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Global Existence of Weak Solutions for the Anisotropic Compressible Stokes System

D. Bresch[*], C. Burtea [†]

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Dedicated to the memory of Geneviève Raugel

Abstract

In this paper, we study the problem of global existence of weak solutions for the quasi-stationary compressible Stokes equations with an anisotropic viscous tensor. The key element of our proof is the control of a particular defect measure associated to the pressure which avoids the use of the effective flux. Using this new tool, we solve an open problem namely global existence of solutions à la Leray for such a system without assuming any restriction on the anisotropy amplitude. It provides a flexible and natural way to treat compressible quasilinear Stokes systems which are important for instance in biology, porous media, supra-conductivity or other applications in the low Reynolds number regime.

Keywords: Compressible Quasi-Stationary Stokes Equations, Anisotropic Viscous Tensor, Global Weak Solutions.

MSC: 35Q35, 35B25, 76T20.

1 Introduction

As explained in [13] or [16], there are various motivations for the study of quasi-stationary Stokes problem. Such study may be used to try to understand how to build solutions of the compressible Navier-Stokes system which exhibit persistence oscillations. The second motivation is that such system may be used for instance in porous media, biology or concerning the dynamics of vortices in supra-conductivity for instance which occurs in the low Reynolds number regime. Such system has been study from a long-time ago for constant isotropic viscosities: see for instance [10], [11], [12], [18], [15], [4] for constant viscosities and see for instance [1] for density dependent viscosities. More complicated quasi-stationary compressible Stokes system has been also studied in [5], [6], [9] and [8] in the multi-fluid setting for instance. Global existence of weak solutions for general anisotropic viscosities for non-stationary compressible barotropic Navier-Stokes equations or even quasi-stationary Stokes equations are open problems. Only recently a positive result has been obtained by B.D. and P.–E. Jabin in [3] assuming some restrictions on the shear and bulk viscosities. The result is not straightforward to prove as it involves a non-local behavior in the compactness characterization explaining in some sense the main tool introduced by the authors: a non-local compactness criterion with the introduction of appropriate weights. In this paper, we solve the open problem for the quasi-stationary compressible Stokes equations with an anisotropic viscous tensor namely for the following system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ -\operatorname{div}\, \tau + \nabla \rho^\gamma = \nabla f. \end{cases} \tag{1.0.1}$$

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where u is the velocity field and ρ is the density and where the strain tensor τ is as follows

$$\tau_{ij}(t,x,D(u)) = A_{ijkl}(t,x)[D(u)]_{kl} \tag{1.0.2}$$

where $D(u) = (\nabla u + {}^t\nabla u)/2$ with

$$A_{ijkl} = A_{ijkl}(t,x) \in W^{1,\infty}((0,T)\times\mathbb{T}^3). \tag{1.0.3}$$

We assume the following extra hypothesis on the strain tensor τ:

$$\bullet\tau(t,x,D(u)) : \nabla u = \tau(t,x,D(u)) : D(u) \tag{1.0.4}$$

$$\bullet D(u) \longmapsto \tau(t,x,D(u)) : D(u) \text{ to be weakly lower semi-continuous} \tag{1.0.5}$$

•There exists $c > 0$ such that

$$E = \int_{\mathbb{T}^3} \tau(t,x,D(u)) : \nabla u \geq c \int_{\mathbb{T}^3} |D(u)|^2 \tag{1.0.6}$$

• The application $\mathcal{A} : v \mapsto -\mathrm{div}\,\tau(t,x,D(v))$
is a second order invertible elliptic operator
such that $\mathcal{A}^{-1}\nabla\mathrm{div}$ is a bounded operator from $L^{\frac{3}{2}-\delta}\left(\mathbb{T}^3\right)$ into $L^{\frac{3}{2}-\delta}\left(\mathbb{T}^3\right)$ for some (1.0.7)
$\delta \in (0,1/2)$. (1.0.8)

We consider the construction of solutions for system (1.0.1) with initial data

$$\rho|_{t=0} = \rho_0 \geq 0. \tag{1.0.9}$$

We present a simple proof for the existence and the weak stability of solutions that consists in introducing a particular defect measure for the pressure which allows to control the oscillation of an approximating sequence of solutions of system (1.0.1)–(1.0.9). The advantage of our method is that we are able to control this defect measure without using the effective flux (presented in the next section on a simple example). To the authors knowledge, this fact is completely new and a mathematical justification of this formal calculation (see subsection 3.3) allows for the first time to get the global existence of weak solutions for compressible quasi-stationary anisotropic Stokes systems. More precisely, we get the following result

Theorem 1. *Let us assume that $f \in H^1((0,T);L^2\left(\mathbb{T}^3\right))$ and the initial data ρ_0 satisfies the bound*

$$\rho_0 \geq 0, \qquad 0 < M_0 = \int_{\mathbb{T}^3} \rho_0 < +\infty, \qquad E_0 = \int_{\mathbb{T}^3} \rho_0^\gamma\,dx < +\infty$$

where $\gamma > 1$ and the strain tensor τ given by (1.0.2) *satisfies* (1.0.4)–(1.0.8). *Then there exists a global weak solution (ρ,u) of the compressible system* (1.0.1) *and* (1.0.9) *with*

$$\rho \in \mathcal{C}([0,T];L^\gamma_{weak}(\mathbb{T}^3)) \cap L^{2\gamma}((0,T)\times\mathbb{T}^3), \quad u \in L^2(0,T;H^1(\mathbb{T}^3) \text{ with } \int_{\mathbb{T}^3} u = 0.$$

A similar result can be obtained for the case of a bounded domain with Dirichlet boundary condition: we have chosen periodic boundary conditions to simplify the presentation. It seems difficult to adapt the method to the non-stationary Navier-Stokes equations in a simple manner: This will be the subject of the forthcoming paper [2]. Note that actually only one result exists for the non-stationary Navier-Stokes equations (see [3]) with an assumption on the coefficients compared to the isotropic case.

We remark however that at least the weak-stability part of our result can be adapted without to much effort to treat the following stationary system

$$\begin{cases} \alpha\rho + \mathrm{div}\,(\rho u) = f, \\ \beta\rho u + \mathrm{div}\,(\rho u \otimes u) - \mathrm{div}\,\tau + \nabla\rho^\gamma = g, \end{cases}$$

where $\alpha, \beta > 0$ and τ is as above. This later system can be viewed as an implicit time discretization of the Navier-Stokes system.

In the sequel, the first part is dedicated to present our new defect measure on the pressure and to show how it is possible to control it if initially it is the case without using the effective flux. Our result uses in a crucial manner compactness properties on the velocity field in $L^2((0,T)\times\mathbb{T}^3)$. For the readers's convenience, we recall the classical method to control defect measures that has been developed by P.–L. Lions and E. Feireisl-A. Novotny-H.Petzeltova and explain why the anisotropic case seems to fall completely out of such strategy as explained in [3] for instance. In the second part we present the derivation of the basic energy estimates and extra control on the density (w.r.t. to the basic energy functional) which are needed in order to justify the weak stability properties of sequences of solutions. In the third part we construct approximate solutions satisfying these estimates uniformly in ε.

2 New approach to control defect measure related to the pressure

To be understandable for the reader, let us present formally on a simple example why the classical approach to control defect measures fails to apply in the case of anisotropic viscosities and how our new way to proceed provides a flexible method for Stokes type systems. More precisely, let us consider $(\rho^\varepsilon, u^\varepsilon)$ a sequence of solutions for the following system.

$$\left\{\begin{array}{l} \partial_t\rho^\varepsilon + \mathrm{div}\,(\rho^\varepsilon u^\varepsilon) = 0, \\ -\Delta_\nu u^\varepsilon + \nabla((\rho^\varepsilon)^\gamma) = \nabla f \end{array}\right. \tag{2.0.1}$$

where

$$\Delta_\nu = \nu_x\partial_x^2 + \nu_z\partial_y^2 + \nu_z\partial_z^2$$

with $\nu_x, \nu_y, \nu_z > 0$ which may be different. Assume

$$\|u^\varepsilon\|_{L^2(0,T;H^1(\mathbb{T}^3))} + \|\rho^\varepsilon\|_{L^{2\gamma}((0,T)\times\mathbb{T}^3)} + \|\rho^\varepsilon\|_{L^\infty(0,T;L^\gamma(\mathbb{T}^3))} \le C < +\infty$$

where C does not depend on ε weak solutions of (2.0.1) and assume that

$$\{u^\varepsilon\}_\varepsilon \text{ is compact in } L^2((0,T)\times\mathbb{T}^3).$$

We denote (ρ, u) the weak limit and, using classical functional analysis arguments it is not hard to see that we have

$$\left\{\begin{array}{l} \partial_t\rho + \mathrm{div}\,(\rho u) = 0, \\ -\Delta_\nu u + \nabla(\overline{\rho^\gamma}) = \nabla f. \end{array}\right. \tag{2.0.2}$$

for some function $\overline{\rho^\gamma} \in L^2((0,T)\times\mathbb{T}^3)$. Of course, the main difficulty is to prove that $\overline{\rho^\gamma} = \rho^\gamma$ and therefore to be able to characterize the possible defect measures.

