From heterogeneous microscopic traffic flow models to macroscopic models

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May 18, 2020

The problem of modelling traffic flow

- By which laws do vehicles interact with each other?
- Temporal evolution of traffic density?

What we address here

- Traffic on a single line
- No overtaking

Two classical models of traffic flow

We study traffic flow models on a single straight road (without overtake).

Two kinds of models:

1) Microscopic models: e.g., the follow-the-leader model is a system of ODEs

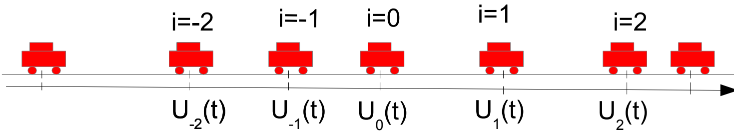
$$\frac{d}{dt}U_i(t) = V(U_{i+1}(t) - U_i(t)), \ t \geqslant 0, \forall i \in \mathbb{Z}.$$

2) Macroscopic models: e.g., the Lighthill-Whitham-Richards (LWR) model is the scalar conservation law

$$\partial_t \rho + (\rho v(\rho))_x = 0 \qquad \text{in } \mathbb{R} \times (0, +\infty),$$

(M. J. Lighthill and G. B. Whitham (1955), P. I. Richards (1956))

The follow-the-leader model



$$\frac{d}{dt}U_i(t) = V(U_{i+1}(t) - U_i(t)),\ t \geqslant 0, \forall i \in \mathbb{Z},$$

where

- $U_i(t)$ denotes the position of car $i \in \mathbb{Z}$ at time $t \geqslant 0$,
- Cars are ordered: $U_i(t) \leqslant U_{i+1}(t)$ for all t, i,
- The velocity $V = V(p) \geqslant 0$ of car i depends in an increasing way on the distance p of car i to car $i+1$.

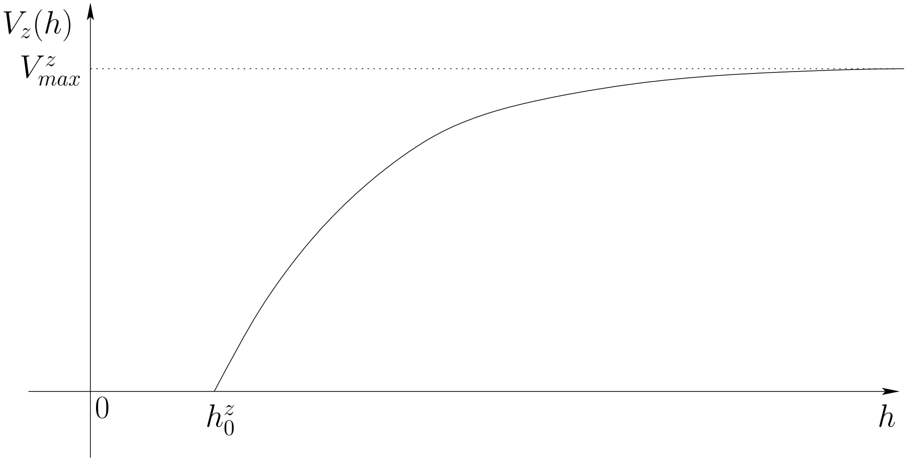


Figure: Typical shape of the optimal velocity function V.

Properties of the follow-the-leader model

$$\boxed{\frac{d}{dt}U_i(t) = V(U_{i+1}(t) - U_i(t)),\ t \geqslant 0, \forall i \in \mathbb{Z},}$$

- Compute trajectories of each vehicle
- Can be extended to multiple lanes
- At the core of most micro-simulators
- Good for simulation

(cf. Seibold, B. (2015). A mathematical introduction to traffic flow theory. IPAM Tutorials.)

