The Dirac equation on spherically symmetric manifolds

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The Dirac equation on flat space-time

Models the free evolution of a pair electron positron.

$$i\gamma^\mu \partial_\mu u = mu$$

with

$$u : \begin{array}{c} \mathbb{R}^{1+3} \to \mathbb{C}^4 \\ (t,x) \mapsto u(t,x). \end{array}$$

and convention $\partial_0 = \partial_t$ and $\nabla = (\partial_1, \partial_2, \partial_3)$.

The matrices γ^0 and $i\gamma^j$ are self-adjoint matrices in $\mathcal{M}_4(\mathbb{C})$ and satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

where $\eta = Diag(1, -1, -1, -1)$ is the Minkowski metrics.

Flat

It is a Schrödinger equation :

$$i\partial_t u = \gamma^0 m u - i\gamma^0\gamma^j\partial_j u = Hu$$

with H self-adjoint.

Standard choice for the γs :

$$\gamma^0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; \gamma^j = \beta\alpha^j = \beta\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$$

where the σs are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Flat

In this representation $u = \begin{pmatrix} e^- \uparrow \\ e^- \downarrow \\ e^+ \uparrow \\ e^+ \downarrow \end{pmatrix}$.

The equation is covariant, that is, it is invariant under change of special relativity referential. Those are the translations plus the linear tranformations that preserves

- the metrics,
- the orientation,
- and causality :

$SO_0(1,3)$. Change variable x into variable x' defined as $(x')^a = L^a_b x^b$, $L \in SO_0(1,3)$.

Flat

There exists a group representation $U : SO_0(1,3) \to \mathcal{U}_4(\mathbb{C})$ such that if u solves the Dirac equation then $u'(x') = U(L)u(x)$ solves the Dirac equation.

To define the Dirac equation, one thus needs to fix equivalently :

- the set of γs,
- the basis of $\mathbb{C}^4$,
- the referential.

Note that U is explicit.

Flat

To the square, we retrieve the Klein-Gordon/ wave (if $m = 0$) equation

$$\left(\partial_t^2 + m^2 - \triangle_{\mathbb{R}^3}\right)u = 0.$$

Consequences :

- Dispersive estimates
- Strichartz
- Local smoothing or Morawetz's estimates
- Local well-posedness for semilinear models.

The Dirac equation on curved space-time

Comes from QFT in curved space-time : approximation of gravitational waves or string theory at low energies.

Now, the equation writes

$$i\underline{\gamma}^{\mu} D_{\mu} u = mu$$

where

- $m \in \mathbb{R}$,
- $u : \mathcal{M}, g, \rightarrow \mathbb{C}^4$,
- g is a Lorentzian metrics (signature (1,-1,-1,-1)) and the manifold has a spin structure,
- $\{\underline{\gamma}^{\mu}, \underline{\gamma}^{\nu}\} = 2g^{\mu\nu}$,
- D_{μ} is the covariant derivative for Dirac spinor fields.

Covariant derivative

The covariant derivatives is defined as covariant derivatives for vectors in the idea that

- they satisfy the Leibniz rule,
- they are compatible with change of variables/ frames for the tangeant space.

It writes

$$D_\mu = \partial_\mu + i\omega_\mu^{ab}\Sigma_{ab}$$

where ω_μ^{ab} is a purely geometric factor and $\Sigma_{ab} = -\frac{i}{8}[\gamma_a, \gamma_b]$ is a purely algebraic factor. They are the generators the matrix representation of the Lie algebra underlying U, as in, at first order ($SO_0(1,3)$ is a connex Lie group)

$$U(\delta^a_{\ b} + \varepsilon^a_{\ b}) = Id + i\varepsilon^{ab}\Sigma_{ab} + o(\varepsilon).$$

Spin connection

The spin connection ω is defined thanks to Cartan's formalism fixing a matrix bundle (vierbein or tetrad) e such that

$$e^{\mu}_{\ a}\eta^{ab}e^{\nu}_{\ b} = g^{\mu\nu} \Leftrightarrow e_{\mu}^{\ a}\eta_{ab}e_{\nu}^{\ b} = g_{\mu\nu}.$$

The spin connection must satisfy the Leibniz rule

$$de^a + \omega^a_{\ b} \wedge e^b = 0$$

with

$$e^a = e_{\mu}^{\ a}dx^{\mu}\ ,\ \omega^a_{\ b} = \omega_{\mu\ b}^{\ a}dx^{\mu}.$$

It also allows to define $\underline{\gamma}^{\mu} = e^{\mu}_{\ a}\gamma^a$.

