

# Polynomial liftings in $H^1$ and $H(\text{div})$

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# Outline

1. (Motivation) **A posteriori error estimates**
2. **Main results** on  $H^1$  and  $\mathbf{H}(\text{div})$  polynomial liftings
3. Main ingredients of **proofs**
4. Numerical results  
(with V. Dolejší, Prague)
5. Global  $\mathbf{H}(\text{div})$  polynomial liftings  
(with I. Smears, London)
6. Local/global best-approximations in  $\mathbf{H}(\text{div})$   
(with T. Gudi, Bangalore, and I. Smears, London)

## A posteriori error estimates

- ▶ Model problem in  $\Omega \subset \mathbb{R}^d$  with data  $f \in L^2(\Omega)$

$$u \in H_0^1(\Omega) \text{ s.t. } (\nabla u, \nabla v)_\Omega = (f, v)_\Omega, \forall v \in H_0^1(\Omega)$$

- ▶  $H_0^1$ -conforming FEM solution  $u_h$  on a simplicial mesh  $\mathcal{T}_h$
- ▶ Many ways to obtain a **computable, global** upper bound

$$\|\nabla(u - u_h)\|_\Omega^2 \leq \sum_{K \in \mathcal{T}_h} \eta_K^2$$

- ▶ Local indicators  $\eta_K$  are **lower bounds** up to data oscillation

$$\eta_K \leq C_{\text{eff}} \|\nabla(u - u_h)\|_{\omega_K} + \text{osc}(f, \omega_K)$$

- ▶ Pioneered by [Babuška, Rheinbolt 78]; recent textbook [Verfürth 13]
  - ▶ classical technique to compute the  $\eta_K$ 's is residual-based
  - ▶ drawback: upper bound features an undetermined constant

## Equilibrated flux reconstruction

- ▶ Exact flux  $\boldsymbol{\sigma} := -\nabla u \in \mathbf{H}(\text{div}, \Omega)$  s.t.  $\nabla \cdot \boldsymbol{\sigma} = f$  (equilibrium)
- ▶ What is **equilibrated flux reconstruction**?

$$\boldsymbol{\sigma}_h \in \mathbf{H}(\text{div}, \Omega) \quad (\nabla \cdot \boldsymbol{\sigma}_h, 1)_K = (f, 1)_K, \quad \forall K \in \mathcal{T}_h$$

- ▶ Setting  $\eta_{\mathbb{F}, K} := \|\nabla u_h + \boldsymbol{\sigma}_h\|_K$  and  $\eta_{\text{osc}, K} := \frac{h_K}{\pi} \|f - \nabla \cdot \boldsymbol{\sigma}_h\|_K$ ,

$$\|\nabla(u - u_h)\|_{\Omega}^2 \leq \sum_{K \in \mathcal{T}_h} \underbrace{(\eta_{\mathbb{F}, K} + \eta_{\text{osc}, K})^2}_{=:\eta_K}$$

- ▶ The upper bound is **guaranteed** (no undetermined constant)
- ▶ Literature: hypercircle method [Prager, Synge 47]; computational mechanics [Ladevèze et al. 75]; textbooks [Ainsworth, Oden 00; Repin 08]

## A simple proof

- ▶ **Residual**  $\rho(u_h) \in H^{-1}(\Omega)$  s.t.

$$\langle \rho(u_h), \varphi \rangle_\Omega := (f, \varphi)_\Omega - (\nabla u_h, \nabla \varphi)_\Omega, \quad \forall \varphi \in H_0^1(\Omega)$$

$$\|\nabla(u - u_h)\|_\Omega = \|\rho(u_h)\|_{H^{-1}(\Omega)} = \sup_{\varphi \in H_0^1(\Omega); \|\nabla \varphi\|_\Omega=1} \langle \rho(u_h), \varphi \rangle_\Omega$$

- ▶ Introduce  **$H(\text{div}, \Omega)$ -flux** and use Green's formula

$$\langle \rho(u_h), \varphi \rangle_\Omega = (f - \nabla \cdot \boldsymbol{\sigma}_h, \varphi)_\Omega - (\nabla u_h + \boldsymbol{\sigma}_h, \nabla \varphi)_\Omega$$

- ▶ Cauchy–Schwarz and Poincaré–Steklov inequalities (**equilibration**)

$$|(f - \nabla \cdot \boldsymbol{\sigma}_h, \varphi)_\Omega| = \sum_{K \in \mathcal{T}_h} |(f - \nabla \cdot \boldsymbol{\sigma}_h, \varphi - \bar{\varphi}_K)_K| \leq \sum_{K \in \mathcal{T}_h} \frac{h_K}{\pi} \|f - \nabla \cdot \boldsymbol{\sigma}_h\|_K \|\nabla \varphi\|_K$$

$$|(\nabla u_h + \boldsymbol{\sigma}_h, \nabla \varphi)_\Omega| \leq \sum_{K \in \mathcal{T}_h} \|\nabla u_h + \boldsymbol{\sigma}_h\|_K \|\nabla \varphi\|_K$$

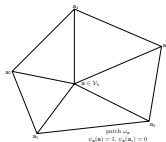
Poincaré (1894) [eigenvalue pb], Steklov (1897) [ $d = 1$ ], Payne, Weinberger (60) [ $d = 2$ ], Bebendorf (03) [ $d \geq 3$ ]

## How to build $\sigma_h$ ?

- ▶ Global flux equilibration on a global Raviart–Thomas space  $\mathbf{V}_h \subset \mathbf{H}(\text{div}, \Omega)$

$$\sigma_h := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h, \nabla \cdot \mathbf{v}_h = \Pi_{(\nabla \cdot \mathbf{v}_h)} f} \|\nabla u_h + \mathbf{v}_h\|_{\Omega} \quad (\dots \text{expensive})$$

- ▶ **Cheap** local flux equilibration by working on **FE stars**
  - ▶  $\mathcal{T}_a \subset \mathcal{T}_h$ : patch of elements sharing vertex  $\mathbf{a} \in \mathcal{V}_h$ ; domain  $\omega_a$
  - ▶ local Raviart–Thomas space  $\mathbf{V}_h^a \subset \mathbf{H}(\text{div}, \omega_a)$
  - ▶ **how to prescribe** the divergence in  $\omega_a$  and the BC's on  $\partial\omega_a$ ?



