La méthode HHO (Hybrid High-Order) dans le cas d'une frontière immergée

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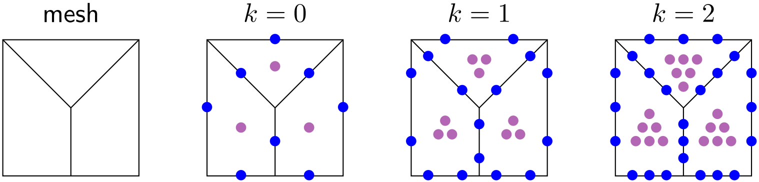
Outline

- General presentation of HHO
- An elliptic interface problem
- The Stokes problem
- Numerical simulations

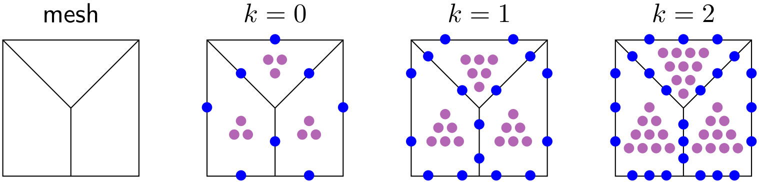
General presentation of HHO

What is HHO?

- Introduced in [Di Pietro, Ern, Lemaire 14; Di Pietro, Ern 15]
- HHO degrees of freedom (dofs) are located on the cells and the faces of the mesh $\longrightarrow$ Polynomials of degree $k \geq 0$



- In the case of **unfitted meshes**, we consider k on the faces and $(k+1)$ on the cells

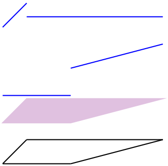


- The discrete problem is assembled cell-wise

What is HHO?

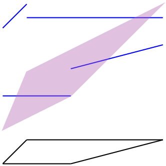
- Representation of 2D unknowns

$k = 0$ (equal-order)



face deg : 0
cell deg : 0

$k = 0$ (mixed-order)

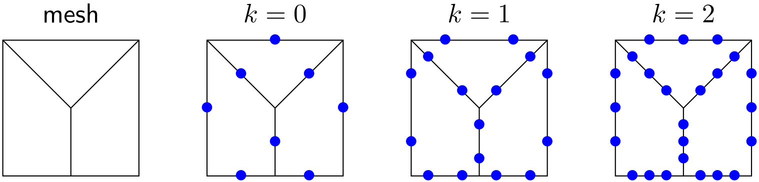


face deg : 0
cell deg : 1

What is HHO?

- Close to the Hybrid Discontinuous Galerkin (HDG) and nonconforming Virtual Element (ncVEM) methods

- The dofs attached to the cells can be **eliminated** by a **local Schur complement** technique (static condensation)



- The global problem comprises only the face dofs

- We can recover the cell dofs by post-processing

Main characteristics

- **General meshes**: polygonal/polyhedral cells, hanging nodes
- **Attractive computational costs**
  - energy error decay $O(h^{k+1})$ with face dofs of order $k \geq 0$
  - global system of size the number of face dofs
- Implementation:
  - open source `diskpp` library [Cicuttin, Di Pietro, Ern 18]
  - available on github `https://github.com/wareHHOuse`

Recent developments in HHO

- **Transport and flows**
  - Stokes [Di Pietro, Ern, Linke, Schieweck 16], NS [Di Pietro, Krell 18]
  - viscoplastic fluids [Cascavita, Bleyer, Chateau, Ern 18]
  - fractured porous media [Chave, Di Pietro, Formaggia 18]

- **Nonlinear mechanics**
  - small defs [Botti, Di Pietro, Sochala 17]
  - hyperelasticity [Abbas, Ern, Pignet 18]
  - elastoplasticity [Abbas, Ern, Pignet 18]
  - Signorini conditions [Cascavita, Chouly, Ern 18]

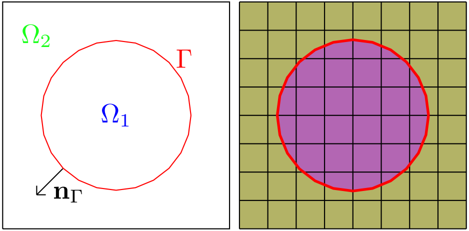
- **Spectral approximation** [Calo, Cicuttin, Deng, Ern 18]

Unfitted HHO

Motivation for unfitted meshes

- Enables the use of simpler meshes to mesh intricate geometries
- Fitted HHO (and other polyhedral methods) is not adapted to treat curvilinear boundaries
- A first work on HHO for elliptic interface problems [Burman, Ern 18]
- Main idea: robustness with respect to bad cuts by **agglomeration of cells** using polyhedral meshes [Johansson, Larson 13]

Elliptic interface problem



- Lipschitz domain $\Omega \subset \mathbb{R}^d$
- Interface Γ, subdomains Ω_1, $\Omega_2 \subset \Omega$

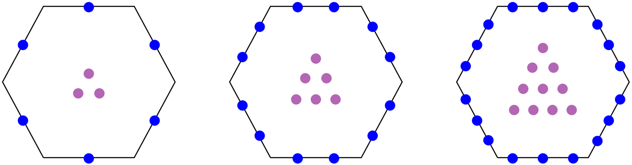
$$\begin{aligned} \kappa_1 \Delta u &= f \text{ in } \Omega_1 \\ \kappa_2 \Delta u &= f \text{ in } \Omega_2 \\ [\![u]\!]_\Gamma &= g_D \text{ on } \Gamma \\ [\![\kappa \nabla u]\!]_\Gamma \cdot \mathbf{n}_\Gamma &= g_N \text{ on } \Gamma \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

with $[\![a]\!]_\Gamma = a_{|\Omega_1} - a_{|\Omega_2}$

Degrees of freedom (1/4)

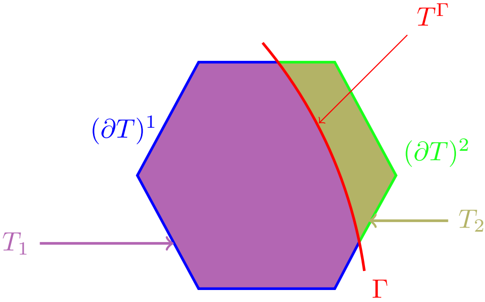
- Let T be a mesh cell in $\mathcal{T}_h$ with unit outward normal $\mathbf{n}_T$

$k=0$ $k=1$ $k=2$



- The local dofs are $u_T \in \mathbb{P}^{k+1}(T)$ on the cell T and the polynomials $u_F \in \mathbb{P}^k(F)$ on every face F composing ∂T
- Generic notation: $\hat{u}_T = (u_T, u_{\partial T})$ with $u_{\partial T} = (u_F)_{F\in\mathcal{F}_T}$