The LWR Model

The Lighthill-Whitham-Richards (LWR) model is the scalar conservation law

$$\partial_t \rho + (\rho v(\rho))_x = 0 \qquad \text{in } \mathbb{R} \times (0, +\infty),$$

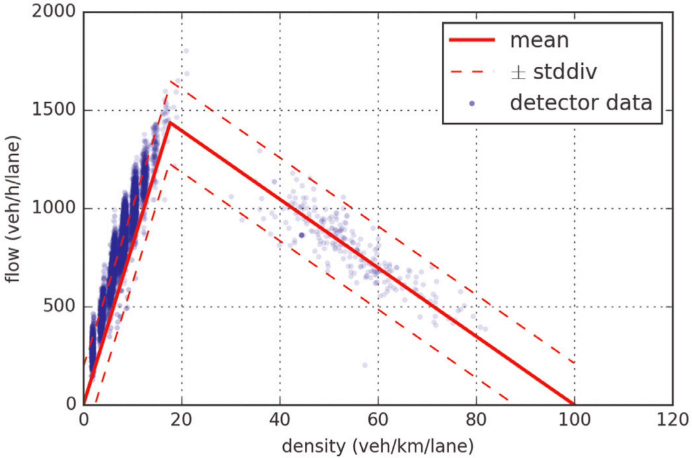
where

- ρ is the density of vehicles on the road,
- $v : \mathbb{R}_+ \to \mathbb{R}_+$. The map $f(\rho) = \rho v(\rho)$ is the so-called "fundamental diagram"

Properties of the LWR model:

- Describe aggregate quantities via PDE
- Natural framework for traveling waves and shocks

The fundamental diagram $f(\rho) = \rho v(\rho)$



(after Seo, T., Kawasaki, Y., Kusakabe, T., & Asakura, Y. (2019))

Goal of the talk

- Discuss how to derive the LWR model

$$\partial_t \rho + (\rho v(\rho))_x = 0 \qquad \text{in } \mathbb{R} \times (0, +\infty),$$

  from the follow-the-leader model

$$\frac{d}{dt} U_i(t) = V(U_{i+1}(t) - U_i(t)), \ t \geqslant 0, \forall i \in \mathbb{Z}.$$

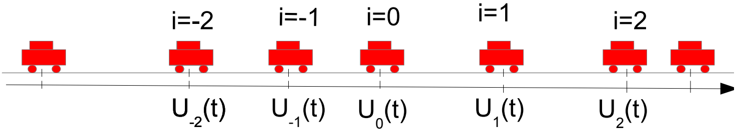
- Well-known when all the vehicles are identical, i.e., V does not depend on i. Then $f(\rho) = \rho v(\rho) = \rho V(1/\rho)$ (Aw, Klar, Materne, and Rascle (2002))
- The fact that the vehicles are identical is a very restrictive (and unatural) assumption.
- Main contribution: we address the case where the vehicles are different:

$$\frac{d}{dt} U_i(t) = V_i(U_{i+1}(t) - U_i(t)), \ t \geqslant 0, \forall i \in \mathbb{Z},$$

  where the distribution of the V_i is "well distributed".

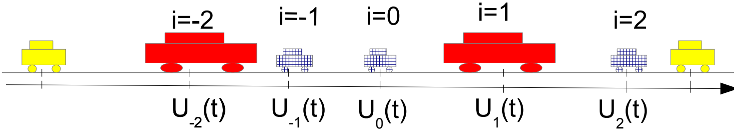
From an homogeneous traffic flow...

$$\frac{d}{dt}U_i(t) = V(U_{i+1}(t) - U_i(t)),\ t \geqslant 0, \forall i \in \mathbb{Z}, \qquad f(\rho) = \rho V(1/\rho)$$



... to an heterogenous one:

$$\frac{d}{dt}U_i(t) = V_i(U_{i+1}(t) - U_i(t)),\ t \geqslant 0, \forall i \in \mathbb{Z},$$



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Heuristic derivation of the macro model from the micro one

The microscopic model: $\frac{d}{dt}U_i(t) = V(U_{i+1}(t) - U_i(t)),\ t \geqslant 0, \forall i \in \mathbb{Z}$,
The macroscopic model (LWR): $\partial_t \rho + (\rho v(\rho))_x = 0 \qquad \text{in } \mathbb{R} \times (0, +\infty)$

- Consider the distribution of vehicles $R(t) = \sum_{i \in \mathbb{Z}} \delta_{U_i(t)}$.
- After an hyperbolic scaling $(x, t) \to (\epsilon^{-1}x, \epsilon^{-1}t)$, we obtain $\rho^\epsilon(t) = \epsilon \sum_{i \in \mathbb{Z}} \delta_{\epsilon U_i(\epsilon^{-1}t)}$.
- Then, for any test function $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} \phi(x)\rho^\epsilon(dx, t) &= \frac{d}{dt}\epsilon \sum_{i \in \mathbb{Z}} \phi(\epsilon U_i(\epsilon^{-1}t)) \\ &= \epsilon \sum_{i \in \mathbb{Z}} \phi_x(\epsilon U_i(\epsilon^{-1}t))\ \frac{d}{dt}U_i(\epsilon^{-1}t) \\ &= \epsilon \sum_{i \in \mathbb{Z}} \phi_x(\epsilon U_i(\epsilon^{-1}t))\ V\left(U_{(i+1)}(\epsilon^{-1}t) - U_i(\epsilon^{-1}t)\right)\end{aligned}$$

- Next we show that $U_{(i+1)}(\epsilon^{-1}t) - U_i(\epsilon^{-1}t) \simeq 1/\left(\rho^\epsilon(U_i(\epsilon^{-1}t)\right)$.

