forall \theta \in \mathbb{R}$

- Asymptotic behavior (that can be proved using a suitable transform reducing $L_u(\lambda)$ to a Zakharov-Shabat spectral problem)

$$\lim_{|\lambda|\to\infty,\, \lambda\in\overline{\Omega}_+} a_u(\lambda) = e^{-\frac{i}{2}\|u\|^2_{L^2(\mathbb{R})}}$$

• We introduce, for $\zeta \in \mathbb{C}$ with $\operatorname{Im}\zeta \geq 0$, $\tilde{a}_u(\zeta) = e^{\frac{i}{2}\|u\|^2_{L^2(\mathbb{R})}} a_u(\sqrt{\zeta})$

$$\lim_{|\zeta|\to\infty,\, \zeta\in\mathbb{C}_+} \tilde{a}_u(\zeta) = 1,\ |\tilde{a}_u(\zeta)| \geq 1 \text{ for } \zeta \in \mathbb{R}_- \text{ and } |\tilde{a}_u(\zeta)| \leq 1 \text{ for } \zeta \in \mathbb{R}_+$$

In particular, $\ln \tilde{a}_u(\zeta)$ (which is holomorphic on ζ in $\mathbb{C}_+$ for $|\zeta|$ sufficiently large) plays an important role in the study of (DNLS)

• Under the hypothesis that (i) $\tilde{a}_u$ does not vanish on the real line and ii) $\tilde{a}_u$ has only simple zeros $\zeta_1, \ldots, \zeta_N$ in $\mathbb{C}_+$ (that hold generically in $\mathcal{S}(\mathbb{R})$ Beals-Coifman,...), one has (by complex analysis arguments)

$$\tilde{a}_u(\zeta) = \prod_{j=1}^{N} \left(\frac{\zeta - \zeta_j}{\zeta - \overline{\zeta}_j}\right) \exp\left(\frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{d\zeta'}{\zeta' - \zeta} \ln|\tilde{a}_u(\zeta')|^2\right)$$

which leads to the following asymptotic expansion as $|\zeta| \to +\infty$, $\operatorname{Im}\zeta \geq 0$:

$$\ln \tilde{a}_u(\zeta) = \sum_{k\geq 1} E_k(u)\zeta^{-k}$$

E_k are, up to a constant, the conservation laws of (DNLS)

$$P(u) = -8\sum_{j=1}^{N} \operatorname{Im}\zeta_j + \frac{2}{\pi}\int_{-\infty}^{\infty} d\xi \, \ln|\tilde{a}_u(\xi)|^2$$

$$M(u) = 4\sum_{j=1}^{N} \arg(\zeta_j) \underbrace{-\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\xi}{\xi} \ln|\tilde{a}_u(\xi)|^2}_{\geq 0}$$

Crucial properties concerning the zeros of $\tilde{a}_u$

• The real parts of the zeros of $\tilde{a}_u(\zeta)$ are low-bounded uniformly/ $\|u\|_{H^{\frac{1}{2}}}$

$$\text{if } \tilde{a}_u(\zeta_0) = 0, \text{ then } \mathrm{Re}(\zeta_0) \geq -C(\|u\|_{H^{\frac{1}{2}}})$$

• Bounds on the number of the zeros of $\tilde{a}_u$ in the angles, using the expression of the mass by means of its zeros and the trace on the real line

$$\sharp\Big\{\zeta \in \mathbb{C}_+ : \tilde{a}_u(\zeta) = 0,\ 0 < \theta_0 < \arg\zeta < \pi\Big\} \leq \frac{1}{4\theta_0}\|u\|^2_{L^2}$$

• Under the hypothesis of Beals-Coifman, if $\tilde{a}_u \neq 0$ on the ray $e^{i\theta_0}\mathbb{R}_+$, then (trace formula and complex analysis)

$$\sharp\Big\{\zeta \in \mathbb{C}_+ : \tilde{a}_u(\zeta) = 0,\ 0 < \theta_0 < \arg\zeta < \pi\Big\} = \frac{1}{2i\pi}\int_0^{+\infty e^{i\theta_0}} \frac{\tilde{a}'_u(s)}{\tilde{a}_u(s)}ds + \frac{1}{4\pi}\|u\|^2_{L^2}$$

Summary

To u solution of (DNLS), we associate $\tilde{a}_u$ holomorphic in $\mathbb{C}_+$:

