De nouvelles idées pour la résolution des problèmes de diffraction combinant décomposition de domaine, opérateurs intégraux et dilatation analytique

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with

in POEMS (CNRS-ENSTA-INRIA):
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in the University of Reading:
Simon Chandler-Wilde
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The general subject of my talk

Conception of appropriate numerical methods for time-harmonic ($e^{-i\omega t}$) scattering problems in "complex" unbounded 2D domains.

The "simple" model problem

\[ \Delta u + k^2 u = 0 \]
\[ u = u_D \]
\[ k = \frac{\omega}{c} \]

Sommerfeld condition $r \to +\infty$

\[ \frac{\partial u}{\partial r} - iku = \mathcal{O}(r^{-3/2}) \]

$u$ is outgoing
Examples of ”complex” scattering problems

Heterogeneous background

\[
\text{div}(\mu \nabla u) + \omega^2 \rho u = 0
\]

\[
\mu = \mu(x) \quad \rho = \rho(x)
\]

Stratified background

Junction of open waveguides
Examples of "complex" scattering problems

Heterogeneous background for various physical models

\[
\begin{align*}
\text{div}(\sigma(u)) + \omega^2 \rho u &= 0 \\
\sigma(u) &= \lambda \text{div } u + 2\mu \varepsilon(u) \\
\rho &= \rho(x) \quad \lambda = \lambda(x) \quad \mu = \mu(x)
\end{align*}
\]

\[
\begin{align*}
\text{curl } (\mu^{-1} \text{curl } E) - \omega^2 \varepsilon E &= 0 \\
\varepsilon &= \varepsilon(x) \quad \mu = \mu(x)
\end{align*}
\]

Stratified background
seismic waves

Junction of open waveguides
integrated optics
Examples of "complex" scattering problems

Anisotropic background

\[
\begin{align*}
\text{div}(A \nabla u) + k^2 u &= 0 \\
\text{curl} (\mu^{-1} \text{curl} E) - \omega^2 \varepsilon E &= 0 \\
\text{div}(\sigma(u)) + \omega^2 \rho u &= 0 \\
\sigma(u) &= C : \varepsilon(u)
\end{align*}
\]

Composite materials
Examples of "complex" scattering problems

"Unbounded" obstacles

or more generally, rough surfaces
There exist several efficient methods for scattering problems in unbounded domains with simple background.

\[ \Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus D \]

\[ \frac{\partial u}{\partial r} - ik u = \mathcal{O}(r^{-3/2}) \]

\[ u = u_D \text{ on } \partial D \]
Standard methods for scattering problems

- **Transparent boundary conditions** on a circle
- **Integral equation methods** on the boundary of the scatterer $\partial D$
- **Perfectly Matched Layers**, cartesian or radial, around the scatterer $D$
Some limits of classical methods

They may be inappropriate or even fail in complex configurations.

**Transparent boundary condition**

Hard to extend to more complex configurations.

**Integral equations**

For a complex background, the Green function (or tensor) is not available or too expensive to compute.

**Perfectly Matched Layers**

PMLs may produce spurious effects or even select a wrong solution.

**Conclusion**

There is a need of a new method

- which does not require the Green function of the whole background
- which works in configurations where PMLs fail
Outline

1. The Half-Space Matching (HSM) method \((k \notin \mathbb{R})\)

2. The complex-scaled HSM method \((k \in \mathbb{R})\)
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1. The Half-Space Matching (HSM) method \((k \notin \mathbb{R})\)

2. The complex-scaled HSM method \((k \in \mathbb{R})\)
The HSM method has been designed for scattering problems such that
- the global Green function is not available,
- but analytic representations in half-spaces are available.

