Non Exchangeability and Synchronization Mechanisms in Multi-Agent Systems

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Multi-agent systems on graphs

Consider N agents, each on the vertex of a weighted symmetric graph G_N and interacting two by two through the kernel K. Denote by $X_i(t) \in \mathbb{R}^d$ the state of the i-th agent, and consider the system

$$\frac{dX_i}{dt} = \sum_{j \neq i} w_{ij} \, K(X_i - X_j),$$

where the $w_{ij} = w_{ij}^N = w_{ji}^N$ are the weight of the edge (i, j) of the graph, with the key assumptions

$$\sup_i \sum_{j=1}^{N} |w_{ij}| \leq C, \quad \sup_{i,j} |w_{ij}| \to 0.$$

Main question: Behavior of the system as $N \to \infty$. .

Multi-agent systems on graphs

Consider N agents, each on the vertex of a weighted symmetric graph G_N and interacting two by two through the kernel K. Denote by $X_i(t) \in \mathbb{R}^d$ the state of the i-th agent, and consider the system

$$dX_i = \sum_{j \neq i} w_{ij} K(X_i - X_j)\, dt + \sigma\, dW_i,$$

where the $w_{ij} = w_{ij}^N = w_{ji}^N$ are the weight of the edge (i, j) of the graph, with the key assumptions

$$\sup_i \sum_{j=1}^{N} |w_{ij}| \leq C, \quad \sup_{i,j} |w_{ij}| \to 0.$$

Possibly with noise, where the W_i are N independent Wiener processes.

An example of application: Biological neurons

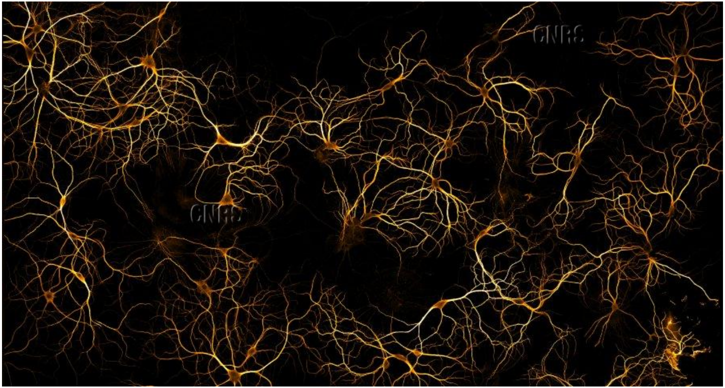

Figure: Credits: CNRS Bordeaux, France; 2D reconstruction of rat hippocampus, marked for cytoskeleton protein

Hebb rule: Neurons wire together if they fire together (S. Löwel).

A simplified multi-agent model for biological neurons

Consider N neurons with activation X_i and connectivity or synaptic weights w_{ij} between neuron i and j,

$$\frac{dX_i}{dt} = \sum_j w_{ij}(t)\, K(X_i, X_j) \quad (+noise), \qquad X_i(t=0) = X_i^0,$$

$$\frac{dw_{ij}}{dt} = w_{ij}\,(L(X_i, X_j) - w_{ij}) \quad (+noise), \qquad w_{ij}(t=0) = w_{ij}^0.$$

Example of Multi-agent systems with basic learning or reinforcement mechanism such as synchronized Kuramoto...,
$\rightarrow$ Creates and amplifies correlations between neurons.

General setting encompasses several common models for neurons dynamics: Integrate and fire, Hodgkin-Huxley, FitzHugh–Nagumo. But those often involve singular kernels K.

Scaling in biological neuron networks

- Mammalian brains contain between $10^8 - 10^{11}$ neurons ($86 \cdot 10^9$ actually in humans). Of course models typically applied only to sub-domains... $\longrightarrow N >> 1$.
- In the human brain, each neuron has on average 7000 synaptic connections to other neurons $\longrightarrow \sup_{i,j} w_{ij} \sim 10^{-3} - 10^{-4} << 1$.

$\longrightarrow$ For analytical and numerical studies, it is desirable to derive macroscopic models.

- Mathematical studies of the macroscopic models have been successful in recovering some of neurons' behavior: See for example, Caceres-Carrillo-Perthame, Caceres-Perthame, Carrillo-Perthame-Salort-Smets, Flandoli-Priola-Zanco, Perthame-Salort...
- How to justify deriving such macroscopic models is more delicate, see among many Compte et al, Mattia-Del Giudice, Omurtag, Pham et al, Renart et al...