Degrees of freedom (2/4)



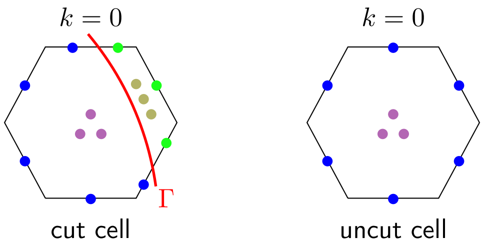
- Decomposition of cut cells

$$\overline{T} = \overline{T_1} \cup \overline{T_2}$$

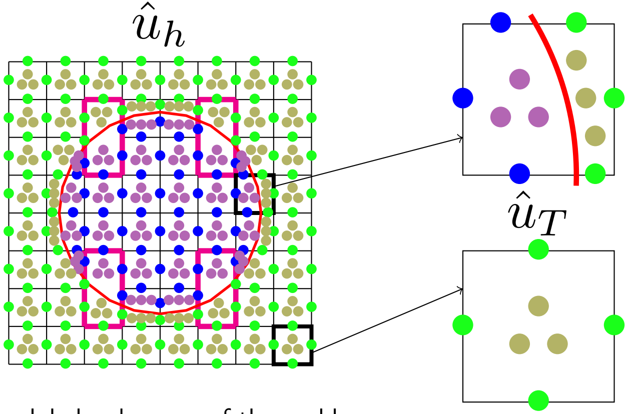
- Decomposition of cut faces

$$\partial(T_1) = (\partial T)^1 \cup T^\Gamma \qquad \partial(T_2) = (\partial T)^2 \cup T^\Gamma$$

Degrees of freedom (3/4)



- We **double** the unknowns on cut cells/faces in the spirit of [Hansbo, Hansbo 02] for cut FEM
- $u_{T_1} \in \mathbb{P}^{k+1}(T_1)$, $u_{T_2} \in \mathbb{P}^{k+1}(T_2)$
- $u_{(\partial T)^1} \in \mathbb{P}^{k}((\partial T)^1)$, $u_{(\partial T)^2} \in \mathbb{P}^{k}((\partial T)^2)$
- $\hat{u}_T = (u_{T_1}, u_{(\partial T)^1}, u_{T_2}, u_{(\partial T)^2})$
- No dofs on T^Γ

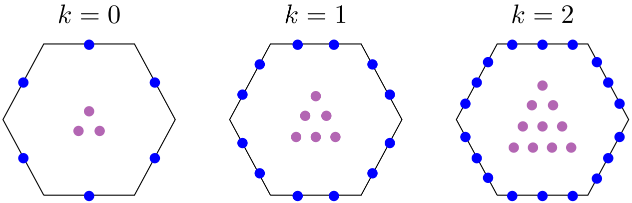


- The global unknowns of the problem are

$$\hat{u}_h \in \prod_{T \in \mathcal{T}_h^1} \mathbb{P}^{k+1}(T_1) \times \prod_{F \in \mathcal{F}_h^1} \mathbb{P}^{k}(F) \times \prod_{T \in \mathcal{T}_h^2} \mathbb{P}^{k+1}(T_2) \times \prod_{F \in \mathcal{F}_h^2} \mathbb{P}^{k}(F)$$

- We collect in $\hat{u}_T$ all the global unknowns related to a mesh cell T

Local discretization: uncut cells (1/2)



Two important ingredients:

- gradient reconstruction $\mathbf{G}_T^k(\hat{u}_T) \in \mathbb{P}^k(T;\mathbb{R}^d)$ s.t. $\forall \mathbf{q} \in \mathbb{P}^k(T;\mathbb{R}^d)$,

$$(\mathbf{G}_T^k(\hat{u}_T), \mathbf{q})_T = -(u_T, \operatorname{div} \mathbf{q})_T + (u_{\partial T}, \mathbf{q}\cdot\mathbf{n}_T)_{\partial T}$$

- stabilization (weakly enforces matching of cell trace and face unknowns)

$$s_T(\hat{u}_T, \hat{v}_T) = h_T^{-1} \sum_{F\in\mathcal{F}_T} (\Pi_F^k(u_F - u_T), v_F - v_T)_F$$

HDG-like stabilization operator (cell unknowns in $\mathbb{P}^{k+1}(T)$)
[Lehrenfeld, Schöberl 16]

Local discretization: uncut cells (2/2)

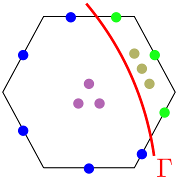
- Local bilinear form

$$a_T(\hat{u}_T, \hat{v}_T) = \kappa_T(\mathbf{G}_T^k(\hat{u}_T), \mathbf{G}_T^k(\hat{v}_T))_T + \kappa_T s_T(\hat{u}_T, \hat{v}_T)$$

- Local right-hand side

$$\ell_T(\hat{v}_T) = (f, v_T)_T$$

Local discretization: cut cells (1/2)



- Two options for the gradient reconstruction
- Option 1: $\mathbf{G}_{T_i}^k(\hat{u}_T) \in \mathbb{P}^k(T_i;\mathbb{R}^d)$ s.t. $\forall \mathbf{q} \in \mathbb{P}^k(T_i;\mathbb{R}^d)$,

$$(\mathbf{G}_{T_i}^k(\hat{u}_T), \mathbf{q})_{T_i} = -(u_{T_i}, \mathrm{div}\, \mathbf{q})_{T_i} + (u_{(\partial T)^i}, \mathbf{q}\cdot\mathbf{n}_T)_{(\partial T)^i} + (u_{T_i}, \mathbf{q}\cdot\mathbf{n}_{T_i})_{T^\Gamma}$$

- Option 2: $\widehat{\mathbf{G}}_{T_i}^k(\hat{u}_T) \in \mathbb{P}^k(T_i;\mathbb{R}^d)$ s.t. $\forall \mathbf{q} \in \mathbb{P}^k(T_i;\mathbb{R}^d)$,

$$(\widehat{\mathbf{G}}_{T_i}^k(\hat{u}_T), \mathbf{q})_{T_i} = -(u_{T_i}, \mathrm{div}\, \mathbf{q})_{T_i} + (u_{(\partial T)^i}, \mathbf{q}\cdot\mathbf{n}_T)_{(\partial T)^i} + (u_{\overline{T_i}}, \mathbf{q}\cdot\mathbf{n}_{T_i})_{T^\Gamma}$$

  where $\overline{T_1} = T_2$, $\overline{T_2} = T_1$
- Stabilization operator

$$s_T(\hat{u}_T, \hat{v}_T) = h_T^{-1} \sum_{i\in\{1,2\}} \kappa_i \sum_{F_i\in\mathcal{F}_{T_i}} (\Pi_{F_i}^k(u_{F_i} - u_{T_i}), v_{F_i} - v_{T_i})_{F_i}$$