Scattering of a quasi-shear wave
A. Tonnoir (POEMS)

Junction of open waveguides
J. Ott (KIT)
Main features of the HSM formulation

We split the infinite domain into 5 overlapping subdomains:
- A rectangle containing the scatterer
- 4 half-spaces which do not contain any scatterer

The unknowns of the HSM formulation are
- The restriction of the solution to the rectangle
- Traces of the solution on 4 infinite lines (boundaries of half-spaces)

The system of equations couples
- a variational formulation in the rectangle
- integral equations on the traces (derived by using half-space representations)
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A similar approach has been first introduced for periodic media in Fliss and Joly, 2009.
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The system of equations couples

- A variational formulation in the rectangle
- *integral equations on the traces (derived by using half-space representations)*

*For simplicity in the sequel, we consider a problem set in the exterior of the rectangle.*
Let us consider the 2D model problem:

\[
\begin{align*}
\Delta u + k^2 u &= 0 \quad (\mathbb{R}^2 \setminus D) \\
 u &= u_D \quad (\partial D)
\end{align*}
\]

with \( k \notin \mathbb{R} \)

where \( D \) is the square \((-a, a)^2\), \( u_D \in H^{1/2}(\partial D) \) and \( u \in H^1(\mathbb{R}^2 \setminus D) \).
Some notations

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\[\Sigma_0 = \{x = (x_1, x_2); x_1 = a\}\]
\[\Sigma_1 = \{x = (x_1, x_2); x_2 = a\}\]
\[\Sigma_2 = \{x = (x_1, x_2); x_1 = -a\}\]
\[\Sigma_3 = \{x = (x_1, x_2); x_2 = -a\}\]

\[\varphi_0 = u|_{\Sigma_0}\]
\[\varphi_1 = u|_{\Sigma_1}\]
\[\varphi_2 = u|_{\Sigma_2}\]
\[\varphi_3 = u|_{\Sigma_3}\]
Half-space representations

\[ \Delta u + k^2 u = 0 \quad (\mathbb{R}^2 \setminus D) \]
\[ u = u_D \quad (\partial D) \]

\[ \Sigma_0 = \{ \mathbf{x} = (x_1, x_2); x_1 = a \} \]
\[ \Omega_0 = \{ \mathbf{x} = (x_1, x_2); x_1 > a \} \]
\[ \varphi_0 = u|_{\Sigma_0} \]
**F**ourier representation of $u$ in $\Omega_0$:

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\varphi}_0(\xi) e^{-\sqrt{\xi^2-k^2}(x_1-a)} e^{i\xi x_2} d\xi \text{ for } x_1 > a$$

where $\Re(e^{\sqrt{\cdot}}) > 0$. 

$$\Delta u + k^2 u = 0 \quad (\Omega_0)$$

$$u = \varphi_0 \quad (\Sigma_0)$$
**Half-space representations**

\[ \Delta u + k^2 u = 0 \quad (\Omega_j) \]
\[ u = \varphi_j \quad (\Sigma_j) \]

**Fourier representation of** \( u \) **in** \( \Omega_0 \):\[
\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\varphi}_0(\xi) e^{-\sqrt{\xi^2 - k^2}(x_1-a)} e^{i\xi x_2} d\xi := U_0(\varphi_0)(x) \text{ for } x_1 > a
\]

and similarly in all half-spaces:\[
u := U_j(\varphi_j) \text{ in } \Omega_j, \ j = 0, 1, 2, 3
\]
The half-space representations must coincide in the overlapping areas: this gives the compatibility equations linking the $\varphi_j$'s.
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For instance in the quarter of plane \( \Omega_0 \cap \Omega_1 \):

\[
    u \equiv U_0(\varphi_0) \equiv U_1(\varphi_1)
\]
Compatibility relations

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$$u \equiv U_0(\varphi_0) \equiv U_1(\varphi_1)$$

In particular, for $x \in \Omega_0 \cap \Sigma_1$

$$\varphi_1(x) = U_0(\varphi_0)(x)$$

which leads to the first integral equation linking $\varphi_0$ and $\varphi_1$: 
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These are the 2 integral equations linking $\varphi_0$ and $\varphi_1$. 
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These are the 2 integral equations linking $\varphi_0$ and $\varphi_1$. Similarly for all the quarter-spaces, we get the 8 equations: $\varphi_j = D^j j^{\pm 1} \varphi_{j\pm 1}$. 
For this model problem, with a suitable parametrization:

\[
\begin{align*}
\varphi_0(t) &= u(a, t) \\
\varphi_1(t) &= u(-t, a) \\
\varphi_2(t) &= u(-a, -t) \\
\varphi_3(t) &= u(t, -a)
\end{align*}
\]
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Half-Space Matching formulation

