The Dirac equation on spherically symmetric manifolds

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The Dirac equation on flat space-time

Models the free evolution of a pair electron positron.

\[ i\gamma^\mu \partial_\mu u = mu \]

with

\[ u : \mathbb{R}^{1+3} \to \mathbb{C}^4 \]

\[ (t, x) \mapsto u(t, x). \]

and convention \( \partial_0 = \partial_t \) and \( \nabla = (\partial_1, \partial_2, \partial_3) \).

The matrices \( \gamma^0 \) and \( i\gamma^j \) are self-adjoint matrices in \( \mathcal{M}_4(\mathbb{C}) \) and satisfy

\[ \{ \gamma^\mu, \gamma^\nu \} = 2\eta^{\mu\nu} \]

where \( \eta = \text{Diag}(1, -1, -1, -1) \) is the Minkowski metrics.
It is a Schrödinger equation:

\[ i \partial_t u = \gamma^0 m u - i \gamma^0 \gamma^j \partial_j u = H u \]

with \( H \) self-adjoint.

Standard choice for the \( \gamma \)s:

\[ \gamma^0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \gamma^j = \beta \alpha^j = \beta \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \]

where the \( \sigma \)s are the Pauli matrices

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \]
In this representation \( u = \begin{pmatrix} e^- & \uparrow \\ e^- & \downarrow \\ e^+ & \uparrow \\ e^+ & \downarrow \end{pmatrix} \).

The equation is covariant, that is, it is invariant under change of special relativity referential. Those are the translations plus the linear transformations that preserves

- the metrics,
- the orientation,
- and causality:

\( SO_0(1, 3) \). Change variable \( x \) into variable \( x' \) defined as

\[(x')^a = L^a_b x^b, \quad L \in SO_0(1, 3).\]
There exists a group representation $U : SO_0(1, 3) \to \mathcal{U}_4(\mathbb{C})$ such that if $u$ solves the Dirac equation then $u'(x') = U(L)u(x)$ solves the Dirac equation.

To define the Dirac equation, one thus needs to fix equivalently:

- the set of $\gamma$s,
- the basis of $\mathbb{C}^4$,
- the referential.

Note that $U$ is explicit.
To the square, we retrieve the Klein-Gordon/wave (if $m = 0$) equation
\[
\left( \partial_t^2 + m^2 - \Delta_{\mathbb{R}^3} \right) u = 0.
\]

Consequences:
- Dispersive estimates
- Strichartz
- Local smoothing or Morawetz’s estimates
- Local well-posedness for semilinear models.
The Dirac equation on curved space-time

Comes from QFT in curved space-time: approximation of gravitational waves or string theory at low energies.

Now, the equation writes

\[ i\gamma^\mu D_\mu u = mu \]

where

- \( m \in \mathbb{R} \),
- \( u : \mathcal{M}, g, \rightarrow \mathbb{C}^4 \),
- \( g \) is a Lorentzian metrics (signature (1,-1,-1,-1)) and the manifold has a spin structure,
- \( \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \),
- \( D_\mu \) is the covariant derivative for Dirac spinor fields.
The covariant derivatives are defined as covariant derivatives for vectors in the idea that

- they satisfy the Leibniz rule,
- they are compatible with change of variables/frames for the tangent space.

It writes

\[ D_{\mu} = \partial_{\mu} + i \omega_{\mu}^{ab} \Sigma_{ab} \]

where \( \omega_{\mu}^{ab} \) is a purely geometric factor and \( \Sigma_{ab} = -\frac{i}{8} [\gamma_a, \gamma_b] \) is a purely algebraic factor. They are the generators the matrix representation of the Lie algebra underlying \( U \), as in, at first order \( (SO_0(1,3) \) is a connex Lie group)

\[ U(\delta^a_b + \varepsilon^a_b) = Id + i \varepsilon^{ab} \Sigma_{ab} + o(\varepsilon). \]
Spin connection

The spin connection $\omega$ is defined thanks to Cartan’s formalism fixing a matrix bundle (vierbein or tetrad) $e$ such that

$$e^\mu_a \eta^{ab} e_\nu^b = g^{\mu\nu} \iff e^a_\mu \eta_{ab} e^b_\nu = g_{\mu\nu}.$$  

The spin connection must satisfy the Leibniz rule

$$d e^a + \omega^a_b \wedge e^b = 0$$

with

$$e^a = e^a_\mu dx^\mu, \quad \omega^a_b = \omega^a_{\mu b} dx^\mu.$$  

It also allows to define $\gamma^\mu = e^\mu_a \gamma^a.$
The Dirac equation is invariant under local Lorentz transform (local $SO_0(1,3)$) up to a local change of basis for $\mathbb{C}^4$.