Local discretization: cut cells (2/2)

- Local bilinear form

$$\widehat{a}_T(\hat{u}_T, \hat{v}_T) = \kappa_1(\widehat{\mathbf{G}}^k_{T_1}(\hat{u}_T), \widehat{\mathbf{G}}^k_{T_1}(\hat{v}_T))_{T_1} + \kappa_2(\mathbf{G}^k_{T_2}(\hat{u}_T), \mathbf{G}^k_{T_2}(\hat{v}_T))_{T_2} + \eta\kappa_1 h_T^{-1}([\![u_T]\!]_\Gamma, [\![v_T]\!]_\Gamma)_{T^\Gamma} + s_T(\hat{u}_T, \hat{v}_T)$$

- Local right-hand side

$$\widehat{\ell}_T(\hat{v}_T) = \sum_{i\in\{1,2\}} (f, v_{T_i})_{T_i} + (g_N, v_{T_2})_{T^\Gamma} - \kappa_1(g_D, \widehat{\mathbf{G}}^k_{T_1}(\hat{v}_T)\cdot\mathbf{n}_\Gamma)_{T^\Gamma} + \eta\kappa_1 h_T^{-1}(g_D, [\![v_T]\!]_\Gamma)_{T^\Gamma}$$

Convergence result

- We set to zero all the face components attached to $\partial\Omega$
- Global problem: Find $\hat{u}_h$ such that

$$\widehat{a}_h(\hat{u}_h, \hat{v}_h) = \widehat{\ell}_h(\hat{v}_h) \text{ for every } \hat{v}_h \tag{1}$$

$$\text{with} \quad \widehat{a}_h(\hat{u}_h, \hat{v}_h) = \sum_{T \in \mathcal{T}_h} \widehat{a}_T(\hat{u}_T, \hat{v}_T), \qquad \widehat{\ell}_h(\hat{v}_h) = \sum_{T \in \mathcal{T}_h} \widehat{\ell}_T(\hat{v}_T)$$

Theorem

For **every** $\eta > 0$, there is a unique discrete solution $\hat{u}_h$ to (1) s.t.

$$\sum_{T} \sum_{i \in \{1,2\}} \kappa_i \|\nabla(u - u_{T_i})\|_{T_i}^2 \leq C h^{2(k+1)} \sum_{i \in \{1,2\}} \kappa_i |u|_{H^{k+2}(\Omega_i)}^2$$

- **No need for η large enough**

Numerical analysis for the elliptic interface problem

Numerical analysis : assumptions

- The interface is well resolved (mesh fine enough)
- There are no small cut cells (agglomeration)
- We are able to integrate over the curved interface

Inverse inequality

Let $\ell \in \mathbb{N}$ and $i \in \{1,2\}$. For every $T \in \mathcal{T}_h$, and $v_{T_i} \in \mathbb{P}^\ell(T_i)$, we have

$$\|\nabla v_{T_i}\|_{T_i} \leq C h_T^{-1} \|v_{T_i}\|_{T_i}$$

Discrete trace inequality

Let $\ell \in \mathbb{N}$ and $i \in \{1,2\}$. For every $T \in \mathcal{T}_h$, and $v_{T_i} \in \mathbb{P}^\ell(T_i)$, we have

$$\|v_{T_i}\|_{(\partial T)^i} + \|v_{T_i}\|_{T^\Gamma} \leq C h_T^{-\frac{1}{2}} \|v_{T_i}\|_{T_i}$$

Numerical analysis : stability (1/2)

$$\|\hat{u}_T\|_*^2 = \kappa_1\|\nabla u_{T_1}\|_{T_1}^2 + \kappa_2\|\nabla u_{T_2}\|_{T_2}^2 + \eta\kappa_1 h_T^{-1}\|[\![u_T]\!]\|_{T^\Gamma}^2 + \kappa_1 h_T^{-1}\|u_{T_1} - u_{(\partial T)^1}\|_{(\partial T)^1}^2 + \kappa_2 h_T^{-1}\|u_{T_2} - u_{(\partial T)^2}\|_{(\partial T)^2}^2$$

Coercivity

For every $\hat{u}_T \in \hat{U}_T^k$, we have $\|\hat{u}_T\|_*^2 \leq C\widehat{a}_T(\hat{u}_T, \hat{u}_T)$

proof:

$$\widehat{a}_T(\hat{u}_T, \hat{u}_T) = \kappa_1\|\widehat{\mathbf{G}}_{T_1}\|_{T_1}^2 + \kappa_2\|\mathbf{G}_{T_2}\|_{T_2}^2 + \eta\kappa_1 h_T^{-1}\|[\![u_T]\!]\|_{T^\Gamma}^2 + h_T^{-1}\sum_{i\in\{1,2\}}\kappa_i\|\Pi_{(\partial T)^i}^k(u_{T_i} - u_{(\partial T)^i})\|_{(\partial T)^i}^2$$

$$\|\nabla u_{T_1}\|_{T_1} \leq C(\|\widehat{\mathbf{G}}_{T_1}\|_{T_1} + h_T^{-1/2}\|\Pi_{(\partial T)^1}^k(u_{(\partial T)^1} - u_{T_1})\|_{(\partial T)^1} + h_T^{-1/2}\|[\![u_T]\!]\|_{T^\Gamma})$$

$$\|\nabla u_{T_2}\|_{T_2} \leq C(\|\mathbf{G}_{T_2}\|_{T_2} + h_T^{-1/2}\|\Pi_{(\partial T)^2}^k(u_{(\partial T)^2} - u_{T_2})\|_{(\partial T)^2})$$

$$h_T^{-1}\|u_{T_i} - u_{(\partial T)^i}\|_{(\partial T)^i}^2 \leq C(h_T^{-1}\|\Pi_{(\partial T)^i}^k(u_{T_i} - u_{(\partial T)^i})\|_{(\partial T)^i}^2 + \|\nabla u_{T_i}\|_{T_i}^2)$$

Numerical analysis : stability (2/2)

$$\underline{\|\nabla u_{T_1}\|_{T_1} \leq C(\|\widehat{\mathbf{G}}_{T_1}\|_{T_1} + h_T^{-1/2}\|\Pi^k_{(\partial T)^1}(u_{(\partial T)^1} - u_{T_1})\|_{(\partial T)^1} + h_T^{-1/2}\|[\![u_T]\!]\|_{T^\Gamma}):}$$

$$\begin{aligned}\|\nabla u_{T_1}\|^2_{T_1} = (\nabla u_{T_1}, \widehat{\mathbf{G}}_{T_1})_{T_1} &- (\nabla u_{T_1} \cdot \mathbf{n}_T, u_{(\partial T)^1} - u_{T_1})_{(\partial T)^1} \\ &- (\nabla u_{T_1} \cdot \mathbf{n}_\Gamma, u_{T_2} - u_{T_1})_{T^\Gamma}\end{aligned}$$