Covariance

The Dirac equation is invariant under local Lorentz transform (local $SO_0(1,3)$) up to a local change of basis for $\mathbb{C}^4$.

If the vierbein changes locally by the transform $L \in SO_0(3,1)$ which it can always, the basis of $\mathbb{C}^4$ changes locally as $U(L)$.

The Dirac equation on spherically sym manifolds

We take g such that time and space decouple

$$g = \begin{pmatrix} 1 & (0) \\ (0) & -h \end{pmatrix}$$

where h is a Riemannian metrics. We assume that there exists a set of coordinates such that

$$h(r,\theta,\phi) = \begin{pmatrix} 1 & & \\ (0) & \varphi^2(r) & (0) \\ & & \varphi^2(r)\sin^2\theta \end{pmatrix}.$$

Remarks:

- $\varphi(r) = r$ corresponds to the flat case in spherical coordinates,
- this is a special occurence of a warped product,
- useful choice of vierbein :

$$e_\mu^{\ a} = \begin{pmatrix} 1 & & \\ (0) & \varphi(r) & (0) \\ & & \varphi(r)\sin\theta \end{pmatrix}.$$

The equation

The equation reduces by bloc as $i\partial_t\psi = H_\varphi\psi$ with

$$H_\varphi = \begin{pmatrix} m & A \\ A & -m \end{pmatrix}$$

with

$$A = -i\sigma_1\Big(\partial_r + \frac{\varphi'(r)}{\varphi(r)}\Big) + \frac{1}{\varphi}\Big(- i\sigma_2\Big(\partial_\theta + \frac{\cot\theta}{2}\Big) - i\frac{\sigma_3}{\sin\theta}\partial_\phi\Big).$$

It is almost the Dirac equation on the sphere $\mathbb{S}^2$.

Assumptions on φ

General : $\varphi \in \mathcal{C}^\infty$, $\varphi > 0$ on $(0, \infty)$, scalar curvature is bounded (may be < 0).

Assumptions at ∞ : $\frac{\varphi'}{\varphi}$, φ^{-1} are bounded on $r > 1$.

Assumptions at 0 : $\varphi^{(even)}(0) = 0$, $\varphi'(0) = 1$.

Consequences :

- OK for the hyperbolic space $\varphi(r) = \mathrm{sh} r$,
- Eq to the square $\left(\partial_t^2 + m^2 - D^j D_j + \frac{1}{4}\mathcal{R}_h\right)\psi = 0$,
- Eq is well-posed on H^s, $s \in [0, 2]$.

Local strichartz estimates

$H^{a,b}$: a radial derivatives, b angular derivatives, in $L^2(\mathcal{M})$.

TH [Cacciafesta, dS] Set $p, q \in [2, \infty]$ and $a, b > 0$. Assume either

$$m \in \mathbb{R} \begin{cases} p > 2 \\ b > \frac{4}{p} \\ \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \\ \frac{2}{pa} + \frac{2}{pb} < 1 \end{cases} \quad \text{or } m \neq 0 \begin{cases} b \geq \frac{3}{p} \\ \frac{2}{p} + \frac{3}{q} = \frac{3}{2} \\ \frac{1}{pa} + \frac{2}{pb} \leq 1 \end{cases} .$$

Take I a bounded interval. There exists C such that for all $\psi_0 \in H^{a,b}$ the solution ψ to the linear equation $i\partial_t \psi = H_\varphi \psi$ with id ψ_0 satisfies

$$\left\| \left(\frac{\varphi(r)}{r} \right)^{1-2/q} \psi \right\|_{L^p(I, L^q(\mathcal{M}))} \leq C \|\psi_0\|_{H^{a,b}}.$$

Ideas of proof

- Littlewood-Paley theory on the sphere
- Partial wave decomposition
- Multiplier trick
- Stricharz estimates on Minkowski's space-time with potential.

Local well-posedness

TH [Cacciafesta, dS] Assume $\inf \frac{\varphi(r)}{r} > 0$. Let $r > 0$ and $r' = \max(r, 2)$. Set $s_1 = \frac{3}{2} - \frac{3}{r'}$. Let $a, b > s_1$ such that

$$a < 2 \,,\ \frac{2}{r'}\left(\frac{1}{a - s_1} + \frac{1}{b - s_1}\right) < 1 \,,\ b > \frac{3}{2} + \frac{1}{r'}.$$

For all $R \geq 0$ there exists $T(R) > 0$ such that for all $u_0 \in H^{a,b}$ such that $\|u_0\|_{H^{a,b}} \leq R$, the Cauchy problem

$$\begin{cases} i\partial_t u - H_\varphi u = |\langle \beta u, u\rangle_{\mathbb{C}^4}|^{r/2} u \\ u_{|t=0} = u_0 \end{cases}$$

has a unique solution in $\mathcal{C}([-T, T], H^{a,b})$ and the flow hence defined is continuous in the initial datum.