- $\tilde{a}_{u(t)} = \tilde{a}_{u_0}$, $\tilde{a}_{u_\mu}(\zeta) = \tilde{a}_u\left(\frac{\zeta}{\mu}\right)$, $\tilde{a}_u(0) = e^{\frac{i}{2}\|u\|_{L^2}^2}$, $\tilde{a}_u(\zeta) \xrightarrow{|\zeta|\to\infty} 1$

- Complex analysis formula (that holds generically)

$$\tilde{a}_u(\zeta) = \prod_{j=1}^{N} \left(\frac{\zeta - \zeta_j}{\zeta - \overline{\zeta}_j}\right) \exp\left(\frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{d\zeta'}{\zeta' - \zeta} \ln|\tilde{a}_u(\zeta')|^2\right)$$

- If $\tilde{a}_u(\zeta_0) = 0$, then there exists ψ_0 with $\|\psi_0\|_{L^2} = 1$ such that

$$L_u(\lambda_0)\psi_0 = 0, \quad \text{with} \quad \zeta_0 = \lambda_0^2$$

- Stability estimates, bounds on the number of the zeros and their real parts
- $\ln \tilde{a}_u$: holomorphic, trace formula,...

First step : Rigidity type theorem

If for $u_0 \in H^{\frac{1}{2}}$, $\mu_n = \|u(t_n)\|^2_{\dot{H}^{\frac{1}{2}}} \to +\infty$, then there is $1 \leq L_0 \leq \frac{\|u_0\|^2_{L^2}}{4\pi}$ such that, up to subsequence extraction $(U_n = \frac{1}{\sqrt{\mu_n}} u(t_n, \frac{\cdot}{\mu_n}))$

$$U_n(y) = \sum_{\ell=1}^{L_0} V^{(\ell)}(y - y_n^{(\ell)}) + \mathrm{r}_n(y)\,, \ \|\mathrm{r}_n\|_{L^p(\mathbb{R})} \overset{n\to\infty}{\longrightarrow} 0, \ \forall 2 < p < \infty$$

with for all $\ell \neq \ell'$, $|y_n^{(\ell)} - y_n^{(\ell')}| \overset{n\to\infty}{\longrightarrow} \infty$, $V^{(\ell)} \neq 0$ in $H^{\frac{1}{2}}(\mathbb{R})$ and $\tilde{a}_{V^{(\ell)}} \equiv 1$ on $\mathbb{C}_+$. Moreover, we have the stability estimates

$$\|U_n\|^2_{L^2} = \sum_{\ell=1}^{L_0} \|V^{(\ell)}\|^2_{L^2} + \|\mathrm{r}_n\|^2_{L^2} + \circ(1), \quad n \to \infty\,,$$

- $\tilde{a}_{V^{(\ell)}} \equiv 1 \Rightarrow \|V^{(\ell)}\|^2_{L^2} = 4k\pi \overset{\text{orthogonality}}{\Rightarrow}$ a finite number of profiles

- If $\|V^{(\ell)}\|^2_{L^2} = 4\pi \overset{\text{(symmetries)}}{\Rightarrow} V^{(\ell)}(x) = \frac{2\sqrt{2}}{\sqrt{1+4x^2}}$ (extremal of some Gagliardo-Nirenberg inequality) Berestycki-Lions (1983)

Scheme of the proof of the profile decomposition

- The standard profile decomposition techniques ensure that (but with $L \geq 0$ and $V^{(\ell)} \in H^{\frac{1}{2}}$) (up to subsequence extraction)

$$U_n(y) = \sum_{\ell=0}^{L} V^{(\ell)}(y - y_n^{(\ell)}) + \mathrm{r}_n^L(y)\,, \ \limsup_{n\to\infty} \|\mathrm{r}_n^L\|_{L^p} \overset{L\to\infty}{\longrightarrow} 0,\ \forall 2 < p < \infty$$

Two additional informations : $L \geq 1$ and $\tilde{a}_{V^{(\ell)}} \equiv 1$

- How to prove that there is at least one profile $V^{(\ell)} \neq 0$?