Cauchy–Schwarz and discrete trace inequalities:

$$\begin{aligned}\|\nabla u_{T_1}\|^2_{T_1} \leq C\|\nabla u_{T_1}\|_{T_1}(\|\widehat{\mathbf{G}}_{T_1}\|_{T_1} &+ h_T^{-1/2}\|\Pi^k_{(\partial T)^1}(u_{(\partial T)^1} - u_{T_1})\|_{(\partial T)^1} \\ &+ h_T^{-1/2}\|[\![u_T]\!]\|_{T^\Gamma})\end{aligned}$$

- We define the interpolation operator (like in [Burman, Ern 18])

$$\hat{I}_T^k(u) = ((\Pi_{T^\dagger}^{k+1} E_1(u))_{|T_1}, \Pi_{(\partial T)^1}^k u, (\Pi_{T^\dagger}^{k+1} E_2(u))_{|T_2}, \Pi_{(\partial T)^2}^k u)$$

  where $E_i : H^{k+1}(\Omega_i) \to H^{k+1}(\mathbb{R}^d)$ is a stable extension, $T^\dagger$ is a simple shape that contains T

- In the analysis, $\Pi_{(\partial T)^i}^k u$ does not play a role, i.e., we do not need robustness w.r.t. cut faces

Approximation

For $v \in H^{k+2}(\Omega_1 \cup \Omega_2)$,

$$\|\mathbf{G}_{T_i}^k(\hat{I}_T^k(v)) - \nabla v\| \leq Ch^{k+1}|u_i|_{H^{k+2}(\Omega_i)}$$
$$\|I_{T_i}^{k+1}(v) - v\| \leq Ch^{k+2}|u_i|_{H^{k+2}(\Omega_i)}$$

Numerical analysis : consistency (1/2)

Consistency

Let $\hat{u}_h$ be the discrete solution and u the exact solution. Assume that u is smooth enough. For every $\hat{v}_h \in \hat{U}_h^k$, we have
$|\widehat{a}_h(\hat{I}_h^k(u) - \hat{u}_h, \hat{v}_h)| \leq C\|\hat{v}_h\|_*(\kappa_1|u_1|^2_{H^{k+2}(\Omega_1)} + \kappa_2|u_2|^2_{H^{k+2}(\Omega_2)})^{1/2}h^{k+1}$

$$\begin{aligned}\widehat{a}_h(\hat{I}_h^k(u) - \hat{u}_h, \hat{v}_h) &= \widehat{a}_h(\hat{I}_h^k(u), \hat{v}_h) - \widehat{\ell}_h(\hat{v}_h) \\ &= \Psi_1 + \Psi_2\end{aligned}$$

$$\begin{aligned}\Psi_1 = \sum_{T\in\mathcal{T}_h} \kappa_1(\widehat{\mathbf{G}}^k_{T_1}(\hat{I}^k_T(u)), \widehat{\mathbf{G}}^k_{T_1}(\hat{v}_T))_{T_1} + \kappa_2(\mathbf{G}^k_{T_2}(\hat{I}^k_T(u)), \mathbf{G}^k_{T_2}(\hat{v}_T))_{T_2} \\ +\kappa_1(\Delta u, v_{T_1})_{T_1} + \kappa_2(\Delta u, v_{T_2})_{T_2} - (g_N, v_{T_2})_{T^\Gamma}\end{aligned}$$

$$\Psi_2 = \sum_{T\in\mathcal{T}_h} s_T(\hat{I}^k_T(u), \hat{v}_T) + \kappa_1 h_T^{-1}([\![I_T^{k+1}(u)]\!]_\Gamma, [\![v_T]\!]_\Gamma)_{T^\Gamma} - \kappa_1 h_T^{-1}(g_D, [\![v_T]\!]_\Gamma)_{T^\Gamma}$$

Numerical analysis : consistency (2/2)

For instance:

$$(\mathbf{G}^k_{T_2}(\hat{I}^k_T(u)), \mathbf{G}^k_{T_2}(\hat{v}_T))_{T_2} = (\mathbf{G}^k_{T_2}(\hat{I}^k_T(u)), \nabla v_{T_2})_{T_2} + (\mathbf{G}^k_{T_2}(\hat{I}^k_T(u))\cdot\mathbf{n}_T, v_{(\partial T)^2} - v_{T_2})_{(\partial T)^2}$$

$$(\Delta u, v_{T_2})_{T_2} = -(\nabla u, \nabla v_{T_2})_{T_2} + (\nabla u\cdot\mathbf{n}_T, v_{T_2})_{(\partial T)^2} - (\nabla u_2\cdot\mathbf{n}_\Gamma, v_{T_2})_{T^\Gamma}$$

$$\begin{aligned}&\kappa_2(\mathbf{G}^k_{T_2}(\hat{I}^k_T(u)), \mathbf{G}^k_{T_2}(\hat{v}_T))_{T_2} + \kappa_2(\Delta u, v_{T_2})_{T_2}\\ &= \kappa_2(\mathbf{G}^k_{T_2}(\hat{I}^k_T(u)) - \nabla u, \nabla v_{T_2})_{T_2} + \kappa_2((\mathbf{G}^k_{T_2}(\hat{I}^k_T(u)) - \nabla u)\cdot\mathbf{n}_T, v_{(\partial T)^2} - v_{T_2})_{(\partial T)^2}\\ &\quad - \kappa_2(\nabla u_2\cdot\mathbf{n}_\Gamma, v_{T_2})_{T^\Gamma}\end{aligned}$$

Convergence result

Theorem

For **every** $\eta > 0$, there is a unique discrete solution $\hat{u}_h$ to (1) s.t.

$$\sum_T \sum_{i\in\{1,2\}} \kappa_i \|\nabla(u - u_{T_i})\|^2_{T_i} \leq Ch^{2(k+1)} \sum_{i\in\{1,2\}} \kappa_i |u|^2_{H^{k+2}(\Omega_i)}$$

$$\sum_T \sum_{i\in\{1,2\}} \kappa_i \|\nabla(u - u_{T_i})\|^2_{T_i}$$
$$\leq 2 \sum_T \sum_{i\in\{1,2\}} \kappa_i (\|\nabla(u - I^{k+1}_{T_i}(u))\|^2_{T_i} + \|\nabla(I^{k+1}_{T_i}(u) - u_{T_i})\|^2_{T_i})$$