- ▶ Literature on FE stars: [Babuška, Miller 87; Veerer, Verfürth 12]

## Local flux equilibration on FE stars

- ▶ **Local PU** by hat basis functions  $\{\psi_{\mathbf{a}}\}_{\mathbf{a} \in \mathcal{V}_h}$  ( $\sum_{\mathbf{a} \in \mathcal{V}_K} \psi_{\mathbf{a}}|_K = 1$ )
- ▶ We focus for simplicity on FE stars around **interior** vertices
  - ▶ flux equilibration for boundary vertices depends on the BC's
- ▶ Raviart–Thomas space  $\mathbf{V}_h^{\mathbf{a}} \subset \mathbf{H}_0(\text{div}, \omega_{\mathbf{a}})$  (Neumann BC's on  $\partial\omega_{\mathbf{a}}$ )

$$\sigma_h^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}}, \nabla \cdot \mathbf{v}_h = g_h^{\mathbf{a}}} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$

$$\sigma_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \sigma_h^{\mathbf{a}}$$

with data  $g_h^{\mathbf{a}} := \Pi_{(\nabla \cdot \mathbf{v}_h^{\mathbf{a}})}(f \psi_{\mathbf{a}} - \nabla u_h \cdot \nabla \psi_{\mathbf{a}})$

- ▶  $\sum_{\mathbf{a} \in \mathcal{V}_K} g_h^{\mathbf{a}} = \Pi_{(\nabla \cdot \mathbf{v}_h^{\mathbf{a}})} f$  (PU),  $(g_h^{\mathbf{a}}, 1)_{\omega_{\mathbf{a}}} = 0$  (Galerkin orthogonality)
- ▶ Literature: [Destuynder, Métivet 99; Braess, Schöberl 08; MV 08; AE, MV 10]

## $p$ -robust local efficiency

- ▶ Let  $p$  be the polynomial degree used to compute  $u_h$
- ▶ Use RT spaces of order  $p$  for local flux equilibration
- ▶ **Key result:** Local lower error bound ( $\omega_K = \cup_{a \in \mathcal{V}_K} \omega_a$ )

$$\eta_{E,K} = \|\nabla u_h + \sigma_h\|_K \leq C_{\text{eff}} \|\nabla(u - u_h)\|_{\omega_K} + \text{osc}(f, \omega_K)$$

- ▶  $C_{\text{eff}}$  depends on patch geometry (mesh regularity)
  - ▶  $C_{\text{eff}}$  is  $p$ -robust for  $d = 2$  [Braess, Pillwein, Schöberl 09]
  - ▶  $C_{\text{eff}}$  is  $p$ -robust for  $d = 3 \Rightarrow$  [AE, MV 19]
- ▶ ... in contrast to residual-based estimators [Melenk, Wohlmuth 01]



## A two-step proof

- ▶ Two-step proof using  $\infty$ -dimensional, local problems
  - ▶ replaces classical bubble-function argument by Verfürth
  - ▶ see [AE, MV 15]
- ▶ We have  $C_{\text{eff}} = C_{\text{st}} C_{\text{PS}}$ 
  - ▶  $C_{\text{st}}$  results from  $p$ -robust stability properties of RT spaces [BPS 09]
  - ▶  $C_{\text{PS}}$  results from Poincaré–Steklov inequalities on FE stars [Carstensen, Funken 00], values estimated in [Veese, Verfürth 12]

# Step 1

- ▶ Recall  $\boldsymbol{\tau}_h^a := \psi_a \nabla u_h$  and  $\mathbf{V}_h^a \subset \mathbf{H}_0(\text{div}, \omega_a) = \mathbf{V}^a$
- ▶ Local PU yields  $\|\nabla u_h + \boldsymbol{\sigma}_h\|_K \leq \sum_{\mathbf{a} \in \mathcal{V}_K} \|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}_h^{\mathbf{a}}\|_{\omega_a}$
- ▶ The local flux equilibration we perform is

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}}, \nabla \cdot \mathbf{v}_h = g_h^{\mathbf{a}}} \|\boldsymbol{\tau}_h^{\mathbf{a}} + \mathbf{v}_h\|_{\omega_a}$$

- ▶ Auxiliary problem ( $\infty$ -dimensional, still polynomial data)

$$\boldsymbol{\sigma}^{\mathbf{a}} := \arg \min_{\mathbf{v} \in \mathbf{V}^{\mathbf{a}}, \nabla \cdot \mathbf{v} = g_h^{\mathbf{a}}} \|\boldsymbol{\tau}_h^{\mathbf{a}} + \mathbf{v}\|_{\omega_a}$$

- ▶ We need to prove the following (nontrivial) result  $\Rightarrow$  [AE, MV 19]

$$\|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}_h^{\mathbf{a}}\|_{\omega_a} \leq C_{\text{st}} \|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}^{\mathbf{a}}\|_{\omega_a}$$

Note the (trivial) converse bound  $\|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}^{\mathbf{a}}\|_{\omega_a} \leq \|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}_h^{\mathbf{a}}\|_{\omega_a}$

## Step 2

- ▶ Evaluate  $\|\tau_h^a + \sigma^a\|_{\omega_a}$  using equivalent problem in **primal form**
  - ▶  $r^a \in H_*^1(\omega_a) = \{v \in H^1(\omega_a) \mid (v, 1)_{\omega_a} = 0\}$  s.t.

$$(\nabla r^a, \nabla v)_{\omega_a} = -(\tau_h^a, \nabla v)_{\omega_a} + (g_h^a, v)_{\omega_a} \quad \forall v \in H_*^1(\omega_a)$$

- ▶ Equivalence of primal/dual energies

$$\|\tau_h^a + \sigma^a\|_{\omega_a} = \min_{v \in \mathbf{V}^a, \nabla \cdot v = g_h^a} \|\tau_h^a + v\|_{\omega_a} = \|\nabla r^a\|_{\omega_a}$$

- ▶ We have [recall  $\tau_h^a := \psi_a \nabla u_h$ ,  $g_h^a := \Pi_{(\nabla \cdot \mathbf{V}_h^a)}(f \psi_a - \nabla u_h \cdot \nabla \psi_a)$ ]