• If all the profiles $V^{(\ell)} = 0$, then by construction $\|U_n\|_{L^4} \overset{n\to\infty}{\longrightarrow} 0$ and since $P(U_n) = \frac{1}{\mu_n} P(u(t_n)) = \frac{1}{\mu_n} P(u_0)$, we deduce that

$$\mathrm{Im} \int_{\mathbb{R}} \overline{U_n(x)} (U_n)_x(x) dx \overset{n\to\infty}{\longrightarrow} 0$$

• One cannot conclude with $\|U_n\|_{\dot{H}^{\frac{1}{2}}} = 1$! The difficulty is that the momentum doest not allow to control the $H^{\frac{1}{2}}$-norm

• In H^1-framework, one can easily conclude using the energy

• We overcome this difficulty by proving that, for R large enough, $\varphi_u(\rho) = \text{Im}(\ln \tilde{a}_u(i\rho))$, $\rho > 0$ belongs to $L^1([R, +\infty[)$ and that

$$\|u\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R})} \lesssim_{R,\|u\|_{L^2}} \|\varphi_u\|_{L^1([R,+\infty[)} + \|u\|^4_{L^4(\mathbb{R})}$$

Trace formula (for $|\zeta|$ large enough, with $T_u(\sqrt{\zeta}) = i\sqrt{\zeta}(i\sigma_3\partial_x - \zeta)^{-1}U$)

$$\ln \tilde{a}_u(\zeta) = \frac{i}{2}\|u\|^2_{L^2(\mathbb{R})} - \sum_{k=2}^{\infty} \frac{\text{Tr}\, T_u^k(\sqrt{\zeta})}{k} = \frac{i}{2}\int_{\mathbb{R}} dp\, \frac{p|\widehat{u}(p)|^2}{p+2\zeta} - \sum_{k=4}^{\infty} \frac{\text{Tr}\, T_u^k(\sqrt{\zeta})}{k},$$

which implies that

$$\varphi_u(\rho) = \underbrace{\int_{\mathbb{R}} dp\, \frac{p^2|\widehat{u}(p)|^2}{p^2+4\rho^2}}_{L^1(\rho \geq R) \gtrsim \int_{|p|\geq R} |p||\widehat{u}(p)|^2 dp} + \mathcal{O}_{\|u\|_{L^2}}\left(\rho^{-1-s}\|u\|_{\dot{H}^{\frac{1}{4}+s}(\mathbb{R})}\right)$$

φ_u used in works in other contexts

Killip-Visan (2019) and Koch-Tataru (2018), Klaus-Schippa (2020),...

• Applying the above estimate to $u(t_n,\cdot)$ and using the conservation of a_u, we get

$$\|u(t_n,\cdot)\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R})} \lesssim_{R,\|u_0\|_{L^2}} \underbrace{\|\varphi_{u(t_n,\cdot)}\|_{L^1}}_{=\|\varphi_{u_0}\|_{L^1}} + \|u(t_n,\cdot)\|^4_{L^4(\mathbb{R})}$$

which implies that

$$\|u(t_n,\cdot)\|^4_{L^4(\mathbb{R})} \geq c\mu_n, \;\; c > 0$$

• Then, by scale invariance

$$\|U_n\|^4_{L^4(\mathbb{R})} \geq c$$

which implies that there is at least one profile $V^{(\ell)} \neq 0$

• A key information : $\tilde{a}_{u(t)} = \tilde{a}_{u_0}$

- Now how to prove that $\tilde{a}_{V(\ell)} \equiv 1$?

1) We prove that, for all $\rho \geq C(\|u_0\|_{L^2})$ $(\varphi_u(\rho) = \mathrm{Im}(\ln \tilde{a}_u(i\rho)))$,

$$\varphi_{V(\ell)}(\rho) = 0$$

- A profile decomposition for $\varphi_{U_n}(\rho)$

$$\varphi_{U_n}(\rho) = \sum_{\ell=1}^{L} \varphi_{V(\ell)}(\rho) + \Phi_{n,L}(\rho) \quad \text{with} \quad \limsup_{n\to\infty} \Phi_{n,L}(\rho) \overset{L\to\infty}{\longrightarrow} 0$$

(i) Realization of a_u as a regularized Fredholm determinant (Brascamp, Gohberg-Krien, Dunford-Schwartz, ...)