- Proof that $U_{(i+1)}(\epsilon^{-1}t) - U_i(\epsilon^{-1}t) \simeq 1/\left(\rho^\epsilon(U_i(\epsilon^{-1}t))\right)$:

  Indeed, if $x = \epsilon U_i(\epsilon^{-1}t))$ and $dx = \epsilon(U_{(i+1)}(\epsilon^{-1}t) - U_i(\epsilon^{-1}t))$, then,

$$\rho^\epsilon([x, x+dx), t) = \epsilon \operatorname{card}\{j \in \mathbb{Z},\ \epsilon U_j(\epsilon^{-1}t) \in [x, x+dx)) = \epsilon,$$

  so that

$$\begin{aligned}\rho^\epsilon(x,t) &\simeq \rho^\epsilon([x, x+dx), t)\ (dx)^{-1} = \epsilon\ (dx)^{-1} \\ &= (U_{(i+1)}(\epsilon^{-1}t) - U_i(\epsilon^{-1}t))^{-1}.\end{aligned}$$

- As $\rho^\epsilon(t) = \epsilon \sum_{i\in\mathbb{Z}} \delta_{\epsilon U_i(\epsilon^{-1}t)}$, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} \phi(x)\rho^\epsilon(dx,t) &\simeq \epsilon \sum_{i\in\mathbb{Z}} \phi_x(\epsilon U_i(\epsilon^{-1}t))\, V\left(\frac{1}{\rho^\epsilon(\epsilon U_i(\epsilon^{-1}t), t)}\right) \\ &= \int_{\mathbb{R}} \phi_x(x)\, V\left(\frac{1}{\rho^\epsilon(x,t)}\right) \rho^\epsilon(dx,t).\end{aligned}$$

- So ρ^ϵ solves (LWR) $\partial_t \rho^\epsilon + (\rho^\epsilon v(\rho^\epsilon))_x = 0$ in the sense of distribution with $v(s) = V(1/s)$.

Some references

Rigorous derivation of the macroscopic model from the microscopic one:

- For one type of vehicles:
  - Argall, Cheleshkin, Greenberg, Hinde, and Lin (2002)
  - Aw, Klar, Materne, and Rascle (2002)
  - Di Francesco and Rosini (2015)
  - Goatin and Rossi (2017)
  - Holden and Risebro (2018)
- For several types of cars:
  - Chiabaut, Leclercq, and Buisson (2010)
    (random model, heuristic derivation)
  - Forcadel and Salazar (2015)
    (periodic setting)

Another heuristic derivation (through Hamilton-Jacobi)

The microscopic model: $\frac{d}{dt}U_i(t) = V_i(U_{i+1}(t) - U_i(t)),\ t \geqslant 0, \forall i \in \mathbb{Z},$
The macroscopic model (LWR): $\partial_t \rho + (\rho v(\rho))_x = 0 \qquad \text{in } \mathbb{R} \times (0, +\infty)$

- Let $u(\cdot, t)$ be the piecewise affine map such that $u(i,t) = U_i(t)$ for all $i \in \mathbb{Z}$.
- We consider the hyperbolic scaling: $u^\epsilon(x,t) = \epsilon u(\epsilon^{-1}x, \epsilon^{-1}t)$.
- If $\rho^\epsilon(t) = \epsilon \sum_{i \in \mathbb{Z}} \delta_{\epsilon U_i(\epsilon^{-1}t)}$, one can show that $\rho^\epsilon(t) = \partial_x (u^\epsilon)^{-1}(\cdot, t)$.
- If $x = \epsilon i$,

$$\begin{aligned}\partial_t u^\epsilon(x,t) &= \frac{d}{dt}\left(\epsilon U_i(\epsilon^{-1}t)\right) = \frac{d}{dt} U_i(\epsilon^{-1}t)) = V_i(U_{i+1}(\epsilon^{-1}t) - U_i(\epsilon^{-1}t)) \\ &= V_{[x/\epsilon]}\left(\epsilon^{-1}(u^\epsilon(x+\epsilon,t) - u^\epsilon(x,t))\right) \simeq V_{[x/\epsilon]}(\partial_x u^\epsilon(x,t)).\end{aligned}$$

- So u^ϵ "solves" the HJ equation: $\qquad \partial_t u^\epsilon(x,t) = V_{[x/\epsilon]}(\partial_x u^\epsilon(x,t))$ from which one expect to derive (LWR).

