Sparse control of Hegselmann-Krause models: Black hole and declustering

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Nassim Nicholas Taleb uses the term “Black Swan” to describe events that
- Are extremely rare
- Have a massive impact
- Are thought to be retrospectively predictable

Bellomo et al. apply this theory to social competition\(^\text{a}\).
- Rational individual behavior
- Non-linear local interactions
- Irrational collective behavior \(\Rightarrow\) Black Swan phenomenon

Hegselmann-Krause dynamics

**HK model: microscopic**

\[
\dot{x}_i = \frac{1}{N} \sum_{j=1}^{N} a_{ij} (x_j - x_i), \quad x_i \in \mathbb{R}^d, \quad i \in \{1, \ldots, N\}
\]

**HK model: macroscopic**

\[
\partial_t \mu_t + \nabla \cdot \left( \left( \int_{\mathbb{R}^d} a(\|x - y\|)(y - x) d\mu_t(y) \right) \mu_t \right) = 0
\]
Hegselmann-Krause dynamics

HK model: microscopic

\[ \dot{x}_i = \frac{1}{N} \sum_{j=1}^{N} a(\|x_i - x_j\|)(x_j - x_i), \quad x_i \in \mathbb{R}^d, \quad i \in \{1, \ldots, N\} \]

Figure 2 shows only single runs. To get a better feeling of what is going on we run systematically simulations walking along the diagonal. The simulations start with \(0.01\), \(0.02\), \ldots, \(0.4\). (For reasons that become obvious below, there is nothing new and interesting in the parameter space for \(\geq 0.4\)). For each of these 40 steps we repeat the simulation 50 times, always starting with a different random start distribution. Each run is continued until the dynamics becomes stable. Figure 3 gives an overview.

(a) \(0.01\)

(b) \(0.15\)

(c) \(0.25\)

Figure 2: Stops while walking along the diagonal

HK model: macroscopic

\[ \partial_t \mu_t + \nabla \cdot \left( \left( \int_{\mathbb{R}^d} a(\|x - y\|)(y - x) d\mu_t(y) \right) \mu_t \right) = 0 \]
Outline

1. Microscopic model (Finite dimension)
2. Macroscopic model (Infinite dimension)
3. Numerical simulations
Consensus and clustering

Definition
The state characterized by $x_1 = \ldots = x_N$ is referred to as **consensus**.

$$M_c := \{(x_i)_{i\in\{1,\ldots,N\}} \mid \forall (j, k) \in \{1, \ldots, N\}^2, x_j = x_k\}.$$ 

Definition
The system is said to be **clustered** if there exist $(i, j) \in \{1, \ldots, N\}^2$ such that $x_i = x_j$.

$$S_{cl} := \{(x_i)_{i\in\{1,\ldots,N\}} \mid \exists (j, k) \in \{1, \ldots, N\}^2 \text{ s.t. } x_j = x_k\}.$$
Controlled System

Controlled dynamics

\[
\dot{x}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|x_i(t) - x_j(t)\|)(x_j(t) - x_i(t)) + u_i(t), \quad x_i \in \mathbb{R}^d, \quad i \in \{1, \ldots, N\}
\]

where \( a : \mathbb{R}^+ \to \mathbb{R}^+ \) is the interaction function

\( u_i