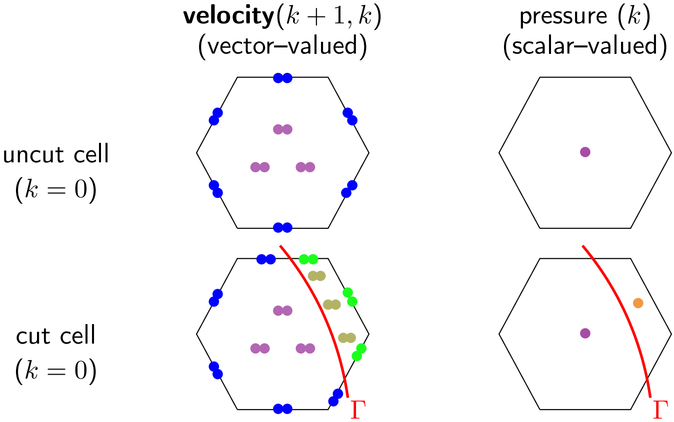
For $\hat{e}_h = \hat{I}^k_h(u) - \hat{u}_h$,

$$\|\hat{e}_h\|^2_* \leq C\widehat{a}_h(\hat{e}_h, \hat{e}_h) \leq C^2 h^{k+1} (\sum_{i\in\{1,2\}} \kappa_i |u|^2_{H^{k+2}(\Omega_i)})^{1/2} \|\hat{e}_h\|_*$$

The Stokes problem

The interface Stokes problem

$$
\begin{aligned}
-\Delta \mathbf{u} + \nabla p &= \mathbf{f} \text{ in } \Omega_1 \cup \Omega_2 \\
\text{div } \mathbf{u} &= 0 \text{ in } \Omega_1 \cup \Omega_2 \\
[\![\mathbf{u}]\!]_\Gamma &= 0 \text{ on } \Gamma \\
[\![\nabla \mathbf{u} - p\mathbb{I}]\!]_\Gamma \mathbf{n}_\Gamma &= \mathbf{g}_N \text{ on } \Gamma \\
\mathbf{u} &= 0 \text{ on } \partial\Omega
\end{aligned}
$$



Local operators

We define:

- Gradient reconstruction operators $\mathbb{G}^k_{T_i}(\hat{\mathbf{u}}_T)$, $\widehat{\mathbb{G}}^k_{T_i}(\hat{\mathbf{u}}_T)$ $\in \mathbb{P}^k(T_i;\mathbb{R}^{d\times d})$ such that for every $\mathbb{Q} \in \mathbb{P}^k(T_i;\mathbb{R}^{d\times d})$,

$$(\mathbb{G}^k_{T_i}(\hat{\mathbf{u}}_T),\mathbb{Q})_{T_i} = -(\mathbf{u}_{T_i},\text{div }\mathbb{Q})_{T_i} + (\mathbf{u}_{(\partial T)^i},\mathbb{Q}\mathbf{n}_T)_{(\partial T)^i} + (\mathbf{u}_{T_i},\mathbb{Q}\mathbf{n}_\Gamma)_{T^\Gamma}$$

$$(\widehat{\mathbb{G}}^k_{T_i}(\hat{\mathbf{u}}_T),\mathbb{Q})_{T_i} = -(\mathbf{u}_{T_i},\text{div }\mathbb{Q})_{T_i} + (\mathbf{u}_{(\partial T)^i},\mathbb{Q}\mathbf{n}_T)_{(\partial T)^i} + (\mathbf{u}_{\overline{T_i}},\mathbb{Q}\mathbf{n}_\Gamma)_{T^\Gamma}$$

- Divergence reconstruction operators $D^k_{T_i}(\hat{\mathbf{u}}_T)$, $\widehat{D}^k_{T_i}(\hat{\mathbf{u}}_T) \in \mathbb{P}^k(T_i)$ such that for every $q \in \mathbb{P}^k(T_i)$,

$$(D^k_{T_i}(\hat{\mathbf{u}}_T),q)_{T_i} = -(\mathbf{u}_{T_i},\nabla q)_{T_i} + (\mathbf{u}_{(\partial T)^i}\cdot\mathbf{n}_T,q)_{(\partial T)^i} + (\mathbf{u}_{T_i}\cdot\mathbf{n}_\Gamma,q)_{T^\Gamma}$$

$$(\widehat{D}^k_{T_i}(\hat{\mathbf{u}}_T),q)_{T_i} = -(\mathbf{u}_{T_i},\nabla q)_{T_i} + (\mathbf{u}_{(\partial T)^i}\cdot\mathbf{n}_T,q)_{(\partial T)^i} + (\mathbf{u}_{\overline{T_i}}\cdot\mathbf{n}_\Gamma,q)_{T^\Gamma}$$

  Note that $D^k_{T_i}(\hat{\mathbf{u}}_T) = \text{Tr}(\mathbb{G}^k_{T_i}(\hat{\mathbf{u}}_T))$, $\widehat{D}^k_{T_i}(\hat{\mathbf{u}}_T) = \text{Tr}(\widehat{\mathbb{G}}^k_{T_i}(\hat{\mathbf{u}}_T))$

- A stabilization operator

$$s_T(\hat{\mathbf{u}}_T,\hat{\mathbf{v}}_T) = h_T^{-1}\sum_{i\in\{1,2\}}\sum_{F_i\in\mathcal{F}_{T_i}}(\Pi^k_{F_i}(\mathbf{u}_{F_i}-\mathbf{u}_{T_i}),\mathbf{v}_{F_i}-\mathbf{v}_{T_i})_{F_i}$$

Local discrete problem

We define the local operators

$$a_T(\hat{\mathbf{u}}_T, \hat{\mathbf{v}}_T) = (\widehat{\mathbb{G}}^k_{T_1}(\hat{\mathbf{u}}_T), \widehat{\mathbb{G}}^k_{T_1}(\hat{\mathbf{v}}_T))_{T_1} + (\mathbb{G}^k_{T_2}(\hat{\mathbf{u}}_T), \mathbb{G}^k_{T_2}(\hat{\mathbf{v}}_T))_{T_2} + s_T(\hat{\mathbf{u}}_T, \hat{\mathbf{v}}_T) + \eta h_T^{-1}([\![\mathbf{u}_T]\!]_\Gamma, [\![\mathbf{v}_T]\!]_\Gamma)_{T^\Gamma}$$

$$b_T(\hat{\mathbf{u}}_T, q_T) = (\widehat{D}^k_{T_1}(\hat{\mathbf{u}}_T), q_{T_1})_{T_1} + (D^k_{T_2}(\hat{\mathbf{u}}_T), q_{T_2})_{T_2}$$

and the local right–hand side

$$\ell_{a_T}(\hat{\mathbf{v}}_T) = (\mathbf{f}, \mathbf{v}_{T_1})_{T_1} + (\mathbf{f}, \mathbf{v}_{T_2})_{T_2} + (\mathbf{g}_N, \mathbf{v}_{T_2})_{T^\Gamma}$$