If the vierbein changes locally by the transform $L \in SO_0(3,1)$ which it can always, the basis of $\mathbb{C}^4$ changes locally as $U(L)$.
The Dirac equation on spherically symmetric manifolds

We take \( g \) such that time and space decouple

\[
g = \begin{pmatrix} 1 & (0) \\ (0) & -h \end{pmatrix}
\]

where \( h \) is a Riemannian metric. We assume that there exists a set of coordinates such that

\[
h(r, \theta, \phi) = \begin{pmatrix} 1 \\ (0) \phi^2(r) \\ (0) \phi^2(r) \sin^2 \theta \end{pmatrix}
\]

Remarks:

- \( \phi(r) = r \) corresponds to the flat case in spherical coordinates,
- this is a special occurrence of a warped product,
- useful choice of vierbein:

\[
e_\mu^a = \begin{pmatrix} 1 \\ (0) \phi(r) \\ (0) \phi(r) \sin \theta \end{pmatrix}
\]
The equation

The equation reduces by bloc as \( i \partial_t \psi = H_\varphi \psi \) with

\[
H_\varphi = \begin{pmatrix}
    m & A \\
    A & -m
\end{pmatrix}
\]

with

\[
A = -i \sigma_1 \left( \partial_r + \frac{\varphi'(r)}{\varphi(r)} \right) + \frac{1}{\varphi} \left( -i \sigma_2 \left( \partial_\theta + \frac{\cot \theta}{2} \right) - i \frac{\sigma_3}{\sin \theta} \partial_\phi \right).
\]

It is almost the Dirac equation on the sphere \( S^2 \).
Assumptions on \( \varphi \)

General: \( \varphi \in C^\infty \), \( \varphi > 0 \) on \((0, \infty)\), scalar curvature is bounded (may be \(< 0\)).

Assumptions at \( \infty \): \( \frac{\varphi'}{\varphi} \), \( \varphi^{-1} \) are bounded on \( r > 1 \).

Assumptions at 0 : \( \varphi^{(even)}(0) = 0 \), \( \varphi'(0) = 1 \).

Consequences:

- OK for the hyperbolic space \( \varphi(r) = \text{sh}r \),
- Eq to the square \( \left( \partial^2_t + m^2 - D^i D_j + \frac{1}{4} \mathcal{R}_h \right) \psi = 0 \),
- Eq is well-posed on \( H^s \), \( s \in [0, 2] \).
Local strichartz estimates

\( H^{a,b} : a \) radial derivatives, \( b \) angular derivatives, in \( L^2(M) \).

**TH** [Cacciafesta, dS] Set \( p, q \in [2, \infty] \) and \( a, b > 0 \). Assume either

\[
\begin{aligned}
m \in \mathbb{R} & \quad \left\{ \begin{array}{l}
p > 2 \\
b > \frac{4}{p} \\
\frac{1}{p} + \frac{1}{q} = \frac{1}{2} \\
\frac{2}{pa} + \frac{2}{pb} < 1
\end{array} \right. \\
or m \neq 0 & \quad \left\{ \begin{array}{l}
b \geq \frac{3}{p} \\
\frac{2}{p} + \frac{3}{q} = \frac{3}{2} \\
\frac{1}{pa} + \frac{2}{pb} \leq 1
\end{array} \right. 
\end{aligned}
\]

Take \( I \) a bounded interval. There exists \( C \) such that for all \( \psi_0 \in H^{a,b} \) the solution \( \psi \) to the linear equation \( i \partial_t \psi = H_\varphi \psi \) with \( \text{id} \psi_0 \) satisfies

\[
\| \left( \frac{\varphi(r)}{r} \right)^{1 - 2/q} \psi \|_{L^p(I, L^q(M))} \leq C \| \psi_0 \|_{H^{a,b}}.
\]
Ideas of proof

- Littlewood-Paley theory on the sphere
- Partial wave decomposition
- Multiplier trick
- Stricharz estimates on Minkowski’s space-time with potential.
Local well-posedness

**TH [Cacciafesta, dS]** Assume $\inf \frac{\psi(r)}{r} > 0$. Let $r > 0$ and $r' = \max(r, 2)$. Set $s_1 = \frac{3}{2} - \frac{3}{r'}$. Let $a, b > s_1$ such that

$$a < 2, \quad \frac{2}{r'} \left( \frac{1}{a - s_1} + \frac{1}{b - s_1} \right) < 1, \quad b > \frac{3}{2} + \frac{1}{r'}.$$

For all $R \geq 0$ there exists $T(R) > 0$ such that for all $u_0 \in H^{a,b}$ such that $\|u_0\|_{H^{a,b}} \leq R$, the Cauchy problem

$$\begin{cases}
i \partial_t u - H\psi u = |\langle \beta u, u \rangle_{C^4}|^{r/2} u \\u_{|t=0} = u_0\end{cases}$$

has a unique solution in $C([-T, T], H^{a,b})$ and the flow hence defined is continuous in the initial datum.
Remarks

▶ $b$ is above the Sobolev threshold but $a$ can be taken as close to the critical regularity $s_c = \frac{3}{2} - \frac{1}{r'}$ as one wants,

▶ for instance, we have well-posedness for “radial” data in $H^a$, for all $a > s_c$,

▶ The Soler model is well-posed in $H^a$, $a > 1$.