For this model problem, with a suitable parametrization

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\end{align*}
\]

the 8 equations become

\[
\begin{align*}
\varphi_j &= S D (\varphi_{j-1}) \text{ for } t < -a \\
\varphi_j &= D S (\varphi_{j+1}) \text{ for } t > a
\end{align*}
\]

where

\[
S(\varphi)(t) = \varphi(-t) \quad \text{and} \quad D(\varphi)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\varphi}(\xi) e^{-\sqrt{\xi^2 - k^2(t-a)}} e^{ia\xi} d\xi
\]
For this model problem, with a suitable parametrization

\[ \varphi_0(t) = u(a, t) \]
\[ \varphi_1(t) = u(-t, a) \]
\[ \varphi_2(t) = u(-a, -t) \]
\[ \varphi_3(t) = u(t, -a) \]

the HSM formulation becomes

Find \( \Phi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3) \in L^2(\mathbb{R})^4 \) such that

\[ \varphi_j = u^j_D \text{ on } \Sigma_j \cap \partial D \quad (|t| < a) \]
\[ \varphi_j = SD(\varphi_{j-1}) \text{ for } t < -a \quad \text{for } j \in \mathbb{Z}/4\mathbb{Z}. \]
\[ \varphi_j = DS(\varphi_{j+1}) \text{ for } t > a \]
Find $\Phi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3) \in L^2(\mathbb{R})^4$ such that

\[
\begin{align*}
\varphi_j &= 0 \text{ on } \Sigma_j \cap \partial D \quad (|t| < a) \\
\varphi_j - SD(\varphi_{j-1}) &= SD(u_D^{j-1}) \text{ for } t < -a \quad \text{for } j \in \mathbb{Z}/4\mathbb{Z}.
\end{align*}
\]

\[
\varphi_j - DS(\varphi_{j+1}) = DS(u_D^{j+1}) \text{ for } t > a
\]

The problem takes the matrix form:

Find $\Phi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3) \in \mathcal{L}$ such that

$$(I - D)\Phi = DF_D$$

where $\mathcal{L} = \{\Phi \in L^2(\mathbb{R})^4; \Phi(t) = 0 \text{ for } |t| < a\}$

**Difficulty**

Due to cross-points, $D$ is not a compact operator!
Analysis of the HSM formulation

Find $\Phi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3) \in L^2(\mathbb{R})^4$ such that

$\varphi_j = 0$ on $\Sigma_j \cap \partial D$ ($|t| < a$)

$\varphi_j - SD(\varphi_{j-1}) = SD(u_{D}^{j-1})$ for $t < -a$ for $j \in \mathbb{Z}/4\mathbb{Z}$.

$\varphi_j - DS(\varphi_{j+1}) = DS(u_{D}^{j+1})$ for $t > a$ for $j \in \mathbb{Z}/4\mathbb{Z}$.

The problem takes the matrix form:

Find $\Phi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3) \in \mathcal{L}$ such that $(I - D)\Phi = DF_D$

where $\mathcal{L} = \{\Phi \in L^2(\mathbb{R})^4; \Phi(t) = 0$ for $|t| < a\}$

**Theorem**

- The essential norm of $D$ on the space $\mathcal{L}$ is $< 1$.
- The problem is coercive + compact, and well-posed.
**Properties of the integral operator**

\[
D(\varphi)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\varphi}(\xi) e^{-\sqrt{\xi^2-k^2}(t-a)} e^{ia\xi} d\xi
\]

\(D(\varphi)\) is the trace on the blue line of the solution \(u\) of

\[
\begin{align*}
\Delta u + k^2 u &= 0 \quad (\Omega_0) \\
u &= \varphi \quad (\Sigma_0)
\end{align*}
\]
Properties of the integral operator

\[ D(\varphi)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\varphi}(\xi) e^{-\sqrt{\xi^2 - k^2} (t-a)} e^{ia\xi} d\xi \]

\( D(\varphi) \) is the trace on the blue line of the solution \( u \) of

\[
\left| \begin{array}{c}
\Delta u + k^2 u = 0 \quad (\Omega_0) \\
u = \varphi \quad (\Sigma_0)
\end{array} \right.
\]

- \( \varphi_- \mapsto D(\varphi_-) \) is compact, due to interior regularity and \( k \notin \mathbb{R} \).