Some notions of complexity

- Observables for the system are quantities of the form

$$\phi_X = \mathbb{E}\,\frac{1}{N}\sum_i \phi(X_i), \quad \text{or } \phi_X = \mathbb{E}\,\frac{1}{N^2}\sum_{i,j} \phi(X_i,\ X_j).$$

- Numerical complexity: Given a fixed ε, what is the numerical cost to calculate ϕ up to ε?
- A variant of the complexity question: Consider 2 systems (X_i), $i = 1 \dots N$, and (Y_j), $j = 1 \dots M$ with $M << N$. How large does M need to be so that $|\phi_X - \phi_Y| \leq \varepsilon$?

$\longrightarrow$ This is at the basis of particles' methods but also essential to understand critical scale in natural systems (how many bacteria to get collective behaviors?).

The classical Mean-field limit

Consider instead the exchangeable system with identical particles

$$\frac{dX_i}{dt} = \frac{1}{N} \sum_{j=1}^{N} K(X_i - X_j),$$

with mean-field limit given by

$$\partial_t f(t,x) + \mathrm{div}_x \left(f(t,x) \int K(x-y)\, f(t,y)\, dy \right) = 0.$$

If $K \in W^{1,\infty}$, now very famous results by Neunzert and Wick, Braun and Hepp, Dobrushin, and later Spohn proving that

$$\|\mu_N - f\|_{W^{-1,1}} \leq \|\mu_N^0 - f\|_{W^{-1,1}}\, e^{t\, \|\nabla K\|_{L^\infty}},$$

where μ_N is the empirical measure for (X_i)

$$\mu_N(t,x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - X_i(t)).$$

A first case of MF limit for non-exchangeable systems

Consider now the original system for $K \in W^{1,\infty}$

$$\frac{dX_i}{dt} = \sum_{j \neq i} w_{ij}\, K(X_i - X_j), \quad X_i(t=0) = X_i^0,$$

with a possible random graph G, and random w_{ij} s.t.

$$\sum_{j=1}^{N} w_{ij} \to 1, \quad \sup_{i,j} |w_{ij}| \to 0, \quad \text{almost surely as } N \to \infty.$$

Assume that the X_i^0 are independent and identically distributed.
Then the X_i remain almost i.i.d. as $\mu_N \to f$ with

$$\partial_t f(t,x) + \mathrm{div}_x \left(f(t,x) \int K(x-y)\, f(t,y)\, dy \right) = 0.$$

See Delattre-Giacomin-Luçon and Coppini-Dietert-Giacomin for asymptotically chaotic initial data.

MF limits with graphons

Assume the X_i^0 are independent, not necessarily i.i.d., $K \in W^{1,\infty}$, that the graph is asymptotically represented by graphons

$$\frac{dX_i}{dt} = \frac{1}{N} \sum_{j \neq i} \bar{w}_{ij} \, K(X_i - X_j), \quad X_i(t=0) = X_i^0,$$

$$0 \leq \bar{w}_{ij}^N \leq 1, \quad \bar{w}_N(\xi,\zeta) = \sum_{i,j} \bar{w}_{ij}^N \, \mathbb{I}_{\frac{i-1}{N} \leq \xi < \frac{i}{N}} \, \mathbb{I}_{\frac{j-1}{N} \leq \zeta < \frac{j}{N}} \longrightarrow \bar{w} \text{ in } L^1,$$

for some $\bar{w} \in L^\infty([0,\ 1]^2)$. Then the limit is given in $\mathbb{R}^{d+1}$ by

$$\partial_t f(t,x,\xi) + \text{div}_x \left(f(t,x,\xi) \int K(x-y) \, \bar{w}(\xi,\zeta) \, f(t,y,\zeta) \, dy \, d\zeta \right) = 0,$$

$$\mu_N(t,x) \longrightarrow \int_0^1 f(t,x,\xi) \, d\xi.$$

See for example Chiba, Medvedev, Mizhura.