Remark 2. *Throughout the paper we denote the weak limit of a sequence $(a^\varepsilon)_{\varepsilon>0}$ by $\bar{a}$.*

Classical approach to control defect measures. As mentioned in [3], the usual method for isotropic viscosities (namely $\nu_x = \nu_y = \nu_z = \nu$) is based on the careful analysis of the defect measures

$$\mathrm{dft}[\rho^\varepsilon - \rho](t) = \int_{\mathbb{T}^3} (\overline{\rho\log\rho})(t) - \rho\log\rho(t))\,dx.$$

More precisely, we can write the two equations

$$\partial_t(\rho\log\rho) + \mathrm{div}(\rho\log\rho u) + \rho\mathrm{div}u = 0 \tag{2.0.3}$$

and

$$\partial_t(\overline{\rho\log\rho}) + \mathrm{div}(\overline{\rho\log\rho} u) + \overline{\rho\mathrm{div}u} = 0 \tag{2.0.4}$$

Note that if $\rho \in L^2((0,T)\times\mathbb{T}^3)$ then using the uniform bound on $u \in L^2(0,T;H^1(\mathbb{T}^3))$, we have $\rho\,\mathrm{div}u \in L^1((0,T)\times\mathbb{T}^3)$ and therefore the third quantity is well defined. At this level comes the so called effective flux comes into play. More precisely, Lions [14] in $'93$ (see also D. Serre [19] for the $1d$ case) observes that the following quantity

$$F^\varepsilon = p(\rho^\varepsilon) - \nu \mathrm{div} u^\varepsilon$$

enjoys the following compactness property:

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{T}^3}(p(\rho^\varepsilon)-\nu\mathrm{div}u^\varepsilon)b(\rho^\varepsilon)\varphi = \int_0^T\int_{\mathbb{T}^3}(\overline{p(\rho)}-\nu\mathrm{div}u)\overline{b(\rho)}\varphi. \tag{2.0.5}$$

This is important at it provides a way to express $\overline{\rho\mathrm{div}u}$ in terms of $\rho\mathrm{div}u$ and an extra term which is signed. Substracting the two equations (2.0.3) and (2.0.4) and using the important property of the effective flux (2.0.5), one gets that

$$\partial_t(\overline{\rho\log\rho}-\rho\log\rho)+\mathrm{div}((\overline{\rho\log\rho}-\rho\log\rho)u) = \frac{1}{\nu}(\overline{p(\rho)}\rho - \overline{p(\rho)\rho})$$

and using the monotonicity of the pressure, one may deduce that

$$\mathrm{dft}[\rho^\varepsilon-\rho](t)\le \mathrm{dft}[\rho^\varepsilon-\rho](0).$$

On the other hand, the strict convexity of the function $s\mapsto s\log s$ with $s\ge 0$ implies that $\mathrm{dft}[\rho^\varepsilon-\rho](t)\ge 0$. If initially this quantity vanishes, it then vanishes at every time. The commutation of the weak convergence with a strictly convex function yields compactness of $\{\rho^\varepsilon\}_\varepsilon$ in $L^1((0,T)\times\mathbb{T}^3)$.

Assuming anisotropic viscosities $\nu_x=\nu_y\ne\nu_z$, the effective flux property reads

$$\overline{\rho\mathrm{div}u}-\rho\mathrm{div}u = \frac{1}{\nu_x}[\rho\overline{A_\nu\rho^\gamma}-\overline{\rho A_\nu\rho^\gamma}]$$

with some non-local anisotropic operator $A_\nu=(\Delta-(\mu_z-\mu_x)\partial_z^2)^{-1}\partial_z^2$ where Δ is the total Laplacian in terms of (X,z) with variables $X=(x,y)$ and z. Unfortunately, we are loosing the structure and in particular the sign of the right-hand side. This explains why the anisotropic case seems to fall completely out the theory developed by P.–L. Lions [13] and E. Feireisl, A. Novotny and H. Petzeltova [7]. The first positive answer has been given by D. Bresch and P.-E. Jabin in [3] for the compressible Navier-Stokes equations developing an other way to characterize compactness in space on the density: it involves a non-local compactness criterion with the introduction of appropriate weights. It allows them to obtain a positive answer assuming the viscosity coefficient ν_x,ν_y, ν_z to be close enough.

New approach to control defect measures in the Stokes regime. Our new approach is based on the careful analysis of the defect measures

$$\mathrm{dft}[\rho^\varepsilon-\rho](t)=\int_{\mathbb{T}^3}\Big((\overline{\rho^\gamma})(t)-\rho^\gamma(t)\Big)^{1/\gamma}dx.$$

The main idea here is to write the equation related to the energy which will not use the effective flux expression but is related to the viscous dissipation in the Stokes regime. More precisely, let us observe that the pressure verifies the following equation :

$$\partial_t(\rho^\varepsilon)^\gamma+\mathrm{div}\left((\rho^\varepsilon)^\gamma u^\varepsilon\right)+(\gamma-1)(\rho^\varepsilon)^\gamma\,\mathrm{div}\,u^\varepsilon=0$$

which rewrites

$$\partial_t(\rho^\varepsilon)^\gamma+\gamma\,\mathrm{div}\left((\rho^\varepsilon)^\gamma u\right)-(\gamma-1)\,u^\varepsilon\nabla(\rho^\varepsilon)^\gamma=0.$$

We observe that with the aid of the second equation of (2.0.1) we may write that

$$\partial_t(\rho^\varepsilon)^\gamma+\gamma\,\mathrm{div}\left((\rho^\varepsilon)^\gamma u\right)-(\gamma-1)\,u^\varepsilon\Delta_\nu u^\varepsilon=(\gamma-1)u^\varepsilon\nabla f$$

which can be put under the following form

$$\partial_t (\rho^\varepsilon)^\gamma + \gamma \operatorname{div}((\rho^\varepsilon)^\gamma u) - (\gamma-1)\Delta_\nu \left(\frac{|u^\varepsilon|^2}{2}\right)$$
$$= -(\gamma-1)\nabla_{\nu^{\frac12}} u^\varepsilon : \nabla_{\nu^{\frac12}} u^\varepsilon + (\gamma-1) u^\varepsilon \nabla f, \tag{2.0.6}$$

where we use the notation

$$\nabla_{\nu^{\frac12}} = \left(\nu_1^{\frac12}\partial_1, \nu_2^{\frac12}\partial_2, \nu_3^{\frac12}\partial_3\right).$$

Of course, we used that

$$\partial_{jj} u_i u_i = \partial_{jj}\left(\frac{(u_i)^2}{2}\right) - (\partial_j u_i)^2$$

Assuming that

$$(u^\varepsilon)_{\varepsilon>0} \text{ is compact in } L^2((0,T)\times\mathbb{T}^3)$$

by passing to the limit in (2.0.6) we obtain that

$$\partial_t \overline{\rho^\gamma} + \gamma \operatorname{div}(\overline{\rho^\gamma} u) - (\gamma-1)\Delta_\nu \left(\frac{|u|^2}{2}\right)$$
$$= -(\gamma-1)\overline{\nabla_{\nu^{\frac12}} u : \nabla_{\nu^{\frac12}} u} + (\gamma-1)\left(\operatorname{div}(f u^\varepsilon) - f \operatorname{div} u^\varepsilon\right). \tag{2.0.7}$$

In the following we will apply the same recipe to the limiting function (ρ, u). Indeed, from (2.0.2) one can deduce that

$$\begin{aligned} 0 &= \partial_t \rho^\gamma + \gamma \operatorname{div}(\rho^\gamma u) - (\gamma-1) u\cdot\nabla\rho^\gamma \\ &= \partial_t \rho^\gamma + \gamma \operatorname{div}(\rho^\gamma u) - (\gamma-1) u\cdot\nabla(\rho^\gamma - \overline{\rho^\gamma}) - (\gamma-1) u\cdot\nabla\overline{\rho^\gamma} \\ &= \partial_t \rho^\gamma + \gamma \operatorname{div}(\rho^\gamma u) - (\gamma-1) u\cdot\nabla(\rho^\gamma - \overline{\rho^\gamma}) - (\gamma-1) u\cdot(\Delta_\nu u + \nabla f) \end{aligned}$$

which rewrites

$$\partial_t \rho^\gamma + \gamma \operatorname{div}(\rho^\gamma u) - (\gamma-1) u\nabla(\rho^\gamma - \overline{\rho^\gamma}) - (\gamma-1)\Delta_\nu\left(\frac{|u|^2}{2}\right)$$
$$= -(\gamma-1)\nabla_{\nu^{\frac12}} u : \nabla_{\nu^{\frac12}} u + (\gamma-1)\left(\operatorname{div}(f u) - f \operatorname{div} u\right). \tag{2.0.8}$$

Let us consider the difference between (2.0.7) and (2.0.8) in order to write that

$$\partial_t(\overline{\rho^\gamma} - \rho^\gamma) + \gamma \operatorname{div}((\overline{\rho^\gamma} - \rho^\gamma) u) - (\gamma-1) u\nabla(\overline{\rho^\gamma} - \rho^\gamma)$$
$$= -(\gamma-1)\left(\overline{\nabla_{\nu^{\frac12}} u : \nabla_{\nu^{\frac12}} u} - \nabla_{\nu^{\frac12}} u : \nabla_{\nu^{\frac12}} u\right).$$

which we put under the form

$$\partial_t(\overline{\rho^\gamma} - \rho^\gamma) + \operatorname{div}((\overline{\rho^\gamma} - \rho^\gamma) u) + (\gamma-1)(\overline{\rho^\gamma} - \rho^\gamma)\operatorname{div} u \tag{2.0.9}$$
$$= -(\gamma-1)\left(\overline{\nabla_{\nu^{\frac12}} u : \nabla_{\nu^{\frac12}} u} - \nabla_{\nu^{\frac12}} u : \nabla_{\nu^{\frac12}} u\right).$$

At this point we observe that owing to the convexity of the pressure function, we have that

$$\overline{\rho^\gamma} \geq \rho^\gamma \text{ a.e.}$$

and

$$\overline{\nabla_{\nu^{\frac12}} u : \nabla_{\nu^{\frac12}} u} - \nabla_{\nu^{\frac12}} u : \nabla_{\nu^{\frac12}} u \geq 0 \tag{2.0.10}$$

at least in the sense of measures. By multiplying (2.0.9) with $\frac{1}{\gamma}(\overline{\rho^\gamma} - \rho^\gamma)^{\frac{1}{\gamma}-1}$ we get that

$$\partial_t(\overline{\rho^\gamma} - \rho^\gamma)^{\frac1\gamma} + \operatorname{div}\left((\overline{\rho^\gamma} - \rho^\gamma)^{\frac1\gamma} u\right) \leq 0$$

such that by integration and using (2.0.10) we end up with

$$\int_0^T \int (\overline{\rho^\gamma} - \rho^\gamma)^{\frac{1}{\gamma}} \le T \int (\overline{\rho^\gamma} - \rho^\gamma)^{\frac{1}{\gamma}}_{|t=0}\,.$$

Therefore if we have compactness initially, we get compactness on ρ_ε. Of course all the previous formal calculations have to be justified because of the weak regularity and of possible vanishing quantity: this will be the subject of Subsection 3.3.

Remark. It interesting to note that our new approach to get characterization of the defect measure on the pressure sequence is related to the energy equation and strongly uses the energy dissipation. We speculate that it has a physical meaning in some sense.