- ▶  $(\nabla r^a, \nabla v)_{\omega_a} = (\nabla(u - u_h), \nabla(\psi_a v))_{\omega_a} + \text{osc}(f, \omega_a)$
- ▶  $\|\nabla(\psi_a v)\|_{\omega_a} \leq (1 + C_{\text{PS}, \omega_a} h_{\omega_a} \|\nabla \psi_a\|_{L^\infty(\omega_a)}) \|\nabla v\|_{\omega_a}$

and we can conclude that

$$\|\tau_h^a + \sigma^a\|_{\omega_a} \leq C_{\text{PS}} \|\nabla(u - u_h)\|_{\omega_a} + \text{osc}(f, \omega_a)$$

with  $C_{\text{PS}} := \max_{a \in \mathcal{V}_h} (1 + C_{\text{PS}, \omega_a} h_{\omega_a} \|\nabla \psi_a\|_{L^\infty(\omega_a)})$

## Nonconforming case ( $u_h \notin H_0^1(\Omega)$ )

- ▶ Error measure w.r.t. **broken gradient**  $\nabla_{\mathcal{T}}$  of discrete solution
- ▶ Additional **nonconformity estimator** in error upper bound
- ▶  $H_0^1$ -**potential reconstruction**  $s_h \in H_0^1(\Omega)$
- ▶ Setting  $\eta_{\text{NC},K} := \|\nabla_{\mathcal{T}}(u_h - s_h)\|$ ,

$$\|\nabla_{\mathcal{T}}(u - u_h)\|^2 \leq \sum_{K \in \mathcal{T}_h} (\eta_{\text{F},K} + \eta_{\text{osc},K})^2 + \sum_{K \in \mathcal{T}_h} (\eta_{\text{NC},K})^2$$

- ▶ Typically,  $s_h$  is built by prescribing its nodal values as averages
  - ▶ see [Achdou, Bernardi, Coquel 03; Karakashian, Pascal, 03] for Crouzeix–Raviart and IPDG residual-based estimates
  - ▶ ... not  $p$ -robust [Burman, AE 07; Houston, Schötzau, Wihler 07]

## Potential reconstruction

- ▶ **Local FEM solves** of order  $(p + 1)$  on vertex-based patches
  - ▶  $s_h^a \in P^{p+1}(\mathcal{T}_a) \cap H_0^1(\omega_a)$
  - ▶ set  $s_h := \sum_{a \in \mathcal{V}_h} s_h^a$
- ▶  **$p$ -robust** local efficiency proved in [AE, MV 15] in 2D
  - ▶ assuming  $\langle \llbracket u_h \rrbracket, 1 \rangle_F = 0$ ,

$$\eta_{\text{NC},K} \leq C_{\text{eff}} \sum_{a \in \mathcal{V}_K} \|\nabla_{\mathcal{T}}(u - u_h)\|_{\omega_a}, \quad C_{\text{eff}} = C_{\text{st}} C_{\text{bPS}}$$

- ▶ two-step proof as above (no oscillation here)
- ▶ 1st step: mixed RT solve of order  $p$  for rotated gradient of  $s_h^a$
- ▶ 2nd step: broken PS inequalities, see also [Carstensen, Merdon 13]
- ▶ jump seminorm added to error and estimator if  $\langle \llbracket u_h \rrbracket, 1 \rangle_F \neq 0$
- ▶ use **discrete gradient** for DG (instead of broken gdt); broken PS and jump seminorm can be avoided for symmetric IPDG [AE, MV 15]
- ▶ **How about 3D?**  $\implies$  [AE, MV 19]

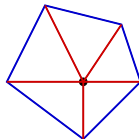
## Main results [AE, MV 19]

- ▶ **3D,  $p$ -robust** flux and potential reconstructions
  - ▶ **Thm. 1**  $p$ -robust,  $H^1$ -stable polynomial lifting for potentials
  - ▶ **Thm. 2**  $p$ -robust,  $H(\text{div})$ -stable polynomial lifting for fluxes
- ▶ Our proofs are constructive (as 2D proofs)
  - ▶ e.g., build  $\tilde{\sigma}_h^a \in \mathbf{V}_h^a$  s.t.  $\nabla \cdot \tilde{\sigma}_h^a = g_h^a$ ,  $\|\tau_h^a + \tilde{\sigma}_h^a\|_{\omega_a} \leq C_{\text{st}} \|\tau_h^a + \sigma^a\|_{\omega_a}$
  - ▶ then  $\|\tau_h^a + \sigma_h^a\|_{\omega_a} \leq \|\tau_h^a + \tilde{\sigma}_h^a\|_{\omega_a} \leq C_{\text{st}} \|\tau_h^a + \sigma^a\|_{\omega_a}$
- ▶ **Main challenges**
  - ▶ how to enumerate tetrahedra in 3D star (triangles in 2D star are enumerated by circling around the vertex)
  - ▶ need to work with potentials (and not with rotated gradients)
- ▶ We focus for simplicity on FE stars around **interior** vertices
  - ▶ adaptations for BC's discussed in [AE, MV 19]

## Some notation

- ▶  $\mathcal{T}_a \subset \mathcal{T}_h$ : FE star (elements sharing vertex  $\mathbf{a} \in \mathcal{V}_h$ ); domain  $\omega_a$
- ▶  $\mathcal{F}_a = \mathcal{F}_a^s \cup \mathcal{F}_a^b$ : faces of the elements in the star  $\mathcal{T}_a$

(2D) skeletal faces  $\mathcal{F}_a^s$   
 (2D) boundary faces  $\mathcal{F}_a^b$



- ▶ Broken  $H^1$ - and  $H(\text{div})$ -spaces (broken gradient  $\nabla_{\mathcal{T}}$ )

$$H^1(\mathcal{T}_a) := \{v \in L^2(\omega_a) \mid v|_K \in H^1(K) \forall K \in \mathcal{T}_a\}$$

$$\mathbf{H}(\text{div}, \mathcal{T}_a) := \{\mathbf{v} \in \mathbf{L}^2(\omega_a) \mid \mathbf{v}|_K \in \mathbf{H}(\text{div}, K) \forall K \in \mathcal{T}_a\}$$

- ▶ Broken polynomial subspaces

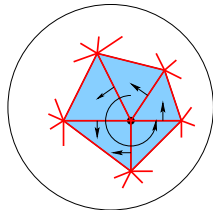
$$\mathbb{P}^p(\mathcal{T}_a) := \{v \in L^2(\omega_a) \mid v|_K \in \mathbb{P}^p(K) \forall K \in \mathcal{T}_a\}$$

$$\mathbf{RT}^p(\mathcal{T}_a) := \{\mathbf{v} \in \mathbf{L}^2(\omega_a) \mid \mathbf{v}|_K \in \mathbf{RT}^p(K) \forall K \in \mathcal{T}_a\}$$