(ii) Factorization of a_{U_n} using the general profile decomposition of (U_n) and general properties of the regularized Fredholm determinants

- Thanks to the conservation of the $\tilde{a}_{u(t)}$, the scale invariance and the asymptotic at infinity, we get

$$\varphi_{U_n}(\rho) = \varphi_{u_0}(\mu_n \rho) \overset{n\to\infty}{\longrightarrow} 0, \quad \forall \rho > 0$$

- One can then conclude by showing that, for all ℓ,

$$\varphi_{V^{(\ell)}}(\rho) \geq 0$$

(i) First, we use the complex analysis formula

$$\tilde{a}(\zeta) = \prod_{j=1}^{N} \left(\frac{\zeta - \zeta_j}{\zeta - \overline{\zeta}_j} \right) \exp\left(\frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{d\zeta'}{\zeta' - \zeta} \ln |\tilde{a}(\zeta')|^2 \right)$$

That holds generically, then $w_n \overset{n \to +\infty}{\longrightarrow} V^{(\ell)}$ such that

$$\varphi_{w_n}(\rho) = \underbrace{\sum_{\substack{1 \leq j \leq N_n \\ \mathrm{Re}\zeta_j^n > 0}} \mathrm{Im} \ln \left(\frac{i\rho - \zeta_j^n}{i\rho - \overline{\zeta_j^n}} \right)}_{\geq 0} \underbrace{- \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\xi}{\xi^2 + \rho^2} \xi \ln |\tilde{a}_{w_n}(\xi)|^2}_{\geq 0}$$

$$+ \sum_{\substack{1 \leq j \leq N_n \\ \mathrm{Re}\zeta_j^n < 0}} \mathrm{Im} \ln \left(\frac{i\rho - \zeta_j^n}{i\rho - \overline{\zeta_j^n}} \right)$$

(ii) Second, we prove that

$$\sum_{\substack{1\leq j\leq N_n\\ \mathrm{Re}\zeta_j^n<0}} \mathrm{Im}\,\mathrm{ln}\left(\frac{i\rho-\zeta_j^n}{i\rho-\overline{\zeta_j^n}}\right) \overset{n\to+\infty}{\longrightarrow} 0$$

which gives the result, thanks to the stability estimates

Now, how to establish the above property ?

Key argument : $\tilde{a}_{V(\ell)}$ does not vanish on Ω_+

By contradiction, assuming that there is ℓ_0 and ψ_0 such that

$$L_{V(\ell_0)}(\lambda_0)\psi_0 = 0, \quad \text{with} \quad \|\psi_0\|_{L^2} = 1$$

Writing $L_{U_n}(\lambda_0)\psi_0(\cdot - y_n^{(\ell_0)}) = \mathcal{R}_n(y)$, we easily find that $\|\mathcal{R}_n\|_{L^2} \overset{n\to+\infty}{\longrightarrow} 0$ the orthogonality between the profiles

By scale invariance and the conservation of the transition coefficient

$$a_{U_n}(\lambda_0) = a_{u(t_n,\cdot)}(\sqrt{\mu}_n\lambda_0) = a_{u_0}(\sqrt{\mu}_n\lambda_0)$$

According to the asymptotic at infinity, this implies that

$$|a_{U_n}(\lambda_0)| \geq \frac{1}{2}, \quad n >> 1 \Leftrightarrow L_{U_n}(\lambda_0) \text{ is invertible}$$

which leads to a contradiction, since $L_{U_n}(\lambda_0)$ is invertible and $\|\psi_0\|_{L^2} = 1$

Then using the stability of the "transmission coefficient" (here involves the information : $\tilde{a}_{V^{(\ell)}}$ does not vanish on Ω_+), we deduce that

$$\sup_{j=1,\ldots,N_n} \frac{\text{Im}\zeta_j^n}{|\text{Re}\zeta_j^n|} \overset{n\to\infty}{\longrightarrow} 0$$

which ensures the result thanks to the mass conservation $\Rightarrow$

$$\sharp\left\{\zeta_i^n, \text{Re}(\zeta_j^n) < 0\right\} \lesssim \|w_n\|_{L^2}^2 \lesssim \|V^{(\ell)}\|_{L^2}^2 \lesssim \|u_0\|_{L^2}^2$$

2) We conclude, by proving the following general result :

Let $u \in H^{\frac{1}{2}}$ such that $\tilde{a}_u$ has no zeros in $\mathbb{C}_+$. If $\varphi_u(\rho_0) = 0$ for some $\rho_0 > 0$, then $\tilde{a}_u \equiv 1$