3 Weak stability of sequences of global weak solutions

3.1 Classical functional analysis tools

This section is devoted to a quick recall of the main results from functional analysis that we need in order to justify the computations done above. First, we introduce a new function

$$g_\varepsilon = g * \omega_\varepsilon(x) \qquad \text{with} \qquad \omega_\varepsilon = \frac{1}{\varepsilon^d}\omega(\frac{x}{\varepsilon}) \tag{3.1.1}$$

with ω a smooth nonnegative even function compactly supported in the space ball of radius 1 and with integral equal to 1. We recall the following classical analysis result

$$\lim_{\varepsilon\to 0} \|g_\varepsilon - g\|_{L^q(0,T;L^p(\mathbb{T}^3))} = 0.$$

Next let us

Proposition 3. *Consider $\beta \in (1,\infty)$ and (a,b) such that $a \in L^\beta\left((0,T)\times\mathbb{T}^3\right)$ and $b, \nabla b \in L^p\left((0,T)\times\mathbb{T}^3\right)$ where $\frac{1}{s} = \frac{1}{\beta} + \frac{1}{p} \le 1$. Then, we have*

$$\lim r_\varepsilon(a,b) = 0 \text{ in } L^s\left((0,T)\times\mathbb{T}^3\right)$$

where

$$r_\varepsilon(a,b) = \partial_t(a_\varepsilon b) - \partial_i\left((ab)_\varepsilon\right). \tag{3.1.2}$$

Next, we state the following

Proposition 4. *Consider $2 \le \beta < \infty$ and λ_0, λ_1 such that $\lambda_0 < 1$ and $-1 \le \lambda_1 \le \beta/2 - 1$. Also, consider $\rho \in L^\beta\left((0,T)\times\mathbb{T}^3\right)$, $\rho \ge 0$ a.e. and $u, \nabla u \in L^2\left((0,T)\times\mathbb{T}^3\right)$ verifying the following transport equation*

$$\partial_t\rho + \operatorname{div}(\rho u) = 0$$

in the sense of distributions. Then, for any function $b \in C^0([0,\infty)) \cap C^1((0,\infty))$ such that

$$\begin{cases} b'(t) \le ct^{-\lambda_0} \text{ for } t \in (0,1], \\ |b'(t)| \le ct^{\lambda_1} \text{ for } t \ge 1 \end{cases}$$

it holds that

$$\partial_t b(\rho) + \operatorname{div}(b(\rho)u) + \left\{\rho b'(\rho) - b(\rho)\right\}\operatorname{div} u = 0 \tag{3.1.3}$$

in the sense of distributions.

The proof of the above results follow by adapting in a straightforward manner lemmas 6.7. and 6.9 from the book of Novotny-Straškraba [16] pages $304 - 308$.

3.2 A priori estimates

In this section we recall the basic *apriori* estimates for (regular) solutions for system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ -\operatorname{div}\tau + \nabla \rho^\gamma = \nabla f \end{cases} \tag{3.2.1}$$

where $\tau_{ij} = A_{ijkl}(t,x) D_{kl}(u)$. First, of course we have the mass conservation:

$$\int_{\mathbb{T}^3} \rho(t) = \int_{\mathbb{T}^3} \rho|_{t=0} = \int_{\mathbb{T}^3} \rho_0, \tag{3.2.2}$$

for all $t > 0$ which follows by integrating the first equation of (3.2.1). Next, by multiplying the velocity equation with u and integrating in space and time we get that

$$\int_{\mathbb{T}^3} \rho^\gamma(t) + \int_0^t \int_{\mathbb{T}^3} \tau : \nabla u \leq \int_{\mathbb{T}^3} \rho_0^\gamma. \tag{3.2.3}$$

The coercivity hypothesis (1.0.8)

$$\int_{\mathbb{T}^3} \tau : \nabla u \geq c \int_{\mathbb{T}^3} |D(u)|^2$$

with $c > 0$, the zero mean value on u, the Körn inequality and Sobolev embedding allows us to conclude that

$$u \in L^2(0,T; H^1(\mathbb{T}^3))$$

Next, following an idea of Lions 's [13] we can get some extra-integrability for the density. Indeed, let us remark that

$$\rho^\gamma - \int_{\mathbb{T}^3} \rho^\gamma = f - \int_{\mathbb{T}^3} f + \Delta^{-1} \operatorname{div}\operatorname{div}\tau$$

and thus, we see that if $f \in L^2\left((0,T)\times\mathbb{T}^3\right)$ and assuming $A(t,x) \in W^{1,\infty}((0,T)\times\mathbb{T}^3))^{3\times 3}$, we get that

$$\rho^\gamma \in L^2\left((0,T)\times\mathbb{T}^3\right). \tag{3.2.4}$$

3.2.1 Estimate for the time derivative of the velocity

We can recover time regularity for u by proceeding in the following way. We write that

$$\begin{aligned} \mathcal{A}\partial_t u &= \operatorname{div}(\partial_t A(t,x) D(u)) - \nabla \partial_t f + \nabla \partial_t \rho^\gamma \\ &= \operatorname{div}(\partial_t A(t,x) D(u)) - \nabla \partial_t f \\ &\quad + \nabla \operatorname{div}\left(\rho^\gamma u - \int \rho^\gamma u\right) + (\gamma - 1)\nabla\left(\rho^\gamma \operatorname{div} u - \int \rho^\gamma \operatorname{div} u\right). \end{aligned}$$

Let us consider ϕ with $\int_{\mathbb{T}^3} \phi = 0$, such that

$$\mathcal{A}\phi = \operatorname{div}(\partial_t A(t,x) D(u)) - \nabla \partial_t f$$

Multiplying by ϕ we get that

$$\begin{aligned} c\int_0^t \int_{\mathbb{T}^3} |\nabla \phi|^2 \leq \int_0^t \int_{\mathbb{T}^3} \phi \mathcal{A}\phi &= \int_0^t \int_{\mathbb{T}^3} \left(\partial_t A(t,x) D(u) - \partial_t f\right) \nabla\phi \\ &\leq \frac{1}{8c}\int_0^t \int_{\mathbb{T}^3} |\partial_t A(t,x) D(u)|^2 + \frac{1}{8c}\int_0^t \int_{\mathbb{T}^3} (\partial_t f)^2 + \frac{c}{2}\int_0^t \int_{\mathbb{T}^3} |\nabla\phi|^2 \end{aligned}$$

and thus, we get that

$$\frac{c}{2}\int_0^t \int_{\mathbb{T}^3} |\nabla\phi|^2 \leq \frac{1}{8c}\int_0^t \int_{\mathbb{T}^3} |\partial_t A(t,x) D(u)|^2 + \frac{1}{8c}\int_0^t \int_{\mathbb{T}^3} (\partial_t f)^2. \tag{3.2.5}$$

Next, we see that, owing to the linearity of the operator $\mathcal{A}$ we may write that

$$\mathcal{A}\left(\partial_t u-\phi\right)=\nabla\operatorname{div}\left(\rho^\gamma u-\int\rho^\gamma u\right)+(\gamma-1)\nabla\left(\rho^\gamma\operatorname{div}u-\int\rho^\gamma\operatorname{div}u\right).$$

We will use a periodic variant of the following result due to Stampacchia and for more general second order elliptic equation to Boccardo-Gallouët that can be found for instance in [17] Proposition 5.1. page 77. Let ψ solution of

$$\begin{cases}\Delta\psi=f,\\ \psi_{|\partial\Omega}=0\end{cases}$$

where $f\in L^1(\Omega)$ with Ω a smooth bounded domain then we have that

$$\left\|\nabla\psi\right\|_{W^{1,\frac{3}{2}-}(\Omega)}\leq C_\delta\left\|f\right\|_{L^1(\Omega)}. \tag{3.2.6}$$

The periodic version reads as follows: Let ψ a solution of

$$\Delta\psi=f\text{ with }f\in L^1(\mathbb{T}^3)\text{ and }\int_{\mathbb{T}^3}f=0$$

then (3.2.6) is satisfied. As $\rho^\gamma\operatorname{div}u\in L^1((0,T)\times\mathbb{T}^3)$, let us consider ψ the solution of

$$\Delta\psi\left(\rho,u\right)=\rho^\gamma\operatorname{div}u$$

which verifies that

$$\left\|\nabla\psi\left(\rho,u\right)\right\|_{L^1(0,T:L^{\frac{3}{2}-}(\mathbb{T}^3))}\leq C_\delta\left\|\rho^\gamma\operatorname{div}u\right\|_{L^1(0,T;L^1(\mathbb{T}^3))}\leq C_\delta\left\|\rho^\gamma\right\|_{L^2((0,T)\times\mathbb{T}^3)}\left\|\operatorname{div}u\right\|_{L^2((0,T)\times\mathbb{T}^3)}.$$

But then, we may write that

$$\begin{aligned}\mathcal{A}\left(\partial_t u-\phi\right)&=\nabla\operatorname{div}\left(\rho^\gamma u\right)+(\gamma-1)\nabla\left(\rho^\gamma\operatorname{div}u\right)\\&=\nabla\operatorname{div}\left(\rho^\gamma u\right)+(\gamma-1)\nabla\operatorname{div}\nabla\psi\left(\rho,u\right)\end{aligned}$$

and using hypothesis (1.0.8) we arrive at

$$\begin{aligned}\left\|(\partial_t u-\phi)\right\|_{L^1(0,T;L^{\frac{3}{2}-}(\mathbb{T}^3))}&\leq\left\|\rho^\gamma u-\int_{\mathbb{T}^3}\rho^\gamma u\right\|_{L^1(0,T;L^{\frac{3}{2}}(\mathbb{T}^3))}+\left\|\nabla\psi\left(\rho,u\right)\right\|_{L^1(0,T;L^{\frac{3}{2}-}(\mathbb{T}^3))}\\&\leq\left\|\rho^\gamma\right\|_{L^2((0,T)\times\mathbb{T}^3)}\left\|u\right\|_{L^2(0,T;L^6(\mathbb{T}^3))}+\left\|\rho^\gamma\right\|_{L^2((0,T)\times\mathbb{T}^3)}\left\|\operatorname{div}u\right\|_{L^2((0,T)\times\mathbb{T}^3)}\\&\leq\left\|\rho^\gamma\right\|_{L^2((0,T)\times\mathbb{T}^3)}\left\|\nabla u\right\|_{L^2((0,T)\times\mathbb{T}^3)}.\end{aligned} \tag{3.2.7}$$