## Some notation (cont'd)

- ▶ In 3D, a FE star  $\omega_a$  is homeomorphic to a ball in  $\mathbb{R}^3$
- ▶ We can look at the star boundary  $\partial\omega_a$ 
  - ▶ the traces of the tetrahedra in  $\mathcal{T}_a$  form a triangulation of  $\partial\omega_a$

every triangle is a boundary face  $F \in \mathcal{F}_a^b$   
 every edge is the trace of a skeletal face  $F \in \mathcal{F}_a^s$   
 every point is the trace of a skeletal edge  $e \in \mathcal{E}_a$



- ▶ Orientation
  - ▶ every skeletal face is oriented so as to define a jump across it
  - ▶ every skeletal edge is oriented so as to circle around it
  - ▶ incidence coefficients  $\iota_{F,e} = \pm 1$ , for all  $F \in \mathcal{F}_e$  and  $e \in \mathcal{E}_a$



# $H^1$ -stable polynomial lifting

- ▶ Let  $p \geq 1$
- ▶ Let  $r_F \in \mathbb{P}^p(F)$  for all  $F \in \mathcal{F}_a^s$  and  $r_F \equiv 0$  for all  $F \in \mathcal{F}_a^b$
- ▶ Assume on all skeletal faces  $F \in \mathcal{F}_a^s$  the compatibility conditions

$$r_F|_{F \cap \partial \omega_a} = 0, \quad \sum_{F \in \mathcal{F}_e} \iota_{F,e} r_F|_e = 0 \quad \forall e \in \mathcal{E}_a$$

- ▶ **Then,**

$$\min_{\substack{v_h \in \mathbb{P}^p(\mathcal{T}_a) \\ v_h = 0 \quad \forall F \in \mathcal{F}_a^b \\ [v_h] = r_F \quad \forall F \in \mathcal{F}_a^s}} \|\nabla_{\mathcal{T}} v_h\|_{\omega_a} \leq C_{\text{st}} \min_{\substack{v \in H^1(\mathcal{T}_a) \\ v = 0 \quad \forall F \in \mathcal{F}_a^b \\ [v] = r_F \quad \forall F \in \mathcal{F}_a^s}} \|\nabla_{\mathcal{T}} v\|_{\omega_a}$$

with  $p$ -robust constant  $C_{\text{st}}$  only depending on mesh regularity

# $H(\text{div})$ -stable polynomial lifting

- ▶ Let  $p \geq 0$
- ▶ Let  $r_K \in \mathbb{P}^p(K)$  for all  $K \in \mathcal{T}_a$  and let  $r_F \in \mathbb{P}^p(F)$  for all  $F \in \mathcal{F}_a$
- ▶ Assume the compatibility condition

$$\sum_{K \in \mathcal{T}_a} (r_K, 1)_K - \sum_{F \in \mathcal{F}_a} (r_F, 1)_F = 0$$

- ▶ Then,

$$\min_{\mathbf{v}_h \in \mathbf{RT}^p(\mathcal{T}_a)} \|\mathbf{v}_h\|_{\omega_a} \leq C_{\text{St}} \min_{\mathbf{v} \in \mathbf{H}(\text{div}, \mathcal{T}_a)} \|\mathbf{v}\|_{\omega_a}$$

$$\begin{array}{l} \mathbf{v}_h \cdot \mathbf{n}_F = r_F \quad \forall F \in \mathcal{F}_a^b \\ \llbracket \mathbf{v}_h \cdot \mathbf{n}_F \rrbracket = r_F \quad \forall F \in \mathcal{F}_a^s \\ \nabla_{\mathcal{T}} \cdot \mathbf{v}_h|_K = r_K \quad \forall K \in \mathcal{T}_a \end{array}$$

$$\begin{array}{l} \mathbf{v} \cdot \mathbf{n}_F = r_F \quad \forall F \in \mathcal{F}_a^b \\ \llbracket \mathbf{v} \cdot \mathbf{n}_F \rrbracket = r_F \quad \forall F \in \mathcal{F}_a^s \\ \nabla_{\mathcal{T}} \cdot \mathbf{v}|_K = r_K \quad \forall K \in \mathcal{T}_a \end{array}$$

with  $p$ -robust constant  $C_{\text{St}}$  only depending on mesh regularity

## Shifted reformulation: potential

- ▶ Let  $\xi_h^a \in P^p(\mathcal{T}_a)$  be any function from the minimization set

$$\xi_h^a = r_F \quad \forall F \in \mathcal{F}_a^b, \quad \llbracket \xi_h^a \rrbracket = r_F \quad \forall F \in \mathcal{F}_a^s$$

- ▶ An equivalent reformulation of Thm. 1 is

$$\min_{v_h \in PP(\mathcal{T}_a) \cap H_0^1(\omega_a)} \|\nabla_{\mathcal{T}}(\xi_h^a - v_h)\|_{\omega_a} \leq C_{\text{st}} \min_{v \in H_0^1(\omega_a)} \|\nabla_{\mathcal{T}}(\xi_h^a - v)\|_{\omega_a}$$

- ▶ Application to a posteriori error analysis:  $\xi_h^a := \psi_a u_h$  and

$$r_F := 0 \quad \forall F \in \mathcal{F}_a^b, \quad r_F := \psi_a \llbracket u_h \rrbracket \quad \forall F \in \mathcal{F}_a^s$$

The compatibility conditions  $\sum_{F \in \mathcal{F}_e} \iota_{F,e} r_F|_e = 0, \quad \forall e \in \mathcal{E}_a$ , follow from algebraic properties of jump operator

## Shifted reformulation: flux

- ▶ Let  $\boldsymbol{\tau}_h^{\mathbf{a}} \in \mathbf{RT}^P(\mathcal{T}_a)$  be any function s.t.

$$\boldsymbol{\tau}_h^{\mathbf{a}} \cdot \mathbf{n}_F = r_F \quad \forall F \in \mathcal{F}_a^{\mathbf{b}}, \quad \llbracket \boldsymbol{\tau}_h^{\mathbf{a}} \rrbracket \cdot \mathbf{n}_F = r_F \quad \forall F \in \mathcal{F}_a^{\mathbf{s}}$$