Global discrete problem

Find $\hat{\mathbf{u}}_h = (\hat{\mathbf{u}}_T)_{T\in\mathcal{T}_h}$ and $p_h = (p_T)_{T\in\mathcal{T}_h}$ such that

$$\sum_{T\in\mathcal{T}_h} a_T(\hat{\mathbf{u}}_T, \hat{\mathbf{v}}_T) - b_T(\hat{\mathbf{v}}_T, p_T) - \gamma_0 h_T([\![\nabla \mathbf{u}_T - p_T\mathbb{I}]\!]_\Gamma \mathbf{n}_\Gamma, [\![\nabla \mathbf{v}_T]\!]_\Gamma \mathbf{n}_\Gamma)_{T^\Gamma}$$

$$= \sum_{T\in\mathcal{T}_h} \ell_{a_T}(\hat{\mathbf{v}}_T) - \gamma_0 h_T(\mathbf{g}_N, [\![\nabla \mathbf{v}_T]\!]_\Gamma \mathbf{n}_\Gamma)_{T^\Gamma}$$

$$\sum_{T\in\mathcal{T}_h} b_T(\hat{\mathbf{u}}_T, q_T) + \gamma_0 h_T([\![\nabla \mathbf{u}_T - p_T\mathbb{I}]\!]_\Gamma \mathbf{n}_\Gamma, [\![q_T]\!]_\Gamma \mathbf{n}_\Gamma)_{T^\Gamma}$$

$$= \sum_{T\in\mathcal{T}_h} \gamma_0 h_T(\mathbf{g}_N, [\![q_T]\!]_\Gamma \mathbf{n}_\Gamma)_{T^\Gamma}$$

for every $\hat{\mathbf{v}}_h = (\hat{\mathbf{v}}_T)_{T\in\mathcal{T}_h}$ and $q_h = (q_T)_{T\in\mathcal{T}_h}$.

An inf-sup condition is satisfied

Theorem

For every $\eta > 0$, for $u \in (H^{k+2}(\Omega_1 \cup \Omega_2))^2$ and $p \in H^{k+1}(\Omega_1 \cup \Omega_2)$, for γ_0 small enough, we have

$$\sum_{T \in \mathcal{T}_h} \sum_{i \in \{1,2\}} \|\nabla(\mathbf{u} - \mathbf{u}_{T_i})\|_{T_i}^2 + \|p - p_{T_i}\|_{T_i}^2$$
$$\leq Ch^{2(k+1)}(\|u\|_{H^{k+2}(\Omega_1 \cup \Omega_2)}^2 + \|p\|_{H^{k+1}(\Omega_1 \cup \Omega_2)}^2)$$

Numerical analysis for Stokes

Stability

- We define the stability norms

$$\|\hat{\mathbf{u}}_T\|_*^2 = \sum_{i\in\{1,2\}} \|\nabla \mathbf{u}_{T_i}\|_{T_i}^2 + h_T^{-1}\|\mathbf{u}_{T_i} - \mathbf{u}_{(\partial T)^i}\|_{(\partial T)^i}^2 + h_T^{-1}\|[\![\mathbf{u}_T]\!]_\Gamma\|_{T^\Gamma}^2$$
$$\|(\hat{\mathbf{u}}_T, p_T)\|_\#^2 = \|\hat{\mathbf{u}}_T\|_*^2 + \|p_{T_1}\|_{T_1}^2 + \|p_{T_2}\|_{T_2}^2$$

Coercivity and continuity of a_T (viscous part)

For every $\hat{\mathbf{u}}_T, \hat{\mathbf{v}}_T \in \hat{U}_T^k$, we have

$$\|\hat{\mathbf{u}}_T\|_*^2 \leq C a_T(\hat{\mathbf{u}}_T, \hat{\mathbf{u}}_T)$$
$$a_T(\hat{\mathbf{u}}_T, \hat{\mathbf{v}}_T) \leq C\|\hat{\mathbf{u}}_T\|_*\|\hat{\mathbf{v}}_T\|_*$$

Stability

- We define the bilinear form

$$A_h((\hat{\mathbf{u}}_h, p_h), (\hat{\mathbf{v}}_h, q_h)) = a_h(\hat{\mathbf{u}}_h, \hat{\mathbf{v}}_h) - b_h(\hat{\mathbf{v}}_h, p_h) + b_h(\hat{\mathbf{u}}_h, q_h) - \gamma_0 \sum_{T \in \mathcal{T}_h} h_T([\![\nabla \mathbf{u}_T - p_T \mathbb{I}]\!]_\Gamma, [\![\nabla \mathbf{v}_T + q_T \mathbb{I}]\!]_\Gamma)_{T^\Gamma}$$

Inf–sup condition

For γ_0 small enough, there exists $c > 0$ such that for every $(\hat{\mathbf{u}}_h, p_h) \in \mathbf{U}_h^k \times P_h^k$,

$$c \|(\hat{\mathbf{u}}_h, p_h)\|_\# \leq \sup_{(\hat{\mathbf{v}}_h, q_h) \in \mathbf{U}_h^k \times P_h^k} \frac{A_h((\hat{\mathbf{u}}_h, p_h), (\hat{\mathbf{v}}_h, q_h))}{\|(\hat{\mathbf{v}}_h, q_h)\|_\#}$$

Stability

- We denote $S = \sup_{(\hat{\mathbf{v}}_h, q_h) \in \mathbf{U}_h^k \times P_h^k} \dfrac{A_h((\hat{\mathbf{u}}_h, p_h), (\hat{\mathbf{v}}_h, q_h))}{\|(\hat{\mathbf{v}}_h, q_h)\|_{\#}}$
- We have

$$\begin{aligned} A_h((\hat{\mathbf{u}}_h, p_h), (\hat{\mathbf{u}}_h, p_h)) = & a_h(\hat{\mathbf{u}}_h, \hat{\mathbf{u}}_h) \\ & + \gamma_0 \sum_{T \in \mathcal{T}_h} h_T (\|[\![p_T]\!]_\Gamma\|_{T^\Gamma}^2 - \|[\![\nabla \mathbf{u}_T]\!]_\Gamma \mathbf{n}_\Gamma\|_{T^\Gamma}^2) \end{aligned}$$

$$\begin{aligned} & c\|\hat{\mathbf{u}}_h\|_*^2 + \gamma_0 \sum_{T \in \mathcal{T}_h} h_T \|[\![p_T]\!]_\Gamma\|_{T^\Gamma}^2 \\ & \leq a_h(\hat{\mathbf{u}}_h, \hat{\mathbf{u}}_h) + \gamma_0 \sum_{T \in \mathcal{T}_h} h_T \|[\![p_T]\!]_\Gamma\|_{T^\Gamma}^2 \\ & \leq A_h((\hat{\mathbf{u}}_h, p_h), (\hat{\mathbf{u}}_h, p_h)) + \gamma_0 \sum_{T \in \mathcal{T}_h} h_T \|[\![\nabla \mathbf{u}_T]\!]_\Gamma \mathbf{n}_\Gamma\|_{T^\Gamma}^2 \end{aligned}$$

and then

$$(c - C\gamma_0)\|\hat{\mathbf{u}}_h\|_*^2 + \gamma_0 \sum_{T \in \mathcal{T}_h} h_T \|[\![p_T]\!]_\Gamma\|_{T^\Gamma}^2 \leq S\|(\hat{\mathbf{u}}_h, p_h)\|_{\#}$$