We get a uniform bound for $\partial_t u$ in $L^1\left(0,T;L^{3/2-}(\mathbb{T}^3)\right)$ by combining estimates (3.2.5) and (3.2.7) in the following manner

$$\begin{aligned}\left\|\partial_t u\right\|_{L^1(0,T;L^{\frac{3}{2}-}(\mathbb{T}^3))}&\leq\left\|(\partial_t u-\phi)\right\|_{L^1(0,T;L^{\frac{3}{2}-}(\mathbb{T}^3))}+\left\|\phi\right\|_{L^1(0,T;L^{\frac{3}{2}-}(\mathbb{T}^3))}\\&\leq\left\|\rho^\gamma\right\|_{L^2((0,T)\times\mathbb{T}^3)}\left\|\nabla u\right\|_{L^2((0,T)\times\mathbb{T}^3)}+\sqrt{t}\left\|\phi\right\|_{L^2(0,T;L^6(\mathbb{T}^3))}\\&\leq\left\|\rho^\gamma\right\|_{L^2((0,T)\times\mathbb{T}^3)}\left\|\nabla u\right\|_{L^2((0,T)\times\mathbb{T}^3)}+\sqrt{t}\left\|\nabla\phi\right\|_{L^2((0,T)\times\mathbb{T}^3)}\\&\leq\left(\left\|\rho^\gamma\right\|_{L^2((0,T)\times\mathbb{T}^3)}+\sqrt{t}\left\|\partial_t A\right\|_{L^\infty((0,T)\times\mathbb{T}^3)}\right)\left\|\nabla u\right\|_{L^2((0,T)\times\mathbb{T}^3)}+\sqrt{t}\left\|\partial_t f\right\|_{L^2((0,T)\times\mathbb{T}^3)}.\end{aligned} \tag{3.2.8}$$

Also, for later purposes it is convenient to observe that we actually proved that if

$$\mathcal{A}u=\operatorname{div}F$$

then Hypothesis (1.0.8) made on the operator $\mathcal{A}$ implies that

$$\left\|\nabla u\right\|_{L^{\frac{3}{2}-}(\mathbb{T}^3)}\leq\left\|F\right\|_{L^1(\mathbb{T}^3)}. \tag{3.2.9}$$

Of course combining this information with the energy inequality (3.2.3) we obtain an uniform bound for

$$u\in L^2(0,T;H^1(\mathbb{T}^3))\cap W^{1,1}(0,T;L^{3/2-}(\mathbb{T}^3)).$$

Remark 5. *The previous estimates are not all available in the case of the full compressible Navier-Stokes system. For instance we do not have control on the time derivative of the velocity and ρ^γ is not square integrable. We control only $\partial_t(\rho u)$ in $L^1(0,T;H^{-1}(\mathbb{T}^3))$ allowing to get compactness on $\sqrt{\rho}u$ in $L^2((0,T)\times\mathbb{T}^3))$ and we gain extra integrability $\rho^{\gamma+\theta}\in L^1((0,T)\times\mathbb{T}^3)$ for $0<\theta<2\gamma/3-1$.*

3.3 Weak stability of solutions of (2.0.1)

The aim of this section is to provide the arguments that make rigorous the computations presented in the introduction.

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ -\operatorname{div}\tau + \nabla\rho^\gamma = \nabla f. \end{cases} \tag{3.3.1}$$

As we saw in Section 3.1 under certain integrability conditions one may conclude that ρ^γ verifies the following equation :

$$\partial_t\rho^\gamma + \operatorname{div}(\rho^\gamma u) + (\gamma-1)\,\rho^\gamma \operatorname{div} u = 0.$$

Of course, the result of Proposition 4 that allows us to write the above equation does not take in account the structure of the system (3.3.1). In the following, we propose a more accurate result taking in consideration the equation of the velocity.

Proposition 6. *Consider $f\in L^{2\gamma}\left((0,T)\times\mathbb{T}^3\right)$ and (ρ,u) a weak solution of (3.3.1) satisfying $\rho\in L^{2\gamma}\left((0,T)\times\mathbb{T}^3\right), \rho\geq 0$ and $u,\nabla u\in L^2\left((0,T)\times\mathbb{T}^3\right)$: Then, one has that*

$$\begin{aligned} \partial_t\rho^\gamma + \gamma\operatorname{div}(\rho^\gamma u) - (\gamma-1)\operatorname{div}(\tau : u) \\ -(\gamma-1)\operatorname{div}(uf) + (\gamma-1)\,f\operatorname{div} u \\ = -(\gamma-1)\,\tau : \nabla u \end{aligned} \tag{3.3.2}$$

in the sense of distributions.

Remark 7. *In order to prove Proposition 6 we do not require regularity on the time derivative of f as it is needed in order to obtain the apriori estimates for $\partial_t u$, see Section 3.2.1*

The proof of the Proposition 6 follows the techniques introduced by Lions in [13], see also the book of Novotny and Straškraba ([16]). Recall the notation introduced in (3.1.1) and (3.1.2) and let us write

$$\partial_t\rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u) = r_\varepsilon(\rho,u)$$

which by multiplying with $\gamma(\rho_\varepsilon)^{\gamma-1}$ yields

$$\partial_t(\rho_\varepsilon)^\gamma + \operatorname{div}((\rho_\varepsilon)^\gamma u) + (\gamma-1)(\rho_\varepsilon)^\gamma\operatorname{div} u = \gamma\, r_\varepsilon(\rho,u)\,(\rho_\varepsilon)^{\gamma-1}.$$

Let us rewrite the above equation in the following manner:

$$\begin{aligned} \partial_t(\rho_\varepsilon)^\gamma + \operatorname{div}((\rho_\varepsilon)^\gamma u) + (\gamma-1)\left\{(\rho_\varepsilon)^\gamma - (\rho^\gamma)_{\varepsilon'}\right\}\operatorname{div} u + (\rho^\gamma)_{\varepsilon'}\left\{\operatorname{div} u - \operatorname{div} u_{\varepsilon'}\right\} \\ +(\rho^\gamma)_{\varepsilon'}\operatorname{div} u_{\varepsilon'} = \gamma r_\varepsilon(\rho,u)\,(\rho_\varepsilon)^{\gamma-1}. \end{aligned}$$

Next, we observe that owing to the second equation of (3.3.1) we get that

$$\begin{aligned} (\rho^\gamma)_{\varepsilon'}\operatorname{div} u_{\varepsilon'} &= \operatorname{div}((\rho^\gamma)_{\varepsilon'}u_{\varepsilon'}) - u_{\varepsilon'}\operatorname{div}\tau_{\varepsilon'} - u_{\varepsilon'}\nabla f_{\varepsilon'} \\ &= \operatorname{div}((\rho^\gamma)_{\varepsilon'}u_{\varepsilon'}) - \operatorname{div}(\tau_{\varepsilon'} : u_{\varepsilon'}) + \tau_{\varepsilon'} : \nabla u_{\varepsilon'} \\ &\quad - \operatorname{div}(u_{\varepsilon'}f_{\varepsilon'}) + f_{\varepsilon'}\operatorname{div} u_{\varepsilon'} \end{aligned}$$

and thus, we may write that

$$\begin{aligned} \partial_t(\rho_\varepsilon)^\gamma + \operatorname{div}((\rho_\varepsilon)^\gamma u) + (\gamma-1)\left\{(\rho_\varepsilon)^\gamma - (\rho^\gamma)_{\varepsilon'}\right\}\operatorname{div} u + (\gamma-1)(\rho^\gamma)_{\varepsilon'}\left\{\operatorname{div} u - \operatorname{div} u_{\varepsilon'}\right\} \\ +(\gamma-1)\operatorname{div}((\rho^\gamma)_{\varepsilon'}u_{\varepsilon'}) - (\gamma-1)\operatorname{div}(\tau_{\varepsilon'} : u_{\varepsilon'}) + (\gamma-1)\,\tau_{\varepsilon'} : \nabla u_{\varepsilon'} \\ -(\gamma-1)\operatorname{div}(u_{\varepsilon'}f_{\varepsilon'}) + (\gamma-1)\,f_{\varepsilon'}\operatorname{div} u_{\varepsilon'} = \gamma r_\varepsilon(\rho,u)\,(\rho_\varepsilon)^{\gamma-1}. \end{aligned}$$

Using the strong convergence properties of the convolution, Proposition 3 and the expression of τ, we get that

$$\begin{cases} (\rho_\varepsilon)^\gamma \to \rho^\gamma \text{ in } L^2\left((0,T)\times\mathbb{T}^3\right) \text{ for } \varepsilon\to 0, \\ (\rho_\varepsilon)^\gamma u \to \rho^\gamma u \text{ in } L^1\left((0,T)\times\mathbb{T}^3\right) \text{ for } \varepsilon\to 0, \\ (\rho^\gamma)_{\varepsilon'}\left\{\operatorname{div} u - \operatorname{div} u_{\varepsilon'}\right\} \to 0 \text{ in } L^1\left((0,T)\times\mathbb{T}^3\right) \text{ for } \varepsilon'\to 0 \\ (\rho^\gamma)_{\varepsilon'} \operatorname{div} u_{\varepsilon'} \to \rho^\gamma \operatorname{div} u \text{ in } L^1\left((0,T)\times\mathbb{T}^3\right) \text{ for } \varepsilon'\to 0, \\ \tau_{\varepsilon'} : u_{\varepsilon'} \to \tau : u \text{ and } \tau_{\varepsilon'} : \nabla u_{\varepsilon'} \to \tau : u \text{ in } L^1\left((0,T)\times\mathbb{T}^3\right) \text{ for } \varepsilon'\to 0, \\ u_{\varepsilon'} f_{\varepsilon'} \to uf \text{ in } L^1\left((0,T)\times\mathbb{T}^3\right) \text{ for } \varepsilon'\to 0, \\ f_{\varepsilon'} \operatorname{div} u_{\varepsilon'} \to f \operatorname{div} u \text{ in } L^1\left((0,T)\times\mathbb{T}^3\right) \text{ for } \varepsilon'\to 0, \\ r_\varepsilon(\rho,u)(\rho_\varepsilon)^{\gamma-1} \to 0 \text{ in } L^1\left((0,T)\times\mathbb{T}^3\right) \text{ for } \varepsilon\to 0. \end{cases}$$

Consequently, we get that

$$\partial_t\rho^\gamma + \gamma \operatorname{div}(\rho^\gamma u) - (\gamma-1)\operatorname{div}(\tau u) - (\gamma-1)\operatorname{div}(uf) + (\gamma-1) f \operatorname{div} u = -(\gamma-1)\tau : \nabla u.$$