- ▶ An equivalent reformulation of Thm. 2 is

$$\min_{\substack{\mathbf{v}_h \in \mathbf{RT}^P(\mathcal{T}_a) \cap \mathbf{H}_0(\text{div}, \omega_a) \\ \nabla \cdot \mathbf{v}_h|_K = r_K - \nabla \cdot \boldsymbol{\tau}_h^{\mathbf{a}}|_K \quad \forall K \in \mathcal{T}_a}} \|\boldsymbol{\tau}_h^{\mathbf{a}} + \mathbf{v}_h\|_{\omega_a} \leq C_{\text{st}} \min_{\substack{\mathbf{v} \in \mathbf{H}_0(\text{div}, \omega_a) \\ \nabla \cdot \mathbf{v}|_K = r_K - \nabla \cdot \boldsymbol{\tau}_h^{\mathbf{a}}|_K \quad \forall K \in \mathcal{T}_a}} \|\boldsymbol{\tau}_h^{\mathbf{a}} + \mathbf{v}\|_{\omega_a}$$

- ▶ Application to a posteriori error analysis:  $\boldsymbol{\tau}_h^{\mathbf{a}} := \psi_a \nabla_{\mathcal{T}} u_h$  and

$$\begin{aligned} r_F &:= 0 \quad \forall F \in \mathcal{F}_a^{\mathbf{b}}, & r_F &:= \psi_a \llbracket \nabla_{\mathcal{T}} u_h \rrbracket \cdot \mathbf{n}_F \quad \forall F \in \mathcal{F}_a^{\mathbf{s}} \\ r_K &:= \psi_a (f + \Delta_{\mathcal{T}} u_h) \quad \forall K \in \mathcal{T}_a \quad (f \text{ pcw. polynomial}) \end{aligned}$$

The compatibility condition  $\sum_{K \in \mathcal{T}_a} (r_K, \mathbf{1})_K - \sum_{F \in \mathcal{F}_a} (r_F, \mathbf{1})_F = 0$  is nothing but Galerkin's orthogonality on the hat basis function  $\psi_a$

# Main ingredients of proofs

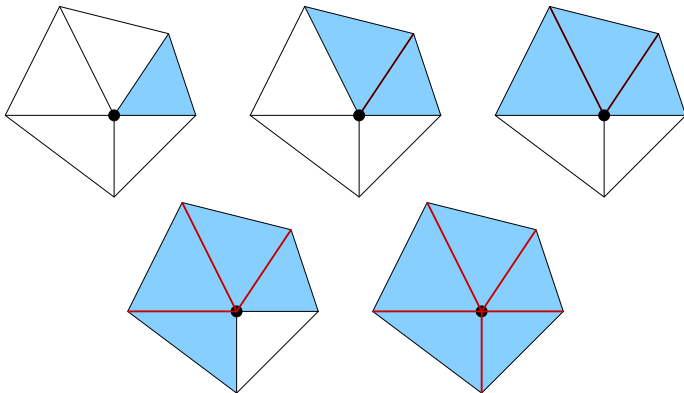
On a **fixed** tetrahedron  $K \in \mathcal{T}_a$ , we can

- ▶ Lift the prescribed divergence of the flux using [Costabel, McIntosh 10] (valid in any space dimension)
- ▶ Lift prescribed polynomials for the flux normal component using [Demkowicz, Gopalakrishnan, Schöberl 12] (proved for  $d = 3$ ) (a compatibility condition is required if the prescription is on all faces)
- ▶ Lift prescribed compatible polynomials for the potential trace using [DGS 09] (proved for  $d = 3$ )

We are left with the  $p$ -robust lifting of the **prescribed jumps** ... but this requires a careful **enumeration of the tetrahedra in the star**

## 2D enumeration

- ▶ Circle around the interior vertex  $\mathbf{a}$ 
  - ▶  $K_1$ : do nothing
  - ▶  $K_n$ : fix jump on face touching  $K_{n-1}$ ,  $n \in \{2 \dots 4\}$
  - ▶  $K_5$ : fix last two jumps (possible owing to **compatibility condition**)



## 3D enumeration: shellability

- ▶ Let  $\mathcal{T}_a$  be a star of tetrahedra around the interior vertex  $\mathbf{a}$
- ▶ Consider the triangulation of the sphere  $S^{(2)} \subset \mathbb{R}^3$  with same connectivity
- ▶ Enumerate surface triangles so that  $\cup_{j \leq i} T_j^{(2)}$  is connected for all  $i$
- ▶ The notion of **shellability of polytopes** shows that this is possible [Ziegler, Lectures on Polytopes, Chap. 8, Springer, 2006]
- ▶ Enumerate patch tetrahedra following surface triangle enumeration and, for each  $K_i \in \mathcal{T}_a$ , fix jump on the skeletal faces of  $K_i$  touching any tetrahedron  $K_j$  for  $j < i$

# Numerical results

- ▶ **Smooth analytical** solution in  $\Omega = (0, 1)^2$ 
  - ▶  $u(x_1, x_2) = \sin(2\pi x_1) \sin(2\pi x_2)$
- ▶ **Uniformly refined, non-nested** unstructured triangulations
- ▶ Discretization by symmetric IPDG method
  - ▶ **asymptotic exactness** observed for pol. degrees  $p \in \{1 \dots 6\}$
  - ▶ similar results for incomplete version of IPDG
  - ▶ slightly larger effectivity indices for nonsymmetric version and even  $p$