Our new result

Theorem

Assume that the X_i^0 are independent, but not necessarily i.i.d., $K \in W^{1,\infty}$, and consider

$$\frac{dX_i}{dt} = \sum_{j \neq i} w_{ij}\, K(X_i - X_j),$$

$$\sup_i \sum_{j=1}^N |w_{ij}| \leq C, \quad \sup_{i,j} |w_{ij}| \to 0.$$

Then there exists $w \in L^\infty([0,\ 1],\ \mathcal{M}([0,\ 1]))$ and the limit is given in an appropriate sense by some $f \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d \times [0,\ 1])$

$$\partial_t f(t,x,\xi) + div_x \left(f(t,x,\xi) \int K(x-y)\, w(\xi,\zeta)\, f(t,y,\zeta)\, dy\, d\zeta \right) = 0.$$

Some preliminary remarks

- The result requires only weak assumptions on the weights w_{ij}. Those could even be relaxed, with for some $p > 1$

$$\frac{1}{N}\sum_i \left(\sum_{j=1}^N |w_{ij}|\right)^p \leq C, \quad \frac{1}{N}\sum_i \sup_j |w_{ij}|^{1/p} \to 0.$$

- Independence is the key to the reduction in complexity. It is not strictly necessary to implement most of the proof though it then leads to measures of measures...
- We do not obtain graphons as $w \in L^\infty([0,\ 1],\ \mathcal{M}([0,\ 1])$. It is not even yet clear in which sense the limiting equation holds.
- The limit holds in the sense that

$$\mu_N(t,x) \longrightarrow \int_0^1 f(t,x,\xi)\, d\xi,$$

but this is only achieved through several intermediary steps.

Propagating independence

Proposition

Assume the X_i^0 are independent, with uniform compact support, and that

$$\sup_i \sum_{j=1}^N |w_{ij}| \leq C.$$

Denote by $f_i(t,x) \in L^\infty(\mathbb{R}_+,\ \mathcal{P}(\mathbb{R}^d))$ the law of the process X_i. Then one has that

$$\left\| f_i(t) - \bar{f}_i(t) \right\|_{W^{-1,1}(\mathbb{R}^d)} \leq C\, e^{t\, \|K\|_{W^{1,\infty}}} \sup_j |w_{ij}|^{1/2}.$$

where the $\bar{f}_i$ solve the coupled PDE system

$$\partial_t \bar{f}_i + div_x \left(\bar{f}_i(x) \sum_j w_{ij} \int K(x-y)\, \bar{f}_j(y)\, dy \right) = 0.$$

Short sketch of the proof

This is not the usual propagation of chaos and it requires a careful extension of the usual arguments...
Define $\mathcal{F}^i$ the σ-algebra generated by X_i^0 and denote $\mathbb{E}^i = \mathbb{E}(.\,|\mathcal{F}^i)$ the corresponding conditional expectation. Define finally $\bar{X}_i = \mathbb{E}^i X_i$ and observe that

$$\frac{d}{dt}(X_i - \bar{X}_i) = \sum_j w_{ij} \left(K(X_i - X_j) - \mathbb{E}^i K(X_i - X_j)\right)$$

$$= \sum_j w_{ij} \left(K(X_i - X_j) - K(\bar{X}_i - \bar{X}_j)\right) + \sum_j w_{ij} \left(K(\bar{X}_i - \bar{X}_j) - \mathbb{E}^i K(\bar{X}_i -\right.$$

Hence

$$\frac{d}{dt}|X_i - \bar{X}_i| \leq C\,\|K\|_{W^{1,\infty}} \sup_j |X_i - \bar{X}_j|$$

$$+ \left|\sum_j w_{ij} \left(K(\bar{X}_i - \bar{X}_j) - \mathbb{E}^i K(\bar{X}_i - \bar{X}_j)\right)\right|.$$

Sketch 2: Law of large number

We remark now that the $\bar{X}_j$ are independent by definition so

$$\mathbb{E}\left|\sum_j w_{ij}\,(K(\bar{X}_i-\bar{X}_j)-\mathbb{E}^i K(\bar{X}_i-\bar{X}_j))\right|^2 \leq C\,\sup_j |w_{ij}|.$$

By Gronwall, we obtain that

$$\mathbb{E}\,|X_i-\bar{X}_i| \leq C\,e^{C\,t\,\|K\|_{W^{1,\infty}}}\,\sup_j |w_{ij}|^{1/2},$$

and we may easily conclude from there by standard arguments.

Only the beginning!