This ends the proof of Proposition 6. Next, we investigate the weak stability of a sequence of solutions of system (3.3.1). Our main results reads

Theorem 8. *Consider a sequence of solutions $(\rho^\varepsilon, u^\varepsilon)_{\varepsilon>0}$ for (3.3.1) with initial data $(\rho_0^\varepsilon)_{\varepsilon>0} \subset L^\gamma\left(\mathbb{T}^3\right)$, i.e.*

$$\begin{cases} \partial_t\rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ -\operatorname{div}\tau^\varepsilon + \nabla(\rho^\varepsilon)^\gamma = \nabla f^\varepsilon, \\ \rho^\varepsilon_{|t=0} = \rho_0^\varepsilon, \end{cases} \tag{3.3.3}$$

with

$$\tau_{ij}^\varepsilon = A_{ijkl}^\varepsilon(t,x)D_{kl}(u^\varepsilon).$$

Assume the following :

$$\begin{cases} \rho_0^\varepsilon \to \rho_0 \text{ in } L^\gamma\left(\mathbb{T}^3\right), \\ \|\rho^\varepsilon\|_{L^\infty(0,T;L^\gamma(\mathbb{T}^3))\cap L^{2\gamma}((0,T)\times\mathbb{T}^3)} + \|u^\varepsilon\|_{L^2(0,T;H^1(\mathbb{T}^3))\cap W^{1,1}(0,T;L^{3/2-}(\mathbb{T}^d))} \le C, \\ \rho^\varepsilon \rightharpoonup \rho \text{ weakly in } L^{2\gamma}\left((0,T)\times\mathbb{T}^3\right), \\ u^\varepsilon \rightharpoonup u \text{ weakly in } L^2(0,T;H^1(\mathbb{T}^3)), \\ u^\varepsilon \to u \text{ in } L^2((0,T)\times\mathbb{T}^3)), \\ A^\varepsilon(t,x) \to A(t,x) \text{ in } W^{1,\infty}((0,T)\times\mathbb{T}^3), \\ f^\varepsilon \to f \text{ in } L^2((0,T)\times\mathbb{T}^3)). \end{cases} \tag{3.3.4}$$

where C is independent of ε. Then (ρ, u) verifies

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ -\operatorname{div}\tau + \nabla\rho^\gamma = \nabla f, \\ \rho_{|t=0} = \rho_0. \end{cases} \tag{3.3.5}$$

with

$$\tau_{ij} = A_{ijkl}(t,x)D_{kl}(u).$$

The assumptions allow us to conclude that there exist three functions (ρ, u) and $\overline{\rho^\gamma}$ such that up to a subsequence we have the following informations :

$$\begin{cases} \rho^\varepsilon \rightharpoonup \rho \text{ weakly in } L^{2\gamma}\left((0,T)\times\mathbb{T}^3\right) \\ (\rho^\varepsilon)^\gamma \rightharpoonup \overline{\rho^\gamma} \text{ weakly in } L^2\left((0,T)\times\mathbb{T}^3\right), \\ \nabla u^\varepsilon \rightharpoonup \nabla u \text{ weakly in } L^2\left((0,T)\times\mathbb{T}^3\right), \\ u^\varepsilon \to u \text{ strongly in } L^2\left((0,T)\times\mathbb{T}^3\right). \end{cases} \tag{3.3.6}$$

Moreover, we may take the above subsequence such as

$$\begin{cases} \tau^\varepsilon : \nabla u^\varepsilon \rightharpoonup \overline{\tau : \nabla u} \text{ in } \mathcal{M}\left((0,T)\times\mathbb{T}^3\right) \text{ and} \\ \tau : \nabla u \le \overline{\tau : \nabla u} \text{ in the sense of measures} \end{cases} \tag{3.3.7}$$

using the weak lower semi-continuity of the viscous work: see hypothesis (1.0.5). The above information allows us to conclude that

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ -\operatorname{div} \tau + \nabla \overline{\rho^\gamma} = \nabla f, \end{cases} \tag{3.3.8}$$

with

$$\tau_{ij} = A_{ijkl}(t,x) D_{kl}(u).$$

Of course, the delicate part is to identify $\overline{\rho^\gamma}$ with ρ^γ. Let us observe that for any $\varepsilon > 0$, $(\rho^\varepsilon, u^\varepsilon)$ verifies the hypothesis of Proposition 6 and thus we infer that

$$\begin{aligned} &\partial_t (\rho^\varepsilon)^\gamma + \gamma \operatorname{div}((\rho^\varepsilon)^\gamma u^\varepsilon) - (\gamma-1) \operatorname{div}(\tau^\varepsilon : u^\varepsilon) \\ &\quad = -(\gamma-1)\, \tau^\varepsilon : \nabla u^\varepsilon + (\gamma-1)[\operatorname{div}(f^\varepsilon u^\varepsilon) - f^\varepsilon \operatorname{div} u^\varepsilon] \end{aligned} \tag{3.3.9}$$

Moreover, using the information of relation (3.3.6) we may pass to the limit in (3.3.9) such as to obtain

$$\begin{aligned} &\partial_t \overline{\rho^\gamma} + \gamma \operatorname{div}((\overline{\rho^\gamma} u) - (\gamma-1) \operatorname{div}(\tau : u) \\ &\quad = -(\gamma-1)\, \overline{\tau : \nabla u} + (\gamma-1)[\operatorname{div}(fu) - f \operatorname{div} u]. \end{aligned} \tag{3.3.10}$$

Observing that we may put the system (3.3.8) under the form

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ -\operatorname{div} \tau + \nabla \rho^\gamma = \nabla(\rho^\gamma - \overline{\rho^\gamma}) + \nabla f \end{cases} \tag{3.3.11}$$

with $\tau_{ij} = A_{ijkl}(t,x) D_{kl}(u)$ and using Proposition 6 we write that

$$\begin{aligned} &\partial_t \rho^\gamma + \gamma \operatorname{div}(\rho^\gamma u) - (\gamma-1) \operatorname{div}(\tau : u) \\ &- (\gamma-1) \operatorname{div}(u(\rho^\gamma - \overline{\rho^\gamma})) + (\gamma-1)(\rho^\gamma - \overline{\rho^\gamma}) \operatorname{div} u \\ &= -(\gamma-1)\, \tau : \nabla u + (\gamma-1)[\operatorname{div}(fu) - f \operatorname{div} u] \end{aligned} \tag{3.3.12}$$

Next, we take the difference between (3.3.12) and (3.3.10) we get that

$$\begin{aligned} &\partial_t (\overline{\rho^\gamma} - \rho^\gamma) + \operatorname{div}((\overline{\rho^\gamma} - \rho^\gamma) u) + (\gamma-1)(\overline{\rho^\gamma} - \rho^\gamma) \operatorname{div} u \\ &= -(\gamma-1) \left\{ \overline{\tau : \nabla u} - \tau : \nabla u \right\} \end{aligned} \tag{3.3.13}$$

We denote by

$$\delta \overset{not.}{=} \overline{\rho^\gamma} - \rho^\gamma \qquad \mu \overset{not.}{=} \overline{\tau : \nabla u} - \tau : \nabla u$$

and thus (3.3.13) rewrites as

$$\partial_t \delta + \operatorname{div}(\delta u) + (\gamma-1)\, \delta \operatorname{div} u = -(\gamma-1)\, \mu.$$

We regularize the above equation in order to obtain (again recall the notations introduced in (3.1.1) and (3.1.2))

$$\partial_t \delta_{\varepsilon'} + \operatorname{div}(\delta_{\varepsilon'} u) + (\gamma-1)(\delta \operatorname{div} u)_{\varepsilon'} = r_{\varepsilon'}(\delta, u) - (\gamma-1)\, \mu_{\varepsilon'}.$$

we multiply with $\frac{1}{\gamma}(h+\delta_{\varepsilon'})^{\frac{1}{\gamma}-1}$ where h is a fixed positive constant. We end up with

$$\begin{aligned} &\partial_t (h+\delta_{\varepsilon'})^{\frac{1}{\gamma}} + \operatorname{div}\left((h+\delta_{\varepsilon'})^{\frac{1}{\gamma}} u\right) + (h+\delta_{\varepsilon'})^{\frac{1}{\gamma}-1} [\left(\frac{1}{\gamma} - 1\right) \delta_{\varepsilon'} - h] \operatorname{div} u \\ &+ \left(1 - \frac{1}{\gamma}\right)(h+\delta_{\varepsilon'})^{\frac{1}{\gamma}-1} (\delta \operatorname{div} u)_{\varepsilon'} \\ &= \frac{1}{\gamma}(h+\delta_{\varepsilon'})^{\frac{1}{\gamma}-1} r_{\varepsilon'}(\delta, u) - \frac{1}{\gamma}(h+\delta_{\varepsilon'})^{\frac{1}{\gamma}-1} (\gamma-1)\, \mu_{\varepsilon'}. \end{aligned}$$

Let us integrate the above relation in order to get that

$$
\begin{aligned}
&\int_{\mathbb{T}^3} (h+\delta_{\varepsilon'})^{\frac{1}{\gamma}}(t)\\
&= \int_{\mathbb{T}^3} (h+\delta_{\varepsilon'})^{\frac{1}{\gamma}}(0) + \int_0^T \int_{\mathbb{T}^3} \left[\frac{1}{\gamma}(h+\delta_{\varepsilon'})^{\frac{1}{\gamma}-1} r_{\varepsilon'}(\delta,u) - \frac{1}{\gamma}(h+\delta_{\varepsilon'})^{\frac{1}{\gamma}-1}(\gamma-1)\,\mu_{\varepsilon'}\right]\\
&\leq \int_{\mathbb{T}^3} (h+\delta_{\varepsilon'})^{\frac{1}{\gamma}}(0) + \int_0^T \int_{\mathbb{T}^3} \frac{1}{\gamma}(h+\delta_{\varepsilon'})^{\frac{1}{\gamma}-1} r_{\varepsilon'}(\delta,u) - \int_0^T\int_{\mathbb{T}^3} R_{\varepsilon'}
\end{aligned}
$$

with

$$
R_{\varepsilon'} = (h+\delta_{\varepsilon'})^{\frac{1}{\gamma}-1}\Big[\Big(\frac{1}{\gamma}-1\Big)\big(\delta_{\varepsilon'}\operatorname{div} u - (\delta \operatorname{div} u)_{\varepsilon'}\big) - h \operatorname{div} u\Big] \tag{3.3.14}
$$