▶ proof Classical+ interpolation + Strichartz
Partial bibliography

- Dirac equation : Thaller,
- QFT in curved space-time : Parker-Toms,
- Partial wave decomposition : Abrikosov, Thaller
- Strichartz estimates : Keel-Tao, Banica-Duyckaerts (also D’Ancona-Zhang)
- Littlewood-Paley theory on the sphere : Dai-Xu,
- Nonlinear models : Escobedo-Vega, Bejinaru-Herr (critical case), Bourneveas-Candy, Candy-Herr, Cacciafesta (Soler model)
Partial wave decomposition

Recall

\[ H_\varphi = \begin{pmatrix} m & A \\ A & -m \end{pmatrix} \]

with

\[ A = -i\sigma_1\left(\partial_r + \frac{\varphi'(r)}{\varphi(r)}\right) + \frac{1}{\varphi}\left(-i\sigma_2\left(\partial_\theta + \frac{\cot \theta}{2}\right) - i\frac{\sigma_3}{\sin \theta}\partial_\phi\right). \]

It is well-know in the Physics literature that

\[ \left(-i\sigma_2\left(\partial_\theta + \frac{\cot \theta}{2}\right) - i\frac{\sigma_3}{\sin \theta}\partial_\phi\right) \]

can be diagonalised. This provides a nice rewriting of the equation.
We have that \( L^2(S^2) \) can be orthogonally decomposed as

\[
L^2(S^2) = \bigoplus_{j, m_j, k_j} \mathcal{H}_{j, m_j, k_j}
\]

with \( j \in \frac{1}{2} + \mathbb{N} \), \( m_j \in \frac{1}{2} + \mathbb{Z} \) and \( -j \leq m_j \leq j \) and \( k_j = \pm (j + \frac{1}{2}) \) with the following properties:

- (sph harmonics of degree \( n \))

\[
S_n \subseteq \bigoplus_{m, k} \mathcal{H}_{n-1/2, m, k} \bigoplus_{m, k} \mathcal{H}_{n+1/2, m, k},
\]

- \( H_\varphi = \sum_{j, m_j, k_j} h_{j, m_j, k_j} \otimes \pi_{j, m_j, k_j} \)

such that \( h_{j, m_j, k_j} \) admits the following representation

\[
\begin{pmatrix}
  m & -(\partial_r + \frac{\varphi'}{\varphi}) - \frac{k_j}{\varphi} \\
  (\partial_r + \frac{\varphi'}{\varphi}) - \frac{k_j}{\varphi} & -m
\end{pmatrix}.
\]
Mutliplier trick

Taking $\sigma = \frac{r}{\varphi(r)}$ we get

$$\sigma^{-1} H_\varphi \sigma = \sum_{j,m_j,k_j} h_{j,m_j,k_j}^{\sigma} \otimes \pi_{j,m_j,k_j}$$

with

$$h_{j,m_j,k_j}^{\sigma} = \begin{pmatrix} m & -\left(\partial_r + \frac{1}{r}\right) - \frac{k_j}{\varphi} \\ \left(\partial_r + \frac{1}{r}\right) - \frac{k_j}{\varphi} & -m \end{pmatrix}$$

or in other words

$$\sigma^{-1} H_\varphi \sigma = \mathcal{D}_{\mathbb{R}^3} - \sum_{j,m_j,k_j} k_j \left(\frac{1}{\varphi} - \frac{1}{r}\right) \sigma_1 \otimes \pi_{j,m_j,k_j}.$$
Strichartz

Allows to use (local) Strichartz estimates for flat Dirac with smooth bounded potential \(-k_j \left( \frac{1}{\varphi} - \frac{1}{r} \right)\) on \(\mathcal{H}_{j,m_j,k_j}\) but not global because of \(\frac{1}{r}\) (scaling potential).

With Littlewood-Paley theory on the sphere, we get Strichartz estimates with loss of angular derivatives and then local well-posedness.

What about global Strichartz (using Morawetz’s estimates) and endpoint?
Thank you for your attention.