Stability

- There exists $\mathbf{w} \in H_0^1(\Omega_1)$ such that $p_h = \mathrm{div}\ \mathbf{w}$

$$\|p_h\|_{L^2}^2 = \sum_{T,i} (p_{T_i}, \mathrm{div}\ \mathbf{w})_{T_i} = \Psi_1 + \Psi_2$$

$$\begin{aligned}
\Psi_1 &= b_h(p_h, \hat{I}_h^k(\mathbf{w})) \\
&= a_h(\hat{\mathbf{u}}_h, \hat{I}_h^k(\mathbf{w})) - A_h((\hat{\mathbf{u}}_h, p_h), (\hat{I}_h^k(\mathbf{w}), 0)) \\
&\qquad - \gamma_0 \sum_T h_T([\![\nabla \mathbf{u}_T - p_T \mathbb{I}]\!]_\Gamma \mathbf{n}_\Gamma, [\![\nabla I_T^{k+1}(\mathbf{w})]\!]_\Gamma \mathbf{n}_\Gamma)_{T^\Gamma} \\
&\leq C(S\|\hat{I}_h^k(\mathbf{w})\|_* + \|\hat{\mathbf{u}}_h\|_* \|\hat{I}_h^k(\mathbf{w})\|_* + \gamma_0 \sum_T h_T^{1/2} \|[\![p_T]\!]_\Gamma\|_{T^\Gamma} \|\hat{I}_h^k(\mathbf{w})\|_*)
\end{aligned}$$

Stability

$$\Psi_2 = \sum_T (p_{T_1}, \text{div } \mathbf{w} - \widehat{D}^k_{T_1}(\hat{I}^k_T(\mathbf{w})))_{T_1} + (p_{T_2}, \text{div } \mathbf{w} - D^k_{T_2}(\hat{I}^k_T(\mathbf{w})))_{T_2}$$

$$= \sum_{T,i} (p_{T_i}, \text{div } (\mathbf{w} - I^{k+1}_{T_i}(\mathbf{w})))_{T_i} - (p_{T_i}\mathbf{n}_T, I^k_{(\partial T)^i}(\mathbf{w}) - I^{k+1}_{T_i}(\mathbf{w}))_{(\partial T)^i}$$

$$+ (p_{T_1}\mathbf{n}_\Gamma, [\![I^{k+1}_T(\mathbf{w})]\!]_\Gamma \mathbf{n}_T)_{T^\Gamma}$$

$$= \sum_{T,i} -(\nabla p_{T_i}, \mathbf{w} - I^{k+1}_{T_i}(\mathbf{w}))_{T_i} + ([\![p_T]\!]_\Gamma \mathbf{n}_\Gamma, \mathbf{w} - I^{k+1}_{T_2}(\mathbf{w}))_{T^\Gamma}$$

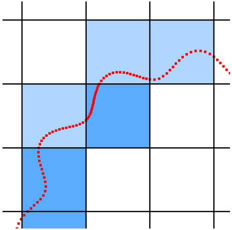
$$\leq C\|\mathbf{w}\|_{H^1}\Big(\sum_{T,i} h_T \|\nabla p_{T_i}\|_{T_i} + \sum_T h_T^{1/2} \|[\![p_T]\!]_\Gamma\|_{T^\Gamma}\Big)$$

$$\|p_h\|^2_{L^2} \leq C\Big(S^2 + \|\mathbf{u}_h\|^2_* + \gamma_0 \sum_T h_T \|[\![p_T]\!]_\Gamma\|^2_{T^\Gamma} + \sum_{T,i} h_T^2 \|\nabla p_{T_i}\|^2_{T_i}\Big)$$

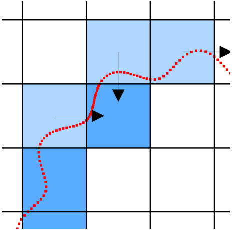
Numerical simulations

Agglomeration procedure (1/4)

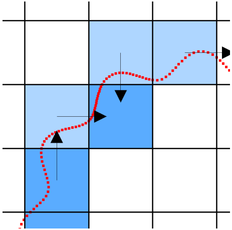
- Small cut cells are agglomerated (see [Johansson, Larson 13])
- The three stages of the procedure