Deriving the system

$$\partial_t \bar f_i + \mathrm{div}_x \left(\bar f_i(x) \sum_j w_{ij} \int K(x-y)\, \bar f_j(y)\, dy \right) = 0$$

is not enough to conclude : So far we have only replaced a coupled system of ODE's by a couple system of PDE's. We do not even have any compactness...

One big exception: If $\sum_j w_{ij} = 1$ and $\bar f_i^0 = \bar f^0$ then $f_i(t,x) = \bar f(t,x)$, the usual propagation of chaos, with

$$\partial_t \bar f + \mathrm{div}_x \left(\bar f(x) \int K(x-y)\, \bar f(y)\, dy \right) = 0.$$

A system on the observables

There is no realistic hope to identify the limit of the individual $\bar f_i$.
But we only care about the limit of observables, such as

$$\frac{1}{N}\sum_i \bar f_i(t,x).$$

Can we obtain equations on such quantities? For example

$$\partial_t \frac{1}{N}\sum_i \bar f_i(x) + \mathrm{div}_x\left(\int K(x-y)\,\frac{1}{N}\sum_{i,j} w_{ij}\,\bar f_i(x)\,\bar f_j(y)\,dy\right) = 0.$$

Can we write a full hierarchy?

A hierarchy indexed by trees

For any tree T, define

$$\tau(T,w,(f_i))(t,x_1,\dots,x_{|T|}) = \frac{1}{N}\sum_{i_1,\dots,i_{|T|}=1}^{N} \Pi_{(k,l)\in T} w_{i_k i_l}\ \Pi_{m=1}^{|T|} f_{i_m}(t,x_m).$$

The $\tau(T,w,(f_i))$ are not symmetric in general and satisfy the generalized, linear, non-exchangeable Vlasov hierarchy

$$\partial_t \tau(T) + \sum_{i=1}^{|T|} \mathrm{div}_{x_i}\left(\int_{\mathbb{R}^d} K(x_i - z)\,\tau(T+i)(t,x_1,\dots,x_{|T|},z)\,dz\right) = 0,$$

$$\tau(T,w,(f_i))(t=0) = \tau(T,w,(f_i^0)),$$

where given a tree T, we denote by $T+i$ the new tree obtained by adding a leaf at vertex $i\in T$.

Uniqueness on the hierarchy with diffusion

Theorem

Consider any sequence (h_T) of solution to the viscous hierarchy

$$\partial_t h_T + \sum_{i=1}^{|T|} div_{x_i} \left(\int_{\mathbb{R}^d} K(x_i - z)\, h_{T+i}(t, x_1, \ldots, x_{|T|}, z)\, dz \right)$$
$$= \nu \sum_{i=1}^{|T|} \Delta_{x_i} h_T.$$

We have the stability result

$$\|h(t)\|_\lambda \leq \exp\left(4^{-\frac{\|K\|_{L^\infty}}{\nu\,\lambda}\,t} \log \|h(0)\|_\lambda\right),$$

$$\|h\|_\lambda = \sup_T \lambda^{|T|/2}\, \|h_T\|_{L^2(\mathbb{R}^{d\,n(T)})}.$$

One step closer...

- Does not require $K \in W^{1,\infty}$...
- Stability on the non-viscous hierarchy is unknown. But still stability on the $\bar{f}_i$ by adding artificial viscosity: Given two solutions $(\bar{f}_i)$ with the w_{ij}, $(\bar{g}_i)$ with $\tilde{w}_{ij}$,

$$\tau(T, w, (\bar{f}_i^0)) = \tau(T, \tilde{w}, (\bar{g}_i^0)) \quad \forall\ T$$
$$\implies \tau(T, w, (\bar{f}_i))(t) = \tau(T, \tilde{w}, (\bar{g}_i))(t) \quad \forall\ T,\ t > 0.$$

- Provides some sense of compactness but not completeness.
- Trying to solve the hierarchy at the limit is unrealistic...
- Remaining question: If $\tau(T, w^N, (\bar{f}_{i,N}^0)) \to \alpha(T)$ for all T as $N \to \infty$, find some suitable representation of the limit.