The last inequality is justified by combining the positiveness of the measure μ (which is obtained using the lower semi-continuity assumption (1.0.5)) along with the fact that the convolution kernel is positive. We integrate the above relation in time in order to recover that

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{T}^3} (h+\delta_{\varepsilon'})^{\frac{1}{\gamma}}(t)\\
&\leq T\int_{\mathbb{T}^3} (h+\delta_{\varepsilon'})^{\frac{1}{\gamma}}(0) + T\int_0^T \int_{\mathbb{T}^3} \frac{1}{\gamma}(h+\delta_{\varepsilon'})^{\frac{1}{\gamma}-1} r_{\varepsilon'}(\delta,u) - \int_0^T\int_{\mathbb{T}^3} R_{\varepsilon'}.
\end{aligned}
$$

with $R_{\varepsilon'}$ given by (3.3.14). Thanks to Proposition 3, we get that

$$
r_{\varepsilon'}(\delta,u) \to 0 \text{ in } L^1\left((0,T)\times\mathbb{T}^3\right).
$$

Thus observing that $(h+\delta_{\varepsilon'})^{1/\gamma-1} \leq h^{1/\gamma-1}$ (because $\gamma > 1$ and $\delta_{\varepsilon'} \geq 0$), we have

$$
\int_0^T \int_{\mathbb{T}^3} (h+\delta_{\varepsilon'})^{\frac{1}{\gamma}-1} r_{\varepsilon'}(\delta,u) \leq h^{\frac{1}{\gamma}-1}\int_0^T\int_{\mathbb{T}^3} |r_{\varepsilon'}(\delta,u)|
$$

and we conclude that

$$
|R_{\varepsilon'}| \leq \left(1-\frac{1}{\gamma}\right) h^{\frac{1}{\gamma}-1}|r_{\varepsilon'}(\delta,u)| + h^{\frac{1}{\gamma}}|\operatorname{div} u|.
$$

Taking in account the last observations, by making $\varepsilon' \to 0$ we get that

$$
\int_0^T\int_{\mathbb{T}^3} (\overline{\rho^\gamma} - \rho^\gamma + h)^{\frac{1}{\gamma}} \leq T\int_{\mathbb{T}^3} (\overline{\rho^\gamma}-\rho^\gamma+h)^{\frac{1}{\gamma}}_{|t=0} + h^{1/\gamma}\int_0^T\int_{\mathbb{T}^3} |\operatorname{div} u|
$$

Letting h go to zero and using the strong convergence at initial time shows that the term in the RHS of the above equation is 0 and the conclusion is that

$$
\overline{\rho^\gamma} = \rho^\gamma \text{ a.e. on } (0,T)\times\mathbb{T}^3.
$$

This ends the proof of Theorem 8.

4 Construction of solutions

In this section, we propose a regularized system with diffusion and drag terms on the density for which we prove global existence and uniqueness of strong solution on $(0,T)$ using a fixed point procedure. Then passing to the limit with respect to the regularization parameter provides a global solution of the quasi-stationary compressible Stokes system with diffusion on the density and drag terms on the density. It remains to show that these extra terms do not perturb the stability procedure, we explained in subsection 3.3, to prove Theorem 8.

4.1 The approximate system

Let us be more precise. For any fixed strictly positive parameter $\varepsilon, \delta, \eta$, we wish to construct a global solution of the following regularized system

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\omega_\delta * u) = \varepsilon\Delta\rho - \eta\rho^{2\gamma} - \eta\rho^3, \\ -\mathcal{A}u + \nabla\omega_\delta * \rho^\gamma = 0, \\ \rho_{|t=0} = \rho_0^{reg} \end{cases} \qquad (S_{\varepsilon,\delta,\eta})$$

with ω_δ a standard regularizing kernel see (3.1.1). This is achieved by a fixed point argument.

In a second time, we pass to the limit δ go to zero to prove global existence of solution of the system

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = \varepsilon\Delta\rho - \eta\rho^{2\gamma} - \eta\rho^3, \\ \mathcal{A}u + \nabla\rho^\gamma = 0, \\ \rho_{|t=0} = \rho_0^{reg} \end{cases} \qquad (4.1.1)$$

which verifies the following uniform estimates (uniformly in ε and η) :

$$\begin{cases} \int_{\mathbb{T}^3}\rho(t) + \eta\int_0^t\int_{\mathbb{T}^3}\rho^{2\gamma} + \eta\int_0^t\int_{\mathbb{T}^3}\rho^3 = \int_{\mathbb{T}^3}\rho_0^{reg}, \\ \int_{\mathbb{T}^3}\rho^\gamma(t) + (\gamma-1)\int_0^t\int_{\mathbb{T}^3}\tau : \nabla u \\ \qquad + \eta\,\gamma[\int_0^t\int_{\mathbb{T}^3}\rho^{3\gamma-1} + \int_0^t\int_{\mathbb{T}^3}\rho^{\gamma+2}] \\ \qquad\qquad + 4\varepsilon[1-\frac{1}{\gamma}]\int_0^t\int\left|\nabla\rho^{\frac{\gamma}{2}}\right|^2 \le \int(\rho_0^{reg})^\gamma, \\ \|\rho^\gamma\|_{L^2((0,T)\times\mathbb{T}^3)} \le C_\gamma\int_{\mathbb{T}^3}(\rho_0^{reg})^\gamma. \end{cases} \qquad (4.1.2)$$

Finally, we show that we can adapt the proof of Theorem 8 in order to pass to the limit ε and $\eta \to 0$ and thus obtaining a solution for the compressible Stokes system.

4.2 Construction of solutions for the regularized system $(S_{\varepsilon,\delta,\eta})$

We consider $T > 0$ to be precised later and we denote by

$$L^2(0,T;\dot{H}^1(\mathbb{T}^3)) = \left\{u \in L^2(0,T;H^1(\mathbb{T}^3)) : \int_{\mathbb{T}^3}u(t) = 0 \text{ a.e. } t \in (0,T)\right\}$$

Consider

$$B : L^2(0,T;\dot{H}^1(\mathbb{T}^3)) \to L^2(0,T;\dot{H}^1(\mathbb{T}^3))$$

defined as

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\omega_\delta * v) = \varepsilon\Delta\rho - \eta\rho^{2\gamma}, \\ \mathcal{A}B(v) + \nabla\omega_\delta * \rho^\gamma = 0, \\ \rho_{|t=0} = \rho_0^{reg} \end{cases} \qquad (4.2.1)$$

Obviously if $v \in L^2(0,T;\dot{H}^1(\mathbb{T}^3))$ then $\omega_\delta * v \in L^2(0,T;C^\infty(\mathbb{T}^3))$ such that the existence of a regular *positive* solution for the first equation of system (4.2.1) follows by classical arguments. Also, $B(v)$ is well-defined as an element of $L^2(0,T;\dot{H}^1(\mathbb{T}^3))$ and

$$\int_0^T\int_{\mathbb{T}^3}A(t,x)D(B(v)) : D(B(v)) = \int_0^T\int_{\mathbb{T}^3}\omega_\delta * \rho^\gamma \operatorname{div}B(v)$$

which provides

$$\|\nabla B(v)\|_{L^2((0,T)\times\mathbb{T}^3)} \le C\,\|\omega_\delta * \rho^\gamma\|_{L^2((0,T)\times\mathbb{T}^3)}, \qquad (4.2.2)$$

with C depending only on the dissipation operator. Let us integrate the equation defining ρ in order to see that

$$\int_{\mathbb{T}^3}\rho(t)+\eta\int_0^t\int_{\mathbb{T}^3}\rho^{2\gamma}+\eta\int_0^t\int_{\mathbb{T}^3}\rho^3=\int_{\mathbb{T}^3}\rho_0^{reg}$$

which, enables us to conclude, that

$$\|\nabla B(v)\|_{L^2((0,T)\times\mathbb{T}^3)}\leq\frac{C}{\eta}\int_{\mathbb{T}^3}\rho_0^{reg}. \tag{4.2.3}$$

Thus, we conclude that for any $T>0$, the operator B (trivially) maps E_T into itself where

$$E_T=\left\{v\in L_T^2(\dot{H}^1(\mathbb{T}^3)):\|\nabla v\|_{L^2((0,T)\times\mathbb{T}^3)}\leq\frac{C}{\eta}\int_{\mathbb{T}^3}\rho_0^{reg}\right\}$$

In the following, we aim at showing that B is a contraction on E_T.

The first observation that we make in towards this direction is that using a maximum principle we get

$$\begin{aligned}\|\rho\|_{L^\infty((0,t)\times\mathbb{T}^3)}&\leq\|\rho_0^{reg}\|_{L^\infty(\mathbb{T}^3)}\exp\left(\int_0^t\|\operatorname{div}\omega_\delta*v\|_{L^\infty(\mathbb{T}^3)}\right)\\&\leq\|\rho_0^{reg}\|_{L^\infty(\mathbb{T}^3)}\exp\left(\sqrt{t}C_{\eta,\delta}\right).\end{aligned}\tag{4.2.4}$$

Next, let us multiply with ρ and integrate in order to obtain that

$$\frac{1}{2}\int_{\mathbb{T}^3}\rho^2+\varepsilon\int_0^t\int_{\mathbb{T}^3}|\nabla\rho|^2+\eta\int_0^t\int_{\mathbb{T}^3}\rho^{2\gamma+1}+\eta\int_0^t\int_{\mathbb{T}^3}\rho^4=\gamma\int_{\mathbb{T}^3}\rho^2\operatorname{div}(\omega_\delta*v)$$

and thus by Gronwall's lemma we get that

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{T}^3}\rho^2&+\varepsilon\int_0^t\int_{\mathbb{T}^3}|\nabla\rho|^2+\eta\int_0^t\int_{\mathbb{T}^3}\rho^{2\gamma+1}+\eta\int_0^t\int_{\mathbb{T}^3}\rho^4\\&\leq\frac{1}{2}\int_{\mathbb{T}^3}(\rho_0^{reg})^2\exp\left(\int_0^t\|\operatorname{div}(\omega_\delta*v)\|_{L^\infty(\mathbb{T}^3)}\right)\\&\leq\frac{1}{2}\int_{\mathbb{T}^3}(\rho_0^{reg})^2\exp\left(tC_\delta\int_0^t\|\nabla v\|^2_{L^2(\mathbb{T}^3)}\right)\\&\leq\frac{1}{2}\int_{\mathbb{T}^3}(\rho_0^{reg})^2\exp\left(tC_{\delta,\eta}\int_{\mathbb{T}^3}\rho_0^{reg}\right)\end{aligned}\tag{4.2.5}$$