Some references (cont'd)

The proof based on Hamilton-Jacobi is related to the analysis of the Frenkel-Kontorova models:

- Aubry (1983), Aubry and Le Daeron (1983),
- Forcadel, Imbert, and Monneau (2009)

Our work is within the framework of (stochastic) homogenization of HJ equations:

- Lions, Papanicolaou, and Varadhan (1987): periodic setting
- Souganidis (1999), Rezakhanlou and Tarver (2000): convergence
- Armstrong, C., Souganidis: convergence rate
- Subsequent works by Armstrong, Ciomaga, Davini, Feldman, Kosygina, Lin, Lions, C., Nolen, Novikov, Schwab, Seeger, Smart, Souganidis, Tran, Varadhan, Yilmaz, Zeitouni...

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Heuristic arguments in the homogeneous case

Main results

Ideas of proof

A random microscopic model

We consider a random version of the follow-the-leader model:

$$\frac{d}{dt} U_i(t) = V_{Z_i}(U_{i+1}(t) - U_i(t)),\ t \geqslant 0, \forall i \in \mathbb{Z},$$

where

- $U_i(t)$ denotes the position of car i at time t,
- Cars are ordered: $U_i(t) \leqslant U_{i+1}(t)$ for all t, i,
- The velocity $V = V_{Z_i}(p)$ of car i depends on the distance p of car i to car $i+1$ and on the "type" Z_i of car i
- The types are (Z_i) are I.I.D. random variables.

(cf. N. Chiabaut, L. Leclercq, and C. Buisson (2010))

Assumptions

On the optimal velocity map $V : \mathcal{Z} \times \mathbb{R}_+ \to \mathbb{R}_+$, we assume the following:

(H_1) The map $(z, p) \to V_z(p)$ is uniformly continuous on $\mathcal{Z} \times \mathbb{R}_+$ and $p \to V_z(p)$ is Lipschitz continuous, uniformly with respect to $z \in \mathcal{Z}$;

(H_2) For any $z \in \mathcal{Z}$, there exists $h_0^z > 0$ (depending in a measurable way on z) such that $V_z(p) = 0$ for all $p \in [0, h_0^z]$;

(H_3) For any $z \in \mathcal{Z}$, $p \to V_z(p)$ is increasing in $[h_0^z, +\infty)$;

(H_4) There exists $V_{\max} > 0$ and, for any $z \in \mathcal{Z}$, there exists $V_{\max}^z \leqslant V_{\max}$, such that $\lim_{p \to +\infty} V_z(p) = V_{\max}^z$.

(H_5) If we set $\underline{V_{\max}} := \inf_{z \in \mathcal{Z}} V_{\max}^z$, then $\lim\limits_{\theta \to \underline{V_{\max}}^-} \mathbb{E}\left[V_{Z_0}^{-1}(\theta)\right] = +\infty$.

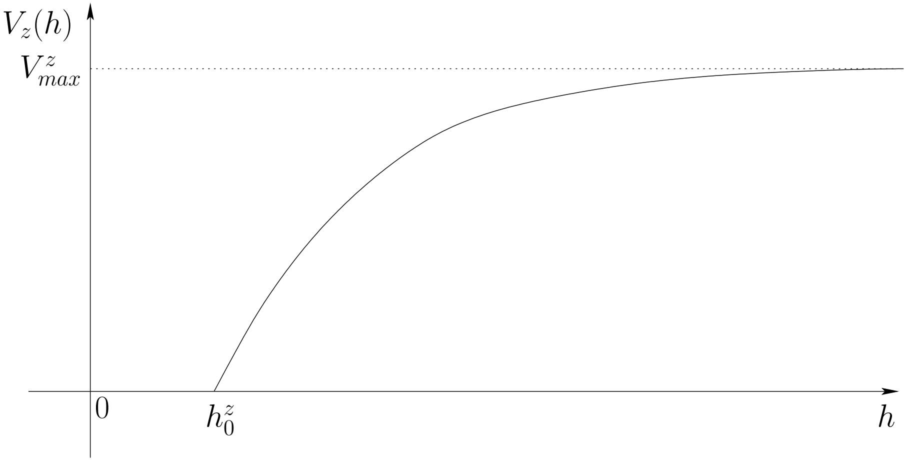

Figure: Schematic representation of the optimal velocity functions.