Graphon-like representations

From w_{ij}, f_i^0, define

$$w_N(\xi,\zeta)=\sum_{i,j} N\, w_{ij}\, \mathbb{I}_{\frac{i-1}{N}\leq\xi<\frac{i}{N}}\, \mathbb{I}_{\frac{j-1}{N}\leq\zeta<\frac{j}{N}},$$

$$f_N(x,\xi)=\sum_{i} f_i(x)\, \mathbb{I}_{\frac{i-1}{N}\leq\xi<\frac{i}{N}}.$$

Then we have the representation

$$\tau(T,w,(f_i))=\tau(T,w_N,f_N)$$

$$=\int_{[0,\ 1]^{|T|}} \Pi_{(k,l)\in T}\, w_N(\xi_k,\xi_l)\, \Pi_{m=1}^{|T|} f_N(x_m,\xi_m)\, d\xi_1\dots d\xi_{|T|}.$$

Remaining question: If $\lim_N \tau(T,w_N,f_N)$ exist for any T, can we find w, f s.t.

$$\tau(T,w,f)=\lim_N \tau(T,w_N,f_N)\quad \forall T.$$

Graphon-like representations

From w_{ij}, , define

$$w_N(\xi,\zeta) = \sum_{i,j} N\, w_{ij}\, \mathbb{I}_{\frac{i-1}{N}\leq \xi < \frac{i}{N}}\, \mathbb{I}_{\frac{j-1}{N}\leq \zeta < \frac{j}{N}},$$

$$\tau(T,w) = \tau(T,w_N) = \int_{[0,\ 1]^{|T|}} \Pi_{(k,l)\in T}\, w_N(\xi_k,\xi_l)\, d\xi_1 \ldots d\xi_{|T|}.$$

Theorem

Assume $\sup_N \sup_\xi \int_0^1 |w_N(\xi,\zeta)|\, d\zeta < \infty$, *and* $\lim_N \tau(T,w_N,f_N)$ *exist for all* T.
Then $\exists w \in L^\infty([0,\ 1],\ \mathcal{M}([0,\ 1]))$ *s.t.*

$$\tau(T,w,f) = \lim_N \tau(T,w_N,f_N) \quad \forall T.$$

Some critical differences with graphons

- Graphons were originally introduced in the context of sparse random graphs where $N\,w_{ij}$ is the probability of having an edge. Hence $0 \leq w_N(\xi,\zeta) \leq 1$ in that case. See Lovász, Lovász-Szegedy.
- Instead here, w_N is not uniformly bounded in $L^\infty_{\xi,\zeta}$ but only in $L^\infty_\xi \mathcal{M}_\zeta$. But we only consider $\tau(T,w)$ for trees T instead of all graphs for standard graphons.

Hence for example

Lemma

For any tree T and any $w \in L^\infty([0,\ 1],\ \mathcal{M}([0,\ 1]))$

$$|\tau(T,w)| \leq \|w\|^{|T|-1}_{L^\infty([0,\ 1],\ \mathcal{M}([0,\ 1]))}.$$

This is obviously false if we consider $\tau(G,w)$ for general graphs.

A key lemma

Lemma

Consider any sequence f_k in $L^\infty([0,\ 1])$ with $0 \le f_k(x) \le 2^{-k+1}$. Then there exists $\Phi : [0,\ 1] \to [0,\ 1]$, measure-preserving, s.t. the $f_k \circ \Phi$ are uniformly compact in L^1 with for some universal constant C

$$\sup_k \int_0^1 |f_k(\xi) - f_k(\xi + h)|\, d\xi \le 2^{-C\sqrt{\log \frac{1}{h}}}.$$

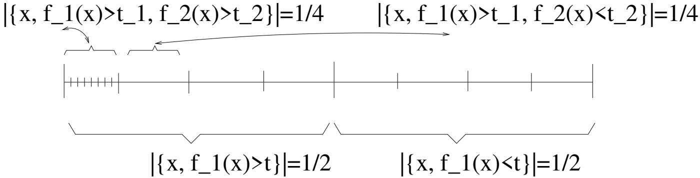

Figure: Hierarchical decomposition

Regularity lemma on graphons

If w is a graphon, then the key lemma implies the classical regularity lemma by Lovász-Szegedy and in turn the famous regularity lemma on graphs by Szemerédi.