Let us consider $v_1,v_2\in E_T$ and let us consider

$$\begin{cases}\partial_t\rho_i+\operatorname{div}(\rho_i\omega_\delta*v_i)=\varepsilon\Delta\rho_i-\eta\rho_i^{2\gamma}-\eta\rho_i^3,\\ \mathcal{A}B(v_i)+\nabla\omega_\delta*\rho_i^\gamma=0,\\ \rho_{i|t=0}=\rho_0^{reg}\end{cases}$$

with $i\in 1,2$. Of course, ρ_1 and ρ_2 verify estimate (4.2.5). We denote by $r=\rho_1-\rho_2$ and $w=v_1-v_2$. We infer that

$$\begin{cases}\partial_t r+\operatorname{div}(r\omega_\delta*v_1)=\varepsilon\Delta r-\eta\left(\rho_1^{2\gamma}+\rho_1^3-\rho_2^{2\gamma}-\rho_2^3\right)-\operatorname{div}(\rho_2V_\delta*w),\\ \mathcal{A}(B(v_1)-B(v_2))+\nabla\omega_\delta*(\rho_1^\gamma-\rho_2^\gamma)=0,\\ r_{|t=0}=0\end{cases}$$

Next, we observe that

$$\int_{\mathbb{T}^3}\frac{r^2(t)}{2}+\varepsilon\int_0^t\int_{\mathbb{T}^3}|\nabla r|^2+\eta\int_0^t\int_{\mathbb{T}^3}\left(\rho_1^{2\gamma}+\rho_1^3-\rho_2^{2\gamma}-\rho_2^3\right)r \tag{4.2.6}$$

$$\begin{aligned}
&\leq \int_0^t\int_{\mathbb{T}^3} r^2\operatorname{div}\omega_\delta * v_1+\int_0^t\int_{\mathbb{T}^3}\operatorname{div}\left(\rho_2\omega_\delta * w\right)r\\
&\leq \int_0^t\int r^2\left\|\operatorname{div}\omega_\delta * v_1\right\|_{L^\infty(\mathbb{T}^3)}+\frac{1}{2\varepsilon}\int_0^t\|\rho_2\|^2_{L^2(\mathbb{T}^3)}\|\omega_\delta * \delta v\|^2_{L^\infty(\mathbb{T}^3)}+\frac{\varepsilon}{2}\int_0^t\int_{\mathbb{T}^3}|\nabla r|^2\\
&\leq \int_0^t\int_{\mathbb{T}^3} r^2\left\|\operatorname{div}\omega_\delta * v_1\right\|_{L^\infty(\mathbb{T}^3)}+C_{\delta,\varepsilon}\exp\left(tC_{\delta,\eta}\int\rho_0^{reg}\right)\int_0^t\|\delta v\|^2_{L^6(\mathbb{T}^3)}+\frac{\varepsilon}{2}\int_0^t\int_{\mathbb{T}^3}|\nabla r|^2\\
&\leq \int_0^t\int_{\mathbb{T}^3} r^2\left\|\operatorname{div}\omega_\delta * v_1\right\|_{L^\infty(\mathbb{T}^3)}+C_{\delta,\varepsilon}\exp\left(tC_{\delta,\eta}\int\rho_0^{reg}\right)\int_0^t\|\nabla\delta v\|^2_{L^2(\mathbb{T}^3)}+\frac{\varepsilon}{2}\int_0^t\int_{\mathbb{T}^3}|\nabla r|^2
\end{aligned} \tag{4.2.7}$$

and thus using Grönwall's lemma we get that

$$\begin{aligned}
&\int_{\mathbb{T}^3}\frac{r^2(t)}{2}+\frac{\varepsilon}{2}\int_0^t\int_{\mathbb{T}^3}|\nabla r|^2+\eta\int_0^t\int_{\mathbb{T}^3}\left(\rho_1^{2\gamma}-\rho_2^{2\gamma}\right)r+\eta\int_0^t\int_{\mathbb{T}^3}\left(\rho_1^3-\rho_2^3\right)r\\
&\leq C_{\delta,\varepsilon}\exp\left(tC_{\delta,\eta}\int_{\mathbb{T}^3}\rho_0^{reg}\right)\int_0^t\|\nabla w\|^2_{L^2(\mathbb{T}^3)}\exp\left(\int_0^t\int_{\mathbb{T}^3}\left\|\operatorname{div}\omega_\delta * v_1\right\|_{L^\infty(\mathbb{T}^3)}\right)\\
&\leq C_{\delta,\varepsilon}\exp\left(C_{\delta,\varepsilon}t\right)\int_0^t\|\nabla w\|^2_{L^2(\mathbb{T}^3)}=C_{\delta,\varepsilon}\exp\left(C_{\delta,\varepsilon}t\right)\int_0^t\|\nabla v_1-\nabla v_2\|^2_{L^2(\mathbb{T}^3)}
\end{aligned} \tag{4.2.8}$$

Finally, recalling that

$$\mathcal{A}\left(B(v_1)-B\left(v_2\right)\right)+\nabla\omega_\delta * \left(\rho_1^\gamma-\rho_2^\gamma\right)=0,$$

we infer that

$$\|\nabla\left(B\left(v_1\right)-B(v_2)\right)\|_{L^2((0,t)\times\mathbb{T}^3)}\leq Ct^{\frac{1}{2}}\|\rho_1^\gamma-\rho_2^\gamma\|_{L^\infty(0,t;L^2(\mathbb{T}^3))} \tag{4.2.9}$$

We use the intermediate value theorem and estimate (4.2.4) in order to asses that

$$\begin{aligned}
|\rho_1^\gamma-\rho_2^\gamma|&\leq\gamma\left|\rho_1-\rho_2\right|\max\left\{\|\rho_1\|^{\gamma-1}_{L^\infty((0,t)\times\mathbb{T}^3)},\|\rho_2\|^{\gamma-1}_{L^\infty((0,t)\times\mathbb{T}^3)}\right\}\\
&\leq\gamma\left|\rho_1-\rho_2\right|\|\rho_0^{reg}\|^{\gamma-1}_{L^\infty(\mathbb{T}^3)}\exp\left(\sqrt{t}C_{\eta,\delta}\right)
\end{aligned} \tag{4.2.10}$$

which, in turn implies that

$$\|\rho_1^\gamma-\rho_2^\gamma\|_{L^\infty(0,t;L^2(\mathbb{T}^3))}\leq\gamma\|\rho_0^{reg}\|^{\gamma-1}_{L^\infty(\mathbb{T}^3))}\exp\left(\sqrt{t}C_{\eta,\delta}\right)\|r\|_{L^\infty(0,t;L^2(\mathbb{T}^3))}.$$

This last estimate along with (4.2.8) gives us

$$\|\nabla\left(B\left(v_1\right)-B(v_2)\right)\|_{L^2((0,t)\times\mathbb{T}^3)}\leq t^{\frac{1}{2}}C_{\delta,\varepsilon}\exp\left(\left(1+t\right)C_{\eta,\delta}\right)\|\nabla v_1-\nabla v_2\|_{L^2((0,t)\times\mathbb{T}^3)}.$$

We conclude that for a small $T^\star$ the operator has a fixed point $u\in E_{T^\star}$ which verifies $(S_{\varepsilon,\delta,\eta})$. As the pair (ρ,u) solution of the above system verifies by integration of the first equation

$$\int_{\mathbb{T}^3}\rho\left(t\right)+\eta\int_0^t\int_{\mathbb{T}^3}\rho^{2\gamma}+\eta\int_0^t\int_{\mathbb{T}^3}\rho^3=\int_{\mathbb{T}^3}\rho_0^{reg},$$

using the second equation of $(S_{\varepsilon,\delta,\eta})$ we see that the last relation implies that

$$\|\nabla u\|_{L^2((0,T^\star)\times\mathbb{T}^3)}\leq\frac{C}{\eta}\int_{\mathbb{T}^3}\rho_0^{reg}.$$

Thus, we may re-iterate the fixed point argument. This implies that the solution (ρ,u) of $(S_{\varepsilon,\delta,\eta})$ is global.

4.3 The limit $\delta \to 0$

We consider (ρ^δ, u^δ) a sequence of solutions to

$$\begin{cases} \partial_t \rho^\delta + \operatorname{div}\left(\rho^\delta \omega_\delta * u^\delta\right) = \varepsilon \Delta \rho^\delta - \eta (\rho^\delta)^{2\gamma} - \eta \left(\rho^\delta\right)^3, \\ \mathcal{A} u^\delta + \nabla \omega_\delta * \left(\rho^\delta\right)^\gamma = 0, \\ \rho_{|t=0} = \rho_0^{reg} \end{cases} \tag{$S_{\varepsilon,\delta,\eta}$}$$

The sequence verifies the following estimates uniformly in δ :

$$\begin{cases} \int_{\mathbb{T}^3} \rho^\delta(t) + \eta \int_0^t \int_{\mathbb{T}^3} \left(\rho^\delta\right)^{2\gamma} + \eta \int_0^t \int_{\mathbb{T}^3} \left(\rho^\delta\right)^3 = \int_{\mathbb{T}^3} \rho_0^{reg}, \\ \int_{\mathbb{T}^3} \left(\rho^\delta\right)^\gamma(t) + (\gamma - 1) \int_0^t \int_{\mathbb{T}^3} u^\delta \mathcal{A} u^\delta \\ \qquad + \eta\gamma \int_0^t \int_{\mathbb{T}^3} \left(\rho^\delta\right)^{3\gamma-1} + \eta\gamma \int_0^t \int_{\mathbb{T}^3} \left(\rho^\delta\right)^{\gamma+2} \\ \qquad\qquad + 4\varepsilon[1 - \frac{1}{\gamma}] \int_0^t \int_{\mathbb{T}^3} \left|\nabla \left(\rho^\delta\right)^{\frac{\gamma}{2}}\right| \leq \int_{\mathbb{T}^3} (\rho_0^{reg})^\gamma, \\ \left\|\omega_\delta * \left(\rho^\delta\right)^\gamma\right\|_{L^2((0,T)\times\mathbb{T}^3)} \leq \left\|\Delta^{-1} \operatorname{div} \mathcal{A} u^\delta\right\|_{L^2((0,T)\times\mathbb{T}^3)} \leq C_\gamma \int_{\mathbb{T}^3} (\rho_0^{reg})^\gamma. \end{cases} \tag{4.3.1}$$