# Errors, estimators, effectivity indices

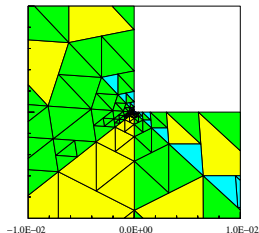
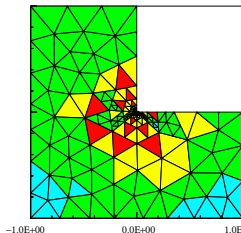
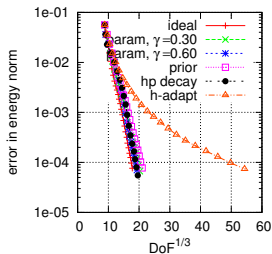
$h/h_0$	$p$	$\ \nabla e\ $	$j(e)$	$\ \nabla e\  + j(e)$	$\eta_F$	$\eta_{osc}$	$\eta_{NC}$	$\eta$	$\eta + j(u_h)$	$j^{eff}$	$j_j^{eff}$
1	1	1.07E-00	1.92E-01	1.09E-00	1.12E-00	5.55E-02	4.16E-01	1.25E-00	1.26E-00	1.17	1.16
1/2	1	5.56E-01	7.28E-02	5.61E-01	5.71E-01	7.42E-03	1.82E-01	6.07E-01	6.11E-01	1.09	1.09
1/4	1	2.92E-01	2.82E-02	2.93E-01	2.96E-01	1.04E-03	8.77E-02	3.10E-01	3.11E-01	1.06	1.06
1/8	1	1.39E-01	9.19E-03	1.39E-01	1.40E-01	1.10E-04	3.85E-02	1.45E-01	1.45E-01	1.04	1.04
1	2	1.54E-01	1.76E-02	1.55E-01	1.55E-01	5.10E-03	3.05E-02	1.63E-01	1.64E-01	1.06	1.06
1/2	2	4.07E-02	4.66E-03	4.09E-02	4.13E-02	3.53E-04	7.55E-03	4.23E-02	4.26E-02	1.04	1.04
1/4	2	1.10E-02	1.26E-03	1.11E-02	1.12E-02	2.51E-05	1.97E-03	1.14E-02	1.15E-02	1.03	1.03
1/8	2	2.50E-03	2.90E-04	2.52E-03	2.54E-03	1.30E-06	4.21E-04	2.57E-03	2.59E-03	1.03	1.03
1	3	1.37E-02	3.96E-04	1.37E-02	1.37E-02	3.58E-04	1.74E-03	1.41E-02	1.41E-02	1.03	1.03
1/2	3	1.85E-03	4.53E-05	1.85E-03	1.85E-03	1.26E-05	2.10E-04	1.88E-03	1.88E-03	1.01	1.01
1/4	3	2.60E-04	4.79E-06	2.60E-04	2.60E-04	4.73E-07	2.54E-05	2.62E-04	2.62E-04	1.01	1.01
1/8	3	2.75E-05	3.75E-07	2.75E-05	2.75E-05	1.15E-08	2.55E-06	2.76E-05	2.76E-05	1.01	1.01
1	4	9.87E-04	2.95E-05	9.87E-04	9.84E-04	2.12E-05	1.11E-04	1.01E-03	1.01E-03	1.02	1.02
1/2	4	6.92E-05	2.06E-06	6.93E-05	6.92E-05	3.96E-07	7.44E-06	7.00E-05	7.00E-05	1.01	1.01
1/4	4	5.04E-06	1.42E-07	5.04E-06	5.04E-06	7.58E-09	4.98E-07	5.07E-06	5.07E-06	1.01	1.01
1/8	4	2.58E-07	7.61E-09	2.59E-07	2.58E-07	8.96E-11	2.47E-08	2.60E-07	2.60E-07	1.01	1.01
1	5	5.64E-05	6.76E-07	5.64E-05	5.63E-05	1.06E-06	4.50E-06	5.75E-05	5.75E-05	1.02	1.02
1/2	5	2.01E-06	2.18E-08	2.01E-06	2.01E-06	9.88E-09	1.46E-07	2.03E-06	2.03E-06	1.01	1.01
1/4	5	7.74E-08	6.04E-10	7.74E-08	7.73E-08	1.01E-10	4.35E-09	7.76E-08	7.76E-08	1.00	1.00
1/8	5	1.86E-09	1.18E-11	1.86E-09	1.86E-09	1.70E-12	1.00E-10	1.86E-09	1.86E-09	1.00	1.00
1	6	2.85E-06	3.70E-08	2.85E-06	2.85E-06	4.70E-08	2.18E-07	2.90E-06	2.90E-06	1.02	1.02
1/2	6	5.42E-08	6.78E-10	5.42E-08	5.42E-08	2.40E-10	4.02E-09	5.46E-08	5.46E-08	1.01	1.01
1/4	6	1.07E-09	1.20E-11	1.07E-09	1.07E-09	1.03E-11	6.90E-11	1.08E-09	1.08E-09	1.01	1.01

## *hp*-adaptivity

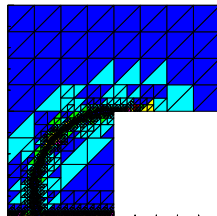
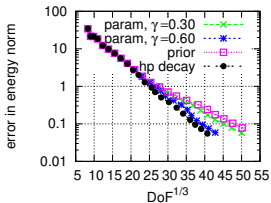
- ▶ Nested simplicial meshes allowing for **hanging nodes**
- ▶ Extension of reconstruction procedures to hanging nodes
  - ▶ only matching refinement of individual patches is needed
- ▶ Bulk chasing criterion based on local  $p$ -robust estimators
- ▶  $hp$ -refinement decision criteria inspired from [Mitchell, McClain 14]
- ▶ **Algebraic convergence w.r.t. dof's** observed on several 2D benchmark problems from [Mitchell 13]

# Numerical examples

- ▶ **Re-entrant corner singularity** ( $u \in H^{1+t}(\Omega)$ ,  $t < \frac{2}{3}$ )

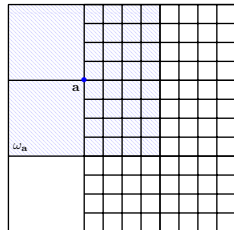
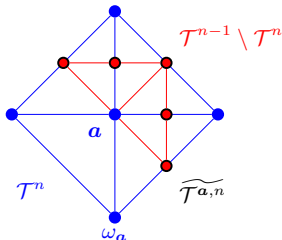


- ▶ **Multiple difficulties** (re-entrant corner, point sing, circular wave)



# Global $H(\text{div})$ -liftings

- ▶ So far, we devised **local** liftings of polynomial data on **FE stars**
- ▶ We now consider **global  $H(\text{div})$ -liftings** of polynomial data
- ▶ One important application is to devise liftings on patches of mesh cells that are **(much) larger than a star**
- ▶ This allows us to remove some theoretical barriers on
  - ▶ number of hanging nodes for elliptic problems
  - ▶ level of mesh coarsening between time-steps in parabolic problems