Main result (1)

Scaling: For $\epsilon > 0$, we consider an initial condition $(U_i^{\epsilon,0})$ such that there exists a Lipschitz continuous function $u_0 : \mathbb{R} \to \mathbb{R}$ with

$$\lim_{\epsilon \to 0,\ \epsilon i \to x} \epsilon U_i^{\epsilon,0} = u_0(x),$$

locally uniformly with respect to x. Let (U_i^ϵ) be the solution of

$$\frac{d}{dt} U_i^\epsilon(t) = V_{Z_i}(U_{i+1}^\epsilon(t) - U_i^\epsilon(t)),\ t \geqslant 0, \forall i \in \mathbb{Z}.$$

with initial condition $(U_i^{\epsilon,0})$.

We want to study the limit

$$u(x,t) := \lim_{\epsilon \to 0,\ \epsilon(i,s) \to (x,t)} \epsilon U_i^\epsilon(s)$$

Main result (2)

Theorem (C.-Forcadel)

Under assumptions $(H1)-(H5)$, the limit

$$u(x,t) := \lim_{\epsilon\to 0,\ \epsilon(i,s)\to(x,t)} \epsilon U_i^\epsilon(s)$$

exists a.s., locally uniformly in (x,t), and u is the unique (deterministic) viscosity solution of

$$\begin{cases} \partial_t u = \bar{F}(\partial_x u) & \text{in } \mathbb{R}\times]0,+\infty[\\ u(x,0) = u_0(x) & \text{in } \mathbb{R} \end{cases}$$

where the effective velocity $\bar{F} : [0,+\infty) \to [0, \underline{V_{\max}})$ is the continuous and increasing map defined by

- $\bar{F}(p) = 0$ if $p \leqslant \bar{h}_0$ where $\bar{h}_0 := \mathbb{E}[h_0^{Z_0}]$,
- and $\bar{F}(p)$ is the unique solution to $\mathbb{E}[V_{Z_0}^{-1}(\bar{F}(p))] = p$ if $p > \bar{h}_0$.

Link with the Lighthill-Whitham-Richards (LWR) model

We consider the (rescaled) empirical density of cars:

$$\rho^\epsilon(t) = \epsilon \sum_{i\in\mathbb{Z}} \delta_{\epsilon U_i^\epsilon(t/\epsilon)}, \qquad t \geqslant 0.$$

Corollary [Convergence to the LWR model]

As $\epsilon \to 0$, $\rho^\epsilon(t)$ converges, a.s., in distribution and locally uniformly in time, to the density of cars

$$\rho(t) := u(\cdot, t)\sharp dx,$$

where u is the solution of the limit HJ equation. If, in addition, there exists $C > 0$ such that

$$C^{-1} \leqslant \partial_x u_0(x) \leqslant C,$$

then ρ has an absolutely continuous density which is locally bounded and is the entropy solution of the LWR model

$$(LWR) \qquad \partial_t \rho + \partial_x(\rho \bar{v}(\rho)) = 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}_+,$$

with $\bar{v}(\rho) = \bar{F}(1/\rho)$.

Sketch of proof of the corollary

- Let $\varphi \in \mathcal{C}_c^0(\mathbb{R})$. Then, for any $t' \geqslant 0$,

$$\int_{\mathbb{R}} \varphi(x)\rho^{\epsilon}(dx, t') = \epsilon \sum_{i \in \mathbb{Z}} \varphi(\epsilon U_i^{\epsilon}(t'/\epsilon)) = \int_{\mathbb{R}} \varphi(\epsilon U_{[x/\epsilon]}^{\epsilon}(t'/\epsilon))dx.$$

- As $\epsilon([x/\epsilon], t'/\epsilon) \to (x, t)$ as $\epsilon \to 0$ and $t' \to t$, the main Theorem implies:

$$\begin{aligned}\lim_{\epsilon \to 0,\ t' \to t} \int_{\mathbb{R}} \varphi(x)\rho^{\epsilon}(dx, t) &= \int_{\mathbb{R}} \varphi(u(x, t))dx \\ &= \int_{\mathbb{R}} \varphi(x) d(u(\cdot, t)\sharp dx) = \int_{\mathbb{R}} \varphi(x)\rho(dx, t).\end{aligned}$$

  This proves that $\rho^{\epsilon}(t)$ converges locally uniformly in time and in the sense of measures to $\rho(t) := u(\cdot, t)\sharp dx$.