Moreover, we have that

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{T}^3} \left(\rho^\delta\right)^2 + \varepsilon \int_0^t \int_{\mathbb{T}^3} \left|\nabla \rho^\delta\right|^2 \\ &+ \eta \int_0^t \int_{\mathbb{T}^3} \left(\rho^\delta\right)^{2\gamma+1} + \eta \int_0^t \int_{\mathbb{T}^3} \left(\rho^\delta\right)^4 = \gamma \int_0^t \int_{\mathbb{T}^3} \left(\rho^\delta\right)^2 \operatorname{div}\left(\omega_\delta * u^\delta\right) \\ &\qquad \leq \frac{\eta}{2} \int_0^t \int_{\mathbb{T}^3} \left(\rho^\delta\right)^4 + \frac{\gamma^2}{2\eta} \int_0^t \int_{\mathbb{T}^3} \left(\omega_\delta * \operatorname{div} u^\delta\right)^2 \end{aligned}$$

and owing to the uniform bound on ∇u^δ ensured by the estimates (4.3.1) we get that

$$\frac{1}{2} \int_{\mathbb{T}^3} \left(\rho^\delta\right)^2 + \varepsilon \int_0^t \int_{\mathbb{T}^3} \left|\nabla \rho^\delta\right|^2 + \eta \int_0^t \int_{\mathbb{T}^3} \left(\rho^\delta\right)^{2\gamma+1} + \frac{\eta}{2} \int_0^t \int_{\mathbb{T}^3} \left(\rho^\delta\right)^4 \leq \frac{C}{\eta} \int_{\mathbb{T}^3} (\rho_0^{reg})^\gamma. \tag{4.3.2}$$

Moreover, we have that

$$\partial_t \rho^\delta \text{ is bounded uniformly in } W^{-1,1}\left((0,T) \times L^1\left(\mathbb{T}^3\right)\right) + L^1\left((0,T) \times \mathbb{T}^3\right) \tag{4.3.3}$$

The estimates (4.3.1), (4.3.2) and (4.3.3) are enough in order to pass to the limit when $\delta \to 0$ such that we obtain the existence of a solution of system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = \varepsilon \Delta \rho - \eta \rho^{2\gamma} - \eta \rho^3, \\ \mathcal{A} u^\delta + \nabla \rho^\gamma = 0, \\ \rho_{|t=0} = \rho_0^{reg} \end{cases}$$

which verifies the following bounds

$$\begin{cases} \int_{\mathbb{T}^3} \rho(t) + \eta \int_0^t \int_{\mathbb{T}^3} \rho^{2\gamma} + \eta \int_0^t \int_{\mathbb{T}^3} \rho^3 = \int_{\mathbb{T}^3} \rho_0^{reg}, \\ \int_{\mathbb{T}^3} \rho^\delta(t) + (\gamma - 1) \int_0^t \int_{\mathbb{T}^3} u \mathcal{A} u \\ \qquad + \eta\gamma \int_0^t \int_{\mathbb{T}^3} \rho^{3\gamma-1} + \eta\gamma \int_0^t \int_{\mathbb{T}^3} \rho^{\gamma+2} \\ \qquad\qquad + 4\varepsilon[1 - \frac{1}{\gamma}] \int_0^t \int_{\mathbb{T}^3} \left|\nabla \rho^{\frac{\gamma}{2}}\right| \leq \int_{\mathbb{T}^3} (\rho_0^{reg})^\gamma, \\ \|\rho^\gamma\|_{L^2((0,T)\times\mathbb{T}^3)} \leq C_\gamma \int_{\mathbb{T}^3} (\rho_0^{reg})^\gamma. \end{cases} \tag{4.3.4}$$

4.4 Weak stability result for the perturbed system with diffusion and drag terms

In view of what was proved in the last section, let us consider a sequence $(\rho^\varepsilon, u^\varepsilon)$ of solutions of

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = \varepsilon \Delta \rho^\varepsilon - \varepsilon (\rho^\varepsilon)^{2\gamma} - \varepsilon (\rho^\varepsilon)^3, \\ \mathcal{A} u^\varepsilon + \nabla (\rho^\varepsilon)^\gamma = 0, \\ \rho_{|t=0} = \rho_0^{reg} \end{cases} \qquad (\mathcal{S}_\varepsilon)$$

which verifies the following estimates uniformly in ε

$$\begin{cases} \int_{\mathbb{T}^3} \rho^\varepsilon(t) + \varepsilon \int_0^t \int_{\mathbb{T}^3} (\rho^\varepsilon)^{2\gamma} + \varepsilon \int_0^t \int_{\mathbb{T}^3} (\rho^\varepsilon)^3 = \int_{\mathbb{T}^3} \rho_0^{reg}, \\ \int_{\mathbb{T}^3} (\rho^\varepsilon)^\gamma(t) + (\gamma - 1) \int_0^t \int_{\mathbb{T}^3} \tau^\varepsilon : \nabla u^\varepsilon \\ \qquad + \varepsilon\gamma \int_0^t \int_{\mathbb{T}^3} (\rho^\varepsilon)^{3\gamma-1} \varepsilon\gamma \int_0^t \int_{\mathbb{T}^3} (\rho^\varepsilon)^{\gamma+2} \\ \qquad\qquad + 4\varepsilon[1 - \frac{1}{\gamma}] \int_0^t \int_{\mathbb{T}^3} \left|\nabla (\rho^\varepsilon)^{\frac{\gamma}{2}}\right|^2 \leq \int_{\mathbb{T}^3} (\rho_0^{reg})^\gamma, \\ \|(\rho^\varepsilon)^\gamma\|_{L^2((0,T)\times\mathbb{T}^3)} \leq C_\gamma \int_{\mathbb{T}^3} (\rho_0^{reg})^\gamma. \end{cases} \tag{4.4.1}$$

In the following we show that it is possible to slightly modify the proof of stability in order to show that the limiting function (ρ, u) is a solution of the semi-stationary Stokes system. Indeed, let us observe that

$$\begin{aligned} &\gamma (h + \omega_{\varepsilon'} * (\rho^\varepsilon))^{\gamma-1} \Delta \omega_{\varepsilon'} * (\rho^\varepsilon) \\ &= \Delta ((h + \omega_{\varepsilon'} * (\rho^\varepsilon))^\gamma) - \nabla (h + \omega_{\varepsilon'} * (\rho^\varepsilon))^{\gamma-1} \nabla \omega_{\varepsilon'} * (\rho^\varepsilon) \\ &= \Delta ((h + \omega_{\varepsilon'} * (\rho^\varepsilon))^\gamma) - (\gamma - 1) (h + \omega_{\varepsilon'} * (\rho^\varepsilon))^{\gamma-2} \nabla \omega_{\varepsilon'} * (\rho^\varepsilon) \nabla \omega_{\varepsilon'} * (\rho^\varepsilon) \\ &= \Delta ((h + \omega_{\varepsilon'} * (\rho^\varepsilon))^\gamma) - \gamma \frac{(\gamma - 1)}{\left(\frac{\gamma}{2}\right)^2} \nabla (h + \omega_{\varepsilon'} * (\rho^\varepsilon))^{\frac{\gamma}{2}} \nabla (h + \omega_{\varepsilon'} * (\rho^\varepsilon))^{\frac{\gamma}{2}}. \end{aligned}$$

Thus, in the sense of distributions, we get that

$$\gamma (h + \omega_{\varepsilon'} * (\rho^\varepsilon))^{\gamma-1} \Delta \omega_{\varepsilon'} * (\rho^\varepsilon) \underset{\varepsilon', h \to 0}{\to} \Delta (\rho^\varepsilon)^\gamma - 4 [1 - \frac{1}{\gamma}] \left|\nabla (\rho^\varepsilon)^{\frac{\gamma}{2}}\right|^2.$$

Also, we have that

$$\begin{cases} (h + \omega_{\varepsilon'} * (\rho^\varepsilon))^{\gamma-1} \omega_{\varepsilon'} * (\rho^\varepsilon)^{2\gamma} \underset{\varepsilon', h \to 0}{\to} (\rho^\varepsilon)^{3\gamma-1} \text{ in } L^1_{t,x} \\ (h + \omega_{\varepsilon'} * (\rho^\varepsilon))^{\gamma-1} \omega_{\varepsilon'} * (\rho^\varepsilon)^3 \underset{\varepsilon', h \to 0}{\to} (\rho^\varepsilon)^{\gamma+2} \text{ in } L^1_{t,x}. \end{cases}$$

We may thus write the renormalized equation for $(\rho^\varepsilon)^\gamma$ which yields

$$\begin{aligned} &\partial_t (\rho^\varepsilon)^\gamma + \gamma \operatorname{div}((\rho^\varepsilon)^\gamma u^\varepsilon) - (\gamma - 1) \operatorname{div}(u^\varepsilon \tau^\varepsilon) \\ &= -(\gamma - 1) \tau^\varepsilon : \nabla u^\varepsilon \\ &+ \varepsilon \Delta (\rho^\varepsilon)^\gamma - 4\varepsilon [1 - \frac{1}{\gamma}] \left|\nabla (\rho^\varepsilon)^{\frac{\gamma}{2}}\right|^2 - \varepsilon (\rho^\varepsilon)^{3\gamma-1} - \varepsilon (\rho^\varepsilon)^{\gamma+2} \end{aligned}$$

Then, we conclude that

$$\begin{aligned} &\partial_t (\overline{\rho^\gamma} - \rho^\gamma) + \operatorname{div}((\overline{\rho^\gamma} - \rho^\gamma) u) + (\gamma - 1)(\overline{\rho^\gamma} - \rho^\gamma) \operatorname{div} u \\ &= -(\gamma - 1) \left\{\overline{\tau : \nabla u} - \tau : \nabla u\right\} - \nu \end{aligned} \tag{4.4.2}$$

where ν is a positive measure i.e.

$$\nu = \lim_{\varepsilon \to 0} \Big(4\varepsilon[1 - \frac{1}{\gamma}] \left|\nabla (\rho^\varepsilon)^{\frac{\gamma}{2}}\right|^2 + \varepsilon (\rho^\varepsilon)^{3\gamma-1} + (\rho^\varepsilon)^{\gamma+2}\Big)$$

which will not perturb the stability procedure which follows exactly the same lines as in Subsection (3.3).

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