## Main result

- ▶ Lipschitz domain  $\Omega \subset \mathbb{R}^d$ ; boundary partition  $\Gamma = \Gamma_D \cup \Gamma_N$
- ▶ Let  $\mathcal{T}_h$  be a simplicial mesh of  $\Omega$ , **possibly locally refined**
- ▶ **Thm.** Let  $p \geq 1$ ,  $f \in P^{p-1}(\mathcal{T}_h)$ ,  $\xi \in RT^{p-1}(\mathcal{T}_h)$  (broken spaces) (with  $(f, 1)_\Omega = 0$  if  $\Gamma_N = \Gamma$ ). Then,

$$\min_{\substack{\mathbf{v}_h \in RT^p(\mathcal{T}_h) \cap \mathbf{H}(\text{div}, \Omega) \\ \nabla \cdot \mathbf{v}_h = f \text{ in } \Omega \\ \mathbf{v}_h \cdot \mathbf{n} = 0 \text{ on } \Gamma_N}} \|\xi + \mathbf{v}_h\|_\Omega \leq C_{\text{st}} \min_{\substack{\mathbf{v} \in \mathbf{H}(\text{div}, \Omega) \\ \nabla \cdot \mathbf{v} = f \text{ in } \Omega \\ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_N}} \|\xi + \mathbf{v}\|_\Omega$$

with  **$p$ -robust** constant  $C_{\text{st}}$  only depending on mesh regularity  
(Note that the discrete minimizer is one polynomial order higher than the data)

- ▶ See [Ainsworth, Guzman, Sayas 16] for zero interior source terms, nonzero boundary traces, and fixed polynomial degree

## Main idea in proof

- ▶ Primal problem with  $H^{-1}$  data

$$u \in H_*^1(\Omega) \text{ s.t. } (\nabla u, \nabla v)_\Omega = (f, v)_\Omega - (\xi, \nabla v)_\Omega, \forall v \in H_*^1(\Omega)$$

$$H_*^1(\Omega) = \{v \in H^1(\Omega) \mid (v, 1)_\Omega = 0\} \text{ if } \Gamma_N = \Gamma$$

$$H_*^1(\Omega) = \{v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0\} \text{ otherwise}$$

- ▶ Equivalence of primal/dual energies

$$\min_{\substack{\mathbf{v} \in \mathbf{H}(\text{div}, \Omega) \\ \nabla \cdot \mathbf{v} = f \text{ in } \Omega \\ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_N}} \|\xi + \mathbf{v}\|_\Omega = \max_{v \in H_*^1(\Omega)} \frac{(f, v)_\Omega - (\xi, \nabla v)_\Omega}{\|\nabla v\|_\Omega} = \|\nabla u\|_\Omega$$

- ▶ Need to construct  $\sigma_h \in \mathbf{RT}^P(\mathcal{T}_h) \cap \mathbf{H}(\text{div}, \Omega)$  s.t.

$$\|\xi + \sigma_h\|_\Omega \leq C_{\text{st}} \|\nabla u\|_\Omega$$

## A posteriori error estimate with $H^{-1}$ data

- ▶ Consider general data  $f \in L^2(\Omega)$  and  $\boldsymbol{\xi} \in \mathbf{L}^2(\Omega)$

$$u \in H_*^1(\Omega) \text{ s.t. } (\nabla u, \nabla v)_\Omega = (f, v)_\Omega - (\boldsymbol{\xi}, \nabla v)_\Omega, \forall v \in H_*^1(\Omega)$$

- ▶  $H_*^1$ -conforming approximation  $u_h \in P^{p'}(\mathcal{T}_h) \cap H_*^1(\Omega)$ ,  $p' \geq 1$
- ▶ Equilibrated flux reconstruction  $\boldsymbol{\sigma}_h \in \mathbf{RT}^p(\mathcal{T}_h) \cap \mathbf{H}(\text{div}, \Omega)$ ,  $p \geq p'$

$$\|\nabla(u - u_h)\|_\Omega^2 \leq \sum_{K \in \mathcal{T}_h} (\|\nabla u_h + \boldsymbol{\xi} + \boldsymbol{\sigma}_h\|_K + \text{osc}(f, K))^2$$

$$\|\nabla u_h + \boldsymbol{\xi} + \boldsymbol{\sigma}_h\|_K \leq C_{\text{eff}} \|\nabla(u - u_h)\|_{\omega_K} + \text{osc}(f, \boldsymbol{\xi}, \omega_K)$$

- ▶  $\boldsymbol{\sigma}_h$  constructed locally from local **shifted** flux equilibration in  $\mathbf{V}_h^{\mathbf{a}} = \mathbf{RT}^p(\mathcal{T}_a) \cap \mathbf{H}(\text{div}, \omega_a)$  (+ Neumann BC's)

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}}, \nabla \cdot \mathbf{v}_h = g_h^{\mathbf{a}}} \|\boldsymbol{\tau}_h^{\mathbf{a}} + \mathbf{v}_h\|_{\omega_a}$$

$$\text{with data } \boldsymbol{\tau}_h^{\mathbf{a}} := \psi_{\mathbf{a}}(\boldsymbol{\xi} + \nabla u_h), \quad g_h^{\mathbf{a}} := \Pi_{(\nabla \cdot \mathbf{V}_h^{\mathbf{a}})}(f\psi_{\mathbf{a}} - (\boldsymbol{\xi} + \nabla u_h) \cdot \nabla \psi_{\mathbf{a}})$$

## Conclusion of proof

- ▶ Consider polynomial data  $f \in P^{p-1}(\mathcal{T}_h)$ ,  $\xi \in RT^{p-1}(\mathcal{T}_h)$ ,  $p \geq 1$
- ▶ Consider  $H_*^1$ -conforming FEM approximation with  $1 = p' \leq p$
- ▶ The **data oscillation term**  $\text{osc}(f, \xi, \omega_K)$  **vanishes** so that

$$\|\nabla u_h + \xi + \sigma_h\|_{\Omega} \leq C_{\text{eff}} \|\nabla(u - u_h)\|_{\Omega}$$

- ▶ Combined with the basic bound  $\|\nabla u_h\|_{\Omega} \leq \|\nabla u\|_{\Omega}$ , we conclude that

$$\|\xi + \sigma_h\|_{\Omega} \leq \|\nabla u_h + \xi + \sigma_h\|_{\Omega} + \|\nabla u_h\|_{\Omega} \leq (2C_{\text{eff}} + 1) \|\nabla u\|_{\Omega}$$