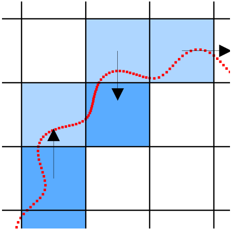
initial mesh



stage 1

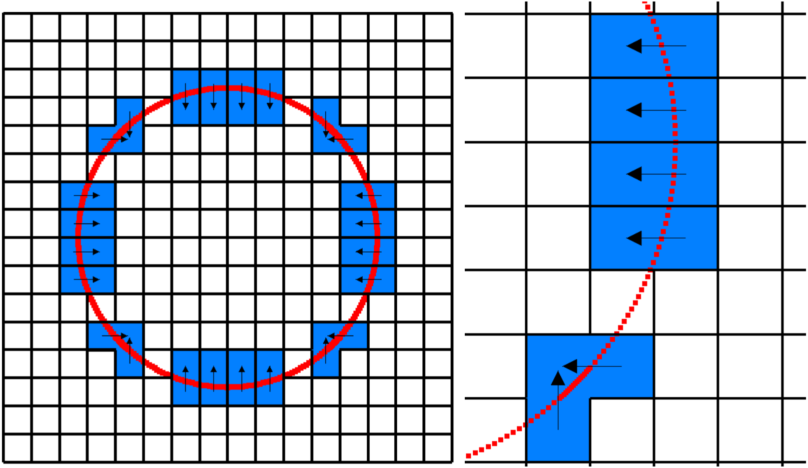


stage 2

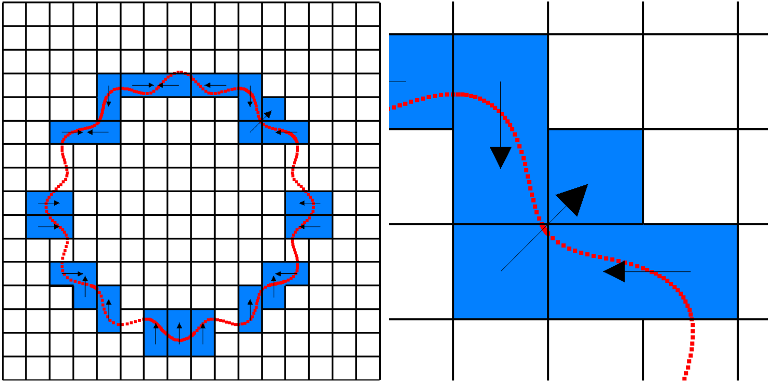


stage 3

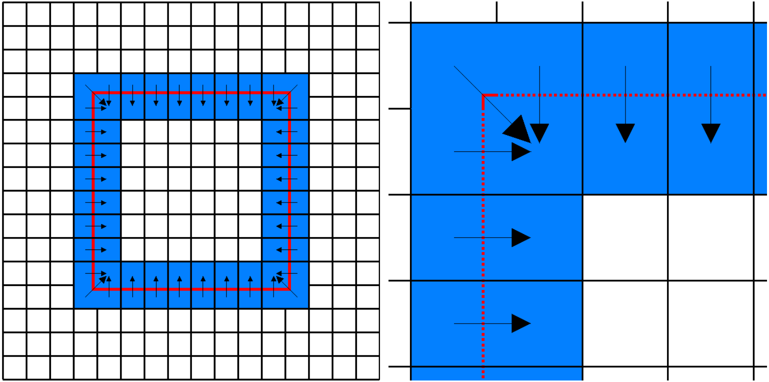
- A 16x16 mesh with a circular interface:



- A 16x16 mesh with a flower-like interface:

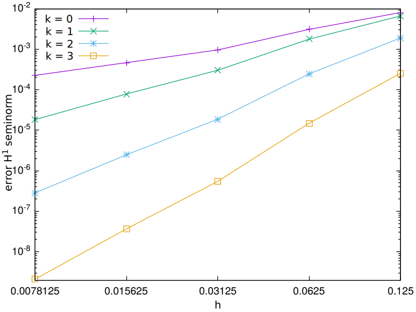


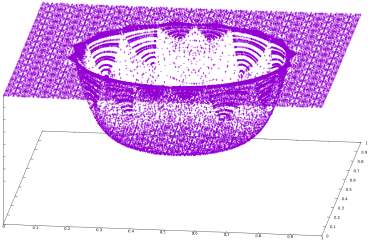
- A 16x16 mesh with a square interface:



- $\kappa_1 = 1$, $\kappa_2 = 10^4$, $g_D = g_N = 0$, $\eta = 1$
- Exact solution $(r^2 = (x_1 - 0.5)^2 + (x_2 - 0.5)^2)$

$$u(x_1, x_2) = \begin{cases} \dfrac{r^6}{\kappa_1} & \text{in } \Omega_1 \\ \dfrac{r^6}{\kappa_2} + R^6\left(\dfrac{1}{\kappa_1} - \dfrac{1}{\kappa_2}\right) & \text{in } \Omega_2 \end{cases}$$

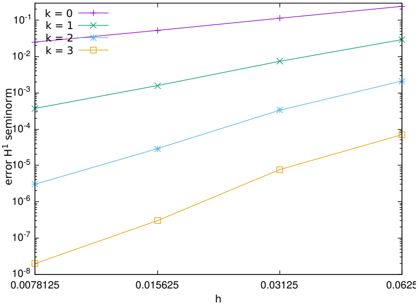


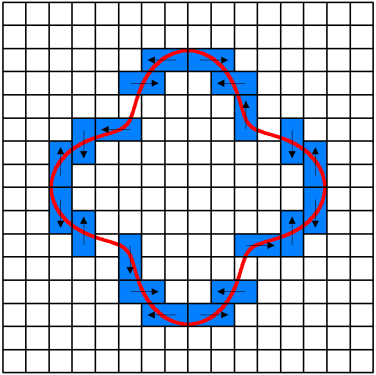


- Exact solution

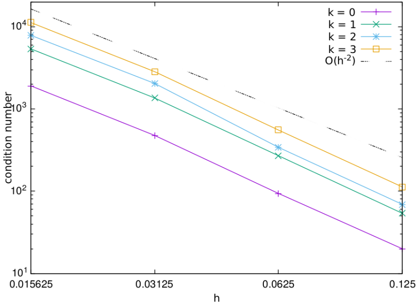
$$u(x_1, x_2) = \begin{cases} \sin(\pi x_1)\sin(\pi x_2) & \text{in } \Omega_1 \\ \sin(\pi x_1)\sin(\pi x_2) + 2 + x^3 y^3 & \text{in } \Omega_2 \end{cases}$$

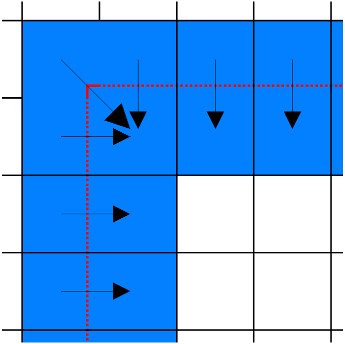
- $\kappa_1 = \kappa_2 = 1$





- Square interface, $\kappa_1 = \kappa_2 = 1$, after static condensation





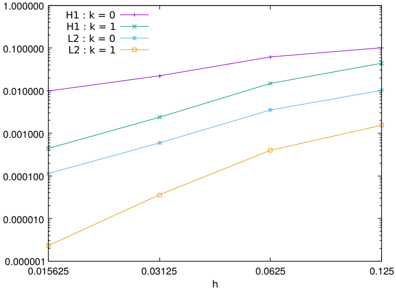
Stokes problem

$$X = x - 0.5 \qquad Y = y - 0.5 \qquad \Omega = \mathcal{C}(0, 0.33)$$

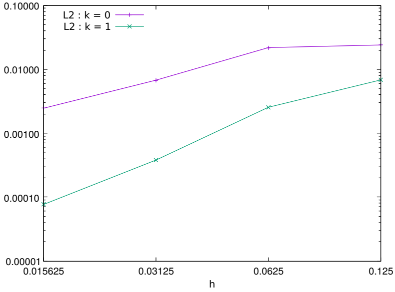
$$u_1 = X^2(X^2 - 2X + 1)Y(4Y^2 - 6Y + 2)$$
$$u_2 = -Y^2(Y^2 - 2Y + 1)X(4X^2 - 6X + 2);$$
$$p = X^5 + Y^5$$

meshes : 8x8, 16x16, 32x32, 64x64; $k = 0, 1$

Velocity error



Pressure error



Wrap up

- Same advantages as fitted HHO
- Usable on curvilinear domains
- Work in progress on interface Stokes problem
- submitted [Burman, Cicuttin, Delay, Ern] (elliptic problem)

Thank you for your attention!