- \( \varphi_+ \mapsto D(\varphi_+) \) is not compact, due to the cross-point \( \bullet \).

- For \( k = 0 \), the norm of \( \varphi_+ \mapsto D(\varphi_+) \) is \( 1/\sqrt{2} \) (Mellin calculus, Chandler 1984).

Theorem

The essential norm of \( \mathbb{D} \) is \( \leq 1/\sqrt{2} \).
Discretization

Find $\Phi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3) \in L^2(\mathbb{R})^4$ such that
$\Phi(t) = 0$ for $|t| < a$ and $(I - D)\Phi = DF_D$

1. Write the variational formulation:
   $$(\Phi, \Psi)_{L^2} - (D\Phi, \Psi)_{L^2} = (DF_D, \Psi)_{L^2}$$

2. Truncate the infinite lines $\Sigma_j$'s

3. Discretize the $\phi_j$'s with 1D Lagrange FE

4. Approximate the Fourier integrals: truncation+quadrature

5. Solve the partially full linear system
Numerical validation (using XLIIFE++)

\[ \Delta u + k^2 u = 0 \quad (\mathbb{R}^2 \setminus D) \]
\[ u = u_D \quad (\partial D) \]

Data:
\[ k = 10 + 0.5i \]
\[ u_D(x) = H_0^{(1)}(k|x|) \]

Truncation of the lines: \( T = 24a \)

The trace \( \varphi_0 = \varphi_1 = \varphi_2 = \varphi_3 \)

\( \text{err}_{\text{rel}} < 0.06\% \)
**Numerical validation (using XLiFE++)**

\[
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u &= u_D \quad (\partial D)
\end{align*}
\]

**Data:**
\[
\begin{align*}
k &= 10 + 0.5i \\
u_D(x) &= H_0^{(1)}(k|x|)
\end{align*}
\]

Truncation of the lines: \( T = 24a \)

The trace \( \varphi_0 = \varphi_1 = \varphi_2 = \varphi_3 \)

A posteriori, the solution can be reconstructed outside
Numerical validation (using XLiFE++)

\[ \Delta u + k^2 u = 0 \quad (\mathbb{R}^2 \setminus D) \]

\[ u = u_D \quad (\partial D) \]

Data:

\[ k = 10 + 0.5i \]

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Truncation of the lines: \( T = 24a \)

The trace \( \varphi_0 = \varphi_1 = \varphi_2 = \varphi_3 \)
Numerical validation (using XLiFE++)

We plot the log of the $L^2$ error on $\varphi_j$ as a function of $T$:

![Graph showing the log of the $L^2$ error on $\varphi_j$ as a function of $T$.]

- The smaller $\Im m k$, the slower the convergence versus $T$...
- What happens in the non-dissipative case?
Outline

1. The Half-Space Matching (HSM) Method ($k \not\in \mathbb{R}$)

2. The Complex-scaled HSM Method ($k \in \mathbb{R}$)
The difficulties in the non-dissipative case

\[ \Delta u + k^2 u = 0 \quad (\mathbb{R}^2 \setminus D) \]

\[ u = u_D \quad (\partial D) \]

\[ k \in \mathbb{R}, \ u \text{ outgoing} \]

Similarly to the dissipative case, one can derive half-space representations of the outgoing half-space solutions:

\[ u(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\varphi}_0(\xi) e^{-\sqrt{\xi^2 - k^2}(x_1 - a)} e^{i\xi x_2} d\xi \text{ for } x_1 > a \]

where \( \sqrt{\xi^2 - k^2} = -i \sqrt{k^2 - \xi^2} \) for \( |\xi| < k \).
The difficulties in the non-dissipative case

\[ \Delta u + k^2 u = 0 \quad (\mathbb{R}^2 \setminus D) \]
\[ u = u_D \quad (\partial D) \]

\( k \in \mathbb{R}, \ u \) outgoing

As in the dissipative case, the compatibility of half-space representations leads to a system of integral equations of the form:

**Native HSM system**

\[ (I - D)\Phi = F \]

with \( \Phi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3)^t \)

**Theoretical difficulty**

- \( \varphi_j \notin L^2(\Sigma_j) \) since \( \varphi_j \sim e^{\pm ikt} \sqrt{|t|} \)
- No functional framework for \( D \)
The difficulties in the non-dissipative case

\[ \Delta u + k^2 u = 0 \quad (\mathbb{R}^2 \setminus D) \]
\[ u = u_D \quad (\partial D) \]

\( k \in \mathbb{R}, \ u \ \text{outgoing} \)

As in the dissipative case, the compatibility of half-space representations leads to a system of integral equations of the form:

**Native HSM system**

\[(\mathbb{I} - \mathbb{D}) \Phi = F\]

with \( \Phi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3)^t \)

**Numerically**

the method works well, even for complex problems!
A numerical illustration (Hankel function)

\[ T \log_{10} \text{err} = \begin{cases} k = 1 \\ k = 1 + 0.05i \\ k = 1 + 0.1i \\ k = 1 + 0.2i \end{cases} \]
A numerical illustration (Hankel function)

The error decays like $\frac{1}{\sqrt{T}}$, in accordance with the behaviour

$$\varphi_j \sim e^{\pm ikt} \frac{\sqrt{|t|}}{\sqrt{T}}$$
The HSM method still works for scattering problems without dissipation!

- even for a strongly anisotropic elastic background,
- even if the $\varphi_j$ do not decay at infinity,
- ...
**A Remedy: The Complex-Scaling**

**Main Idea** (in the spirit of PMLs):

for each $x_1$, the function $x_2 \rightarrow u(x_1, x_2)$ has an analytic extension

- from $x_2 \in (a, +\infty)$ to $\Re(x_2) > a$,
- from $x_2 \in (-\infty, -a)$ to $\Re(x_2) < -a$.

and the same exchanging $x_1$ and $x_2$. 

Benefit: $\phi^\theta_0(t)$ is exponentially decaying at infinity.
**Main idea** (in the spirit of PMLs):

for each $x_1$, the function $x_2 \rightarrow u(x_1, x_2)$ has an analytic extension

- from $x_2 \in (a, +\infty)$ to $\Re(x_2) > a$,
- from $x_2 \in (-\infty, -a)$ to $\Re(x_2) < -a$.

Applied to $\varphi_0(\cdot) = u(a, \cdot)$, this allows to define for $0 < \theta < \pi/2$:

$$
\varphi_0^\theta(t) = \begin{cases} 
\varphi_0(t) & -a < t < a \\
\varphi_0(a + (t - a)e^{i\theta}) & t > a \\
\varphi_0(-a + (t + a)e^{i\theta}) & t < -a 
\end{cases}
$$

**Benefit:** $\varphi_0^\theta$ is exponentially decaying at infinity.
A remedy: the complex-scaling

**Native HSM system**

\[(I - D)\Phi = F\]

with \(\Phi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3)^t\)

**Difficulty**

- \(\varphi_j \notin L^2(\Sigma_j)\) since \(\varphi_j \sim e^{\pm ikt} \pm \infty \sqrt{|t|}\)
- No functional framework for \(D\)

**The complex-scaled HSM formulation**

1. Consider as unknowns the complex-scaled traces \(\varphi_j^\theta\):

   \[
   0 < \theta < \pi/2 \quad \varphi_j^\theta(t) = \begin{cases} 
   \varphi_j(t) & -a < t < a \\
   \varphi_j(a + (t - a)e^{i\theta}) & t > a \\
   \varphi_j(-a + (t + a)e^{i\theta}) & t < -a 
   \end{cases}
   \]

2. Establish "half-space" representations with the \(\varphi_j^\theta\)

3. Derive a complex-scaled HSM equation: \((I - D^\theta)\Phi^\theta = F^\theta\)

**Why doing that?**

To recover an \(L^2\) framework: \(\varphi_j^\theta \in L^2(\mathbb{R})\)

| \(\varphi_0^\theta\) | \(\sim\) | \(\frac{e^{-k|t|\sin \theta}}{\sqrt{|t|}}\) |
**Half-space representations**

\[ \Delta u + k^2 u = 0 \quad (\Omega_0) \]
\[ u = \varphi_0 \quad (\Sigma_0) \]