Remarks

- b is above the Sobolev threshold but a can be taken as close to the critical regularity $s_c = \frac{3}{2} - \frac{1}{r'}$ as one wants,
- for instance, we have well-posedness for "radial" data in H^a, for all $a > s_c$,
- The Soler model is well-posed in H^a, $a > 1$.
- proof Classical+ interpolation + Strichartz

Partial bibliography

- Dirac equation : Thaller,
- QFT in curved space-time : Parker-Toms,
- Partial wave decomposition : Abrikosov, Thaller
- Strichartz estimates : Keel-Tao, Banica-Duyckaerts (also D'Ancona-Zhang)
- Littlewood-Paley theory on the sphere : Dai-Xu,
- Nonlinear models : Escobedo-Vega, Bejinaru-Herr (critical case), Bournaveas-Candy, Candy-Herr, Cacciafesta (Soler model)

Partial wave decomposition

Recall

$$H_\varphi = \begin{pmatrix} m & A \\ A & -m \end{pmatrix}$$

with

$$A = -i\sigma_1\Big(\partial_r + \frac{\varphi'(r)}{\varphi(r)}\Big) + \frac{1}{\varphi}\Big(- i\sigma_2\Big(\partial_\theta + \frac{\cot\theta}{2}\Big) - i\frac{\sigma_3}{\sin\theta}\partial_\phi\Big).$$

It is well-know in the Physics litterature that

$$\Big(- i\sigma_2\Big(\partial_\theta + \frac{\cot\theta}{2}\Big) - i\frac{\sigma_3}{\sin\theta}\partial_\phi\Big)$$

can be diagonalised. This provides a nice rewriting of the equation.

PWD

We have that $L^2(\mathbb{S}^2)$ can be orthogonally decomposed as

$$L^2(\mathbb{S}^2) = \bigoplus_{j,m_j,k_j} \mathcal{H}_{j,m_j,k_j}$$

with $j \in \frac{1}{2} + \mathbb{N}$, $m_j \in \frac{1}{2} + \mathbb{Z}$ and $-j \le m_j \le j$ and $k_j = \pm(j + \frac{1}{2})$ with the following properties :

- (sph harmonics of degree n) $\mathcal{S}_n \subseteq \oplus_{m,k} \mathcal{H}_{n-1/2,m,k} \oplus_{m,k} \mathcal{H}_{n+1/2,m,k}$,
- $$H_\varphi = \sum_{j,m_j,k_j} h_{j,m_j,k_j} \otimes \pi_{j,m_j,k_j}$$

  such that h_{j,m_j,k_j} admits the following representation

$$\begin{pmatrix} m & -\left(\partial_r + \frac{\varphi'}{\varphi}\right) - \frac{k_j}{\varphi} \\ \left(\partial_r + \frac{\varphi'}{\varphi}\right) - \frac{k_j}{\varphi} & -m \end{pmatrix}.$$

Mutliplier trick

Taking $\sigma = \frac{r}{\varphi(r)}$ we get

$$\sigma^{-1} H_\varphi \sigma = \sum_{j,m_j,k_j} h^\sigma_{j,m_j,k_j} \otimes \pi_{j,m_j,k_j}$$

with

$$h^\sigma_{j,m_j,k_j} = \begin{pmatrix} m & -\left(\partial_r + \frac{1}{r}\right) - \frac{k_j}{\varphi} \\ \left(\partial_r + \frac{1}{r}\right) - \frac{k_j}{\varphi} & -m \end{pmatrix}$$

or in other words

$$\sigma^{-1} H_\varphi \sigma = \mathcal{D}_{\mathbb{R}^3} - \sum_{j,m_j,k_j} k_j \left(\frac{1}{\varphi} - \frac{1}{r}\right) \sigma_1 \otimes \pi_{j,m_j,k_j}.$$

Strichartz

Allows to use (local) Strichartz estimates for flat Dirac with smooth bounded potential $-k_j\left(\frac{1}{\varphi} - \frac{1}{r}\right)$ on $\mathcal{H}_{j,m_j,k_j}$ but not global because of $\frac{1}{r}$ (scaling potential).

With Littlewood-Paley theory on the sphere, we get Strichartz estimates with loss of angular derivatives and then local well-posedness.

What about global Strichartz (using Morawetz's estimates) and endpoint?

Thank you for your attention.