Lemma (Regularity lemma on graphons)

If $w \in L^\infty([0,\ 1]^2)$ is symmetric with $0 \le w \le 1$, then there exists $\Phi : [0,\ 1] \to [0,\ 1]$, measure-preserving, s.t. in the cut-distance

$$\delta_\square(w(\Phi(.), \Phi(.)), w(\Phi(. + h), \Phi(.))) \le \frac{C}{\sqrt{\log \frac{1}{h}}},$$

where the cut-distance is given by

$$\delta_\square(w, \tilde{w}) = \sup_{\Phi, \tilde{\Phi}\ m.p.} \sup_{S,T} \left| \int_{S \times T} (w(\Phi(\xi), \Phi(\zeta)) - \tilde{w}(\tilde{\Phi}(\xi), \tilde{\Phi}(\zeta)) \, d\xi \, d\zeta \right|.$$

Back to our main proof: Sketch 1

Consider now any w_N symmetric, uniformly bounded in $L^\infty_\xi \mathcal{M}_\zeta$ s.t.

$$\lim_{N\to\infty} \tau(T, w_N) \text{ exists } \forall T.$$

$\longrightarrow$ Construct recursively the sequences $f_{k,N}$ uniformly in $L^\infty([0,\ 1])$ by building the countable algebra $M(w_N)$

$$1 \in M(w_N),$$
$$\phi,\ \psi \in M(w_N) \implies \phi\,\psi \in M(w_N),$$
$$\phi \in M(w_N) \implies \int_0^1 w_N(x,y)\,\phi(y)\,dy \in M(w_N).$$

$\longrightarrow$ Observe that for any tree T, there exists $f_{k,N} \in M(w_N)$ s.t.

$$\tau(T, w_N) = \int_0^1 f_{k,N}(\xi)\,d\xi.$$

Sketch of the main proof 2

$\longrightarrow$ Use the key lemma to obtain Φ_N s.t. the $f_{k,N} \circ \Phi_N$ all converge strongly to f_k in L^1.
$\longrightarrow$ Extract a weak-* converging subsequence of $w_N(\Phi_N(.), \Phi_N(.))$

$$w_N(\Phi_N(.), \Phi_N(.)) \rightharpoonup w_\Phi \quad \text{weak-* in } L^\infty_\xi \mathcal{M}_\zeta.$$

$\longrightarrow$ For every $f_{k,N} \in M(w_N)$, note that

$$\int_0^1 w_N(\Phi_N(\xi), \Phi_N(\zeta))\, f_{k,N}(\Phi_N(\zeta))\, d\zeta \longrightarrow \int_0^1 w_\Phi(\xi,\zeta)\, f_k(\zeta)\, d\zeta,$$

first in weak-* in $\mathcal{M}_\xi$ and hence also strongly in L^1_ξ.
$\longrightarrow$ Identify the limits f_k as elements of $M(w)$ and conclude that

$$\lim_{N\to\infty} \tau(T, w_N) = \tau(T, w).$$

A more precise result

Theorem

Assume that the X_i^0 are independent, $K \in W^{1,\infty}$, and consider

$$\frac{dX_i}{dt} = \sum_{j \neq i} w_{ij}\, K(X_i - X_j), \quad \sup_i \sum_{j=1}^N |w_{ij}| \leq C, \quad \sup_i |X_i^0| \leq C.$$

Then there exists $w \in L^\infty([0,\ 1],\ \mathcal{M}([0,\ 1])$ and $f \in L^\infty$ satisfying

$$\partial_t f(t,x,\xi) + div_x \left(f(t,x,\xi) \int K(x-y)\, w(\xi,\zeta)\, f(t,y,\zeta)\, dy\, d\zeta \right) = 0,$$

$$\int \left| \int_0^1 (w(\xi,\zeta) - w(\xi+h,\zeta))\, f(x,\zeta)\, d\zeta \right| d\xi\, dx \leq 2^{-C\sqrt{\log \frac{1}{h}}},$$

$$\left\| \mu_N - \int_0^1 f(t,x,\xi)\, d\xi \right\|_{W^{-1,1}} \leq C \sup_{i,j} |w_{ij}|^{1/2}.$$

Conclusion and some open questions

- Demands minimal assumptions on the graph.
- What if $K \notin W^{1,\infty}$? Only required for the 1st step: Independence.
- Can $w \in L^\infty\mathcal{M}$ be used to represent any solution to the hierarchy? For any sequence α_T indexed by trees T, can we find $w : [0,\ 1]^2 \to \mathbb{C}$, $w \in L^\infty\mathcal{M}$, s.t.

$$\tau(T, w) = \alpha(T) \quad \forall T.$$

Thank you!