**Fourier representation:**
\[
u(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\varphi}_0(\xi) e^{-\sqrt{\xi^2-k^2}(x_1-a)} e^{i\xi x_2} d\xi
\]

\[\uparrow\quad (\text{Plancherel})\]

**Integral representation:**
\[
u(x) = \int_{\mathbb{R}} \varphi_0(t) g(x_1, x_2, t) dt \quad \text{with}
\]
\[
g(x_1, x_2, t) = \frac{ik(x_1-a)}{2} \frac{H_1^{(1)}(kR)}{R}, \quad R = [(x_1-a)^2 + (x_2-t)^2]^{1/2}.
\]
**Half-space representations**

Integral representation: \( u(x) = \int_{\mathbb{R}} \varphi_0(t) g(x_1, x_2, t) dt \) with

\[
g(x_1, x_2, t) = \frac{ik(x_1 - a) \, H_1^{(1)}(kR)}{2R}, \quad R = [(x_1 - a)^2 + (x_2 - t)^2]^{1/2}.
\]

Remark: The function \( t \to g(x_1, x_2, t) \) is analytic in \( t \) with branch points at \( t = x_2 \pm i(x_1 - a) \).
**Moving the path of integration**

Integral representation: \( u(x) = \int_{\mathbb{R}} \varphi_0(t) g(x_1, x_2, t) \, dt \) for \( x_1 > a \)

The first step consists in moving the path of integration in the complex plane.
Moving the path of integration

Integral representation: \( u(x) = \int_{\mathbb{R}} \varphi_0(t) g(x_1, x_2, t) dt \) for \( x_1 > a \)

Parametrization of the new path:

\[
t = J_\theta(s) := \begin{cases} 
  s & -a < s < a \\
  a + (s - a)e^{i\theta} & s > a \\
  -a + (s + a)e^{i\theta} & s < -a 
\end{cases}
\]
Moving the path of integration

By Cauchy theorem:

\[ u(x) = \int_{\mathbb{R}} \varphi_0(t) g(x_1, x_2, t) dt = \int_{\mathbb{R}} \varphi_0(\mathcal{J}_\theta(s)) g(x_1, x_2; \mathcal{J}_\theta(s)) \mathcal{J}'_\theta(s) ds \]

and \( \varphi_0(\mathcal{J}_\theta(s)) = \varphi_0^\theta(s) \) which is exactly the new unknown!
Moving the path of integration

By Cauchy theorem:

\[ u(x) = \int_{\mathbb{R}} \varphi_0(t) g(x_1, x_2, t) dt = \int_{\mathbb{R}} \varphi_0(J_\theta(s)) g(x_1, x_2; J_\theta(s)) J'_\theta(s) ds \]

and \( \varphi_0(J_\theta(s)) = \varphi^\theta_0(s) \) which is exactly the new unknown!

**WARNING**: due to branch points, this does not hold for all \( x \in \Omega_0 \)!
Moving the path of integration

We get the new exact formula

\[ u(x) = \int_{\mathbb{R}} \varphi_0^\theta(s) g^\theta(x_1, x_2, s) ds \]

where we have set \( g^\theta(x_1, x_2, s) = g(x_1, x_2; \mathcal{J}^\theta(s))\mathcal{J}'^\theta(s) \),

valid only for \( x \) such that

\[ x_1 - a > (|x_2| - a) \tan \theta \]
Before complex-scaling:

\[ u(x) = \int_{\mathbb{R}} \varphi_0(t)g(x_1, x_2; t)dt \]

\( \varphi_0(\cdot)g(x_1, x_2; \cdot) \) slowly decaying

After complex-scaling:

\[ u(x) = \int_{\mathbb{R}} \varphi_0^\theta(t)g^\theta(x_1, x_2, t)dt \]

\( \varphi_0^\theta(\cdot)g^\theta(x_1, x_2, \cdot) \) exp. decaying
We consider \( u(x) = \frac{i}{4} H_0^{(1)}(k|x|) \).