- This implies that ρ is an entropy solution to (LWR) (Caselles (1992)).

□

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Preliminary results on the micro model

Lemma [Uniform bounds]

Let U_i be a solution of

$$(Micro) \qquad \frac{d}{dt}U_i(t) = V_{Z_i}(U_{i+1}(t) - U_i(t)), \;\; t \geqslant 0, \forall i \in \mathbb{Z}.$$

Then, for all $t \geqslant 0$,

$$0 \leqslant U_i(t) - U_i(0) \leqslant V_{\max} t.$$

We also have the following comparison principle:

Proposition [Comparison]

Let U_i and $\tilde{U}_i$ be two solutions of (Micro) such that there exists $i_0 \in \mathbb{Z}$ with

$$U_i(0) \leqslant \tilde{U}_i(0) \quad \forall i \geqslant i_0.$$

Then

$$U_i(t) \leqslant \tilde{U}_i(t) \quad \forall t \geqslant 0 \text{ and } i \geqslant i_0.$$

Construction of the effective velocity

Recall that $\bar{h}_0 := \mathbb{E}[h_0^{Z_0}]$. Given $p > \bar{h}_0$, we consider the solution $\bar{U}^p$ to the problem with linear initial condition:

$$\frac{d}{dt}\bar{U}_i^p(t) = V_{Z_i}(\bar{U}_{i+1}^p(t) - \bar{U}_i^p(t)),\ t \geqslant 0, \qquad \bar{U}_i^p(0) = p\ i \qquad \forall i \geqslant 0.$$

Proposition [Convergence for linear initial conditions]

There exists $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that for every $p \geqslant 0$, $i \in \mathbb{N}$ and $\omega \in \Omega_0$

$$\lim_{t\to+\infty} \frac{\bar{U}_i^p(t)}{t} = \bar{F}(p) \qquad \forall i \geqslant 0,$$

where the continuous and non-decreasing map $\bar{F} : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

- $\bar{F}(p) = 0$ if $p \leqslant \bar{h}_0$ where $\bar{h}_0 := \mathbb{E}[h_0^{Z_0}]$,
- $\mathbb{E}[V_{Z_0}^{-1}(\bar{F}(p))] = p$ if $p > \bar{h}_0$.

Main argument of the proof of the proposition: Correctors

Given $\theta \in (0, \underline{V_{\max}})$, we consider the random sequence $(c_i^\theta)_{i \geqslant 0}$ defined by

$$c_0^\theta = 0, \qquad c_{i+1}^\theta = c_i^\theta + V_{Z_i}^{-1}(\theta) \qquad i \geqslant 0.$$

In other words,

$$V_{Z_i}(c_{i+1}^\theta - c_i^\theta) = \theta \qquad \forall i \geqslant 0.$$

Thus, if we set $\tilde{U}_i^\theta(t) = c_i^\theta + t\theta$, we have

$$\frac{d}{dt}\tilde{U}_i^\theta(t) = \theta = V_{Z_i}(c_{i+1}^\theta - c_i^\theta) = V_{Z_i}(\tilde{U}_{i+1}^\theta(t) - \tilde{U}_i^\theta(t)).$$

So the $(\tilde{U}_i^\theta)$ are the correctors of the problem.

By the law of large numbers, there exists Ω_0 with $\mathbb{P}(\Omega_0) = 1$ such that for every $\omega \in \Omega_0$, we have

$$\frac{c_i^\theta}{i} = \frac{1}{i}\sum_{j=0}^{i-1} V_{Z_j}^{-1}(\theta) \to \mathbb{E}\left[V_{Z_0}^{-1}(\theta)\right] \qquad \text{as } i \to +\infty.$$

Open problems

- Convergence rate

- Models with local perturbations
(cf. Forcadel-Salazar-Zaydan (2017), deterministic setting)

- Models with several roads
(cf. Forcadel-Salazar (2019), two outgoing roads)