## Local/global best-approximation in $H(\text{div})$

- ▶ Let  $p \geq 0$  and set for all  $\mathbf{v} \in \mathbf{H}_{0,\Gamma_N}(\text{div}; \Omega)$ ,

$$[E_{\mathcal{T}_h,p}(\mathbf{v})]^2 := \min_{\substack{\mathbf{v}_h \in \text{RTN}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,\Gamma_N}(\text{div}; \Omega) \\ \nabla \cdot \mathbf{v}_h = \Pi_{\mathcal{T}_h}^p(\nabla \cdot \mathbf{v})}} \|\mathbf{v} - \mathbf{v}_h\|_{\Omega}^2 + \sum_{K \in \mathcal{T}_h} \left[ \frac{h_K}{p+1} \|\nabla \cdot \mathbf{v} - \Pi_{\mathcal{T}_h}^p(\nabla \cdot \mathbf{v})\|_K \right]^2$$

$$[e_{K,p}(\mathbf{v})]^2 := \min_{\mathbf{v}_K \in \text{RTN}_p(K)} \|\mathbf{v} - \mathbf{v}_K\|_K^2 + \left[ \frac{h_K}{p+1} \|\nabla \cdot \mathbf{v} - \Pi_{\mathcal{T}_h}^p(\nabla \cdot \mathbf{v})\|_K \right]^2 \quad \forall K \in \mathcal{T}_h$$

- ▶ [AE, Gudi, Smears, MV 19] There is  $C$ , depending on mesh regularity and  $p$ , s.t. for all  $\mathbf{v} \in \mathbf{H}_{0,\Gamma_N}(\text{div}; \Omega)$ ,

$$\sum_{K \in \mathcal{T}_h} [e_{K,p}(\mathbf{v})]^2 \leq [E_{\mathcal{T}_h,p}(\mathbf{v})]^2 \leq C \sum_{K \in \mathcal{T}_h} [e_{K,p}(\mathbf{v})]^2$$

- ▶ Remarks

- ▶ prescription on the divergence can be included locally
- ▶  $C$  becomes  $p$ -robust in  $[E_{\mathcal{T}_h,p}(\mathbf{v})]^2 \leq C \sum_{K \in \mathcal{T}_h} [e_{K,p-1}(\mathbf{v})]^2$
- ▶ generalizes result for local/global best-approx. in  $H^1$  [Veiser 16]

## Optimal $hp$ -approximation

- ▶ [AE, Gudi, Smears, MV 19] Let  $\mathbf{v} \in \mathbf{H}_{0,\Gamma_N}(\text{div}; \Omega)$  be pcw. in  $\mathbf{H}^s$ ,  $s \in (0, 1)$ . There is  $C$ , only depending on mesh regularity ( $C$  is  $p$ -robust), s.t.

$$[E_{T_h,p}(\mathbf{v})]^2 \leq C \left\{ \sum_{K \in \mathcal{T}_h} \left[ \frac{h_K^{\min(s,p+1)}}{(p+1)^s} \|\mathbf{v}\|_{\mathbf{H}^s(K)} \right]^2 + \left[ \frac{h_K}{p+1} \|\nabla \cdot \mathbf{v}\|_K \right]^2 \right\}$$

- ▶ Remarks
  - ▶ also valid for  $s \geq 1$ , see also [Melenk, Rojik 19]
  - ▶ improves on [Demkowicz, Buffa 05] by removing logarithmic factor in  $p$  and reducing regularity requirements
  - ▶ application to a priori error analysis for mixed FEM
  - ▶ see also [AE, Guermond 17] for averaging operators (not  $p$ -robust)

## Stable, local, commuting projector

- ▶ There is  $P_{\mathcal{T}_h}^p : \mathbf{H}_{0,\Gamma_N}(\text{div}; \Omega) \rightarrow \mathbf{RTN}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,\Gamma_N}(\text{div}; \Omega)$  s.t.  
(constructed locally from local minimizations on stars)

$$\nabla \cdot P_{\mathcal{T}_h}^p(\mathbf{v}) = \Pi_{\mathcal{T}_h}^p(\nabla \cdot \mathbf{v})$$

$$P_{\mathcal{T}_h}^p(\mathbf{v}) = \mathbf{v} \quad \text{for all } \mathbf{v} \in \mathbf{RTN}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,\Gamma_N}(\text{div}; \Omega)$$

$$\|\mathbf{v} - P_{\mathcal{T}_h}^p(\mathbf{v})\|_K^2 + \left[ \frac{h_K}{p+1} \|\nabla \cdot (\mathbf{v} - P_{\mathcal{T}_h}^p(\mathbf{v}))\|_K \right]^2 \leq C \sum_{K' \in \mathcal{T}_K} [e_{K',p}(\mathbf{v})]^2 \quad \forall K \in \mathcal{T}_h$$

where  $C$  depends on mesh regularity and on  $p$

- ▶ Remarks
  - ▶ **non-local**, stable, commuting projectors in [Schöberl 01; Christiansen, Winther 08; AE, Guermond 16; Licht 19]
  - ▶ **local** construction in [Falk, Winther 14], stability **only** in graph-norm, **no approximation** discussed

# Conclusions

- ▶ Equilibrated-flux estimates offer several benefits
  - ▶ **guaranteed** (fully computable) upper bounds
  - ▶  **$p$ -robust** local efficiency
  - ▶ **adaptive inexact Newton solvers** [AE, MV 13]
- ▶ **Unified analysis** for  $p$ -robust  $H^1$ - and  $\mathbf{H}(\text{div})$ -polynomial liftings
- ▶ New local efficiency proofs for **arbitrary level** of hanging nodes (elliptic PDEs) and **no coarsening restriction** (parabolic PDEs)
- ▶ New stable commuting projectors and a priori error estimates for mixed FEM

## ► References for this talk

1. AE, MV, *Math. Comp.* (2019), submitted in 2016
2. AE, I. Smears, MV, *Calcolo* (2017), **54** 1009–1035
3. V. Dolejší, AE, MV, *SISC* (2016), **38** A3220–A3246
4. AE, T. Gudi, I. Smears, MV (2019) [hal-02268960](#)

## ► New Finite Element book(s) (3 volumes, Spring(er) 2020)

10 chapters of 50 pages → 75 chapters of 14/16 pages plus exercises



Thank you for your attention