(i) Using the complex analysis formula

$$\tilde{a}(\zeta) = \prod_{i=1}^{N} \left(\frac{\zeta - \zeta_i}{\zeta - \overline{\zeta}_i} \right) \exp \left(\frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{d\zeta'}{\zeta' - \zeta} \ln |\tilde{a}(\zeta')|^2 \right)$$

and an appropriate approximation (w_n), one can prove that for all $\rho > 0$

$$\tilde{a}_{w_n}(i\rho) \overset{n \to \infty}{\longrightarrow} 1$$

(ii) By stability estimates, we deduce that

$$\tilde{a}_u \equiv 1 \text{ on } i\,\mathbb{R}_+$$

The analyticity of $\tilde{a}_u$ allows to conclude the proof

Second step : End of the proof

- To complete the proof of the theorem, and with such decomposition at hand, we prove that, up to a subsequence extraction and an appropriate approximation (Beals-Coifman,...)

The coefficient $\tilde{a}_{U_n}$ admits at least a zero ζ_n such that

$$\text{Re}(\zeta_n) \leq -c_0, \; n >> 1 \quad \text{with} \quad c_0 > 0$$

- This leads to a contradiction, since by scale invariance and in view of the conservation of the transition coefficient

$$\tilde{a}_{U_n}(\zeta_n) = \tilde{a}_{u_0}(\mu_n \zeta_n)$$

and we know that the real parts of the zeros of $\tilde{a}_{u_0}$ are lower bounded

$$\text{Re}(\mu_n \zeta_n) \geq -C$$

Therefore

$$\underbrace{-\frac{C}{\mu_n}}_{\overset{n\to+\infty}{\longrightarrow} 0} \leq \text{Re}(\zeta_n) \leq \underbrace{-c_0}_{<0}$$

Idea of proof of the fact that $\widetilde{a}_{U_n}$ admits at least a zero ζ_n

Two key ingredients :

- The closeness of $\widetilde{a}_{U_n}$ and $\widetilde{a}_{\mathrm{r}_n}$

$$\widetilde{a}_{U_n}(\zeta) - \widetilde{a}_{\mathrm{r}_n}(\zeta) \overset{n \to +\infty}{\longrightarrow} 0$$

(i) Regularized Fredholm determinants

(ii) $\widetilde{a}_{V(\ell)} \equiv 1$

- By the analyticity and the asymptotic at infinity, $\exists\, \frac{\pi}{2} < \theta_0 < \pi$ such that

$$\tilde{a}_{u_0}(\zeta) \neq 0\,,\ \forall \zeta\,,\ \theta_0 \leq \arg(\zeta) < \pi\,.$$

By scale invariance, conservation of the transmission coefficients, the asymptotic and the closeness of $\widetilde{a}_{U_n}$ and $\widetilde{a}_{\mathrm{r}_n}$, we have

$$\sup_{\substack{\zeta \in \mathbb{C}_+ \\ \arg \zeta = \theta_0}} \left| 1 - \frac{\widetilde{a}_{r_n}(\zeta)}{\widetilde{a}_{U_n}(\zeta)} \right| \overset{n \to +\infty}{\longrightarrow} 0$$

Then, using the bounds on the number of zeros of $\tilde{a}$, we get

$$\sharp\left\{\zeta \in \mathbb{C}_+ : \tilde{a}_{U_n}(\zeta) = 0,\ \theta_0 < \arg\zeta < \pi\right\} \geq \frac{1}{4\pi} \underbrace{\left(\|U_n\|^2_{L^2} - \|\mathrm{r}_n\|^2_{L^2}\right)}_{\overset{n\to+\infty}{\longrightarrow} \sum_{\ell=1}^{L_0} \|V^{(\ell)}\|^2_{L^2(\mathbb{R})}}$$

Here involves the fact that there is at least a profile $V^{(\ell)} \neq 0$

Idea of proof of $\mathrm{Re}(\zeta_n) \leq -c_0,\ n >> 1$ with $c_0 > 0$

By contradiction, assuming that (up to a suitable approximation)

$$\liminf_{n\to\infty} \mathrm{Re}(\zeta_n) \geq 0$$

Then applying the Bäcklund transformation, we get a contradiction

The key property of the Bäcklund transformation is that it allows to add or to remove eigenvalues of the Kaup-Newell spectral problem