The trace \( \varphi_0 \)

Before complex-scaling

\( u \) in \( \Omega_0 \)
Numerical illustration

We consider \( u(\mathbf{x}) = \frac{i}{4} H_0^{(1)}(k|\mathbf{x}|) \).

The complex-scaled trace \( \varphi_0^\theta \)

After complex-scaling: \( \theta = \pi/9 \)

\( u \) in \( \Omega_0^\theta \)
We consider \( u(x) = \frac{i}{4} H_0^{(1)}(k|x|) \).

The complex-scaled trace \( \varphi_0^\theta \) in \( \Omega_0^\theta \)

After complex-scaling: \( \theta = \pi/4 \)
Derivation of the complex-scaled HSM system

Classical unknowns:

\[ \varphi_0(t) = u(a, t) \]
\[ \varphi_1(t) = u(-t, a) \]
\[ \varphi_2(t) = u(-a, -t) \]
\[ \varphi_3(t) = u(t, -a) \]

New unknowns:

\[ \varphi^\theta_j(t) = \varphi_j(J_\theta(t)) \]

where for \( 0 < \theta < \pi/2 \)

\[ J_\theta(t) = \begin{cases} 
  t & -a < t < a \\
  a + (t-a)e^{i\theta} & t > a \\
  -a + (t+a)e^{i\theta} & t < -a 
\end{cases} \]
Let us derive an equation linking $\varphi_0^\theta$ and $\varphi_1^\theta$.

By definition of $\varphi_1^\theta$:

$$\varphi_1^\theta(s) = u(-\mathcal{J}_\theta(s), a)$$

and for $x \in \Omega_0^\theta$:

$$u(x_1, x_2) = \int_{\mathbb{R}} \varphi_0^\theta(t)g_\theta(x_1, x_2, t)dt$$

On $\Sigma_1 \cap \Omega_0^\theta$, we get the new complex-scaled compatibility relation:

$$\varphi_1^\theta(s) = \int_{\mathbb{R}} \varphi_0^\theta(t)g_\theta(-\mathcal{J}_\theta(s), a, t)dt \quad \text{for } s < -a$$
The complex-scaled HSM system

The initial problem with \( k \in \mathbb{R} \)

\[
\Delta u + k^2 u = 0 \quad (\mathbb{R}^2 \setminus D) \\
u = u_D \quad (\partial D)
\]

has been reformulated as follows:

Find \( \Phi = (\varphi_0^\theta, \varphi_1^\theta, \varphi_2^\theta, \varphi_3^\theta) \in L^2(\mathbb{R})^4 \) such that

\[
\varphi_j^\theta = u_D^j \text{ on } \Sigma_j \cap \partial D \quad (|t| < a) \\
\varphi_j^\theta = S D^\theta (\varphi_{j-1}^\theta) \text{ for } t < -a \quad \text{for } j \in \mathbb{Z}/4\mathbb{Z}. \\
\varphi_j^\theta = D^\theta S (\varphi_{j+1}^\theta) \text{ for } t > a
\]
Find $\Phi = (\varphi_0^\theta, \varphi_1^\theta, \varphi_2^\theta, \varphi_3^\theta) \in L^2(\mathbb{R})^4$ such that

$$\varphi_j^\theta = 0 \text{ on } \Sigma_j \cap \partial D \quad (|t| < a)$$

$$\varphi_j^\theta - SD^\theta(\varphi_{j-1}^\theta) = SD^\theta(u_D^{j-1}) \text{ for } t < -a \quad \text{for } j \in \mathbb{Z}/4\mathbb{Z}.$$ 

$$\varphi_j^\theta - D^\theta S(\varphi_{j+1}^\theta) = D^\theta S(u_D^{j+1}) \text{ for } t > a$$

where $S(\varphi)(t) = \varphi(-t)$ and $D^\theta \varphi(s) \overset{\text{def}}{=} \int_{\mathbb{R}} \varphi(t)K_\theta(s, t)dt$

with a kernel $K_\theta(s, t)$ which is

- rapidly decaying at infinity,
- smooth except when $s = t = a$. 
Properties of the integral operator

\[
D^\theta \varphi(s) = \int_{\mathbb{R}} \varphi(t) K_\theta(s, t) dt
\]

- \( \varphi_- \mapsto D^\theta(\varphi_-) \) is compact, due to the regularity of the kernel.
- \( \varphi_+ \mapsto D^\theta(\varphi_+) \) is not compact, due to the cross-point \((s = t = a)\).
- The operator \( \varphi_+ \mapsto D^\theta(\varphi_+) \) for the wavenumber \( k \in \mathbb{R} \) is equal to the operator \( \varphi_+ \mapsto D(\varphi_+) \) with wavenumber \( ke^{i\theta} \)!

**Theorem**

The essential norm of \( D^\theta \) in \( \mathcal{L} \) is \( \leq 1/\sqrt{2} \).
One can prove, exactly as in the dissipative case, that the complex-scaled HSM formulation

$$(\mathbb{I} - D^\theta)\Phi^\theta = F^\theta$$

is coercive+compact in $L^2$.

The proof of uniqueness is subtle.
One can prove, exactly as in the dissipative case, that the complex-scaled HSM formulation

\[(\mathbb{I} - D^\theta)\Phi^\theta = F^\theta\]

is coercive and compact in $L^2$.

The proof of uniqueness is subtle.

Last but not least, if $\theta < \pi/4$: the solution $u$ can be recovered everywhere a posteriori from the knowledge of $\Phi^\theta$. 
Some straightforward generalizations

One can start the complex-scaling at some distance of the Dirichlet boundary ($b > a$):

\[ \varphi_0^\theta(t) = u(a, \mathcal{J}_\theta(t)) \]

with

\[ \mathcal{J}_\theta(t) = \begin{cases} 
  t & -b < t < b \\
  b + (t - b)e^{i\theta} & t > b \\
  -b + (t + b)e^{i\theta} & t < -b 
\end{cases} \]
More generally, truncation and complex-scaling distances can be chosen independently on each of the 8 semi-infinite lines, depending of the expected behavior of the solution:
Validation on the Hankel function

This is a result obtained with $\theta = \pi/6$: 

\[ \Sigma_1 \quad \Sigma_2 \quad \Sigma_3 \quad \Sigma_0 \]

\[ 2T \]
Validation on the Hankel function

Convergence of the standard and complex-scaled HSM versus $T$:

As expected, the convergence is now exponential!
Application to an anisotropic background

\[ \text{div}(\mathbf{A} \nabla u) + k^2 u = 0 \quad \text{with} \quad \mathbf{A} = \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix} \quad (\alpha, \beta > 0, \alpha \beta - \gamma^2 = 1) \]

Compared to the isotropic case:

- The angle \( \theta \) must satisfy the condition: \( |\gamma| \tan \theta < 1 \)
- The domain of validity of the complex-scaled representation is no longer symmetric:

Isotropic case

Anisotropic case
Numerical illustration

\[ A = \begin{pmatrix} 1 & .5 \\ .5 & 2 \end{pmatrix} \]

Condition on \( \theta \):

\( \theta < \pi/2.6 \)
**Numerical Illustration**

\[ \theta = \pi/4 \]

\[ \theta = \pi/6 \]
Reminder of the **OBJECTIVES**: find a new method

- which does not require the Green function of the whole background
- which works in configurations where PMLs fail

*Combining HSM and complex-scaling offers a good framework.*

**RESULTS FOR THE MODEL PROBLEM**

- The complex-scaled HSM formulation for real $k$ has the same **good properties** as the standard HSM one for complex $k$.
- The complex-scaled HSM method **converges exponentially** with respect to the truncation of the lines.

**ONGOING WORK**

- Numerical analysis of the complex-scaled HSM method
- Explicit bounds versus $k$
- Derive far-field formulae
A good surprise

The complex-scaled HSM method also works in configurations where it is commonly admitted that PMLs fail (anisotropic background). This must be checked in elastodynamics.

An open question

In presence of inverse waves ($\vec{V}_G$ and $\vec{V}_\Phi$ with opposite directions):
- PMLs provide a wrong solution.
- The standard HSM method works!
- Can we adapt the complex-scaled HSM method?

A related subject: the complex-scaled boundary integral equations (PML-BIE) Wangtao Lu et al., 2018
MERCI !

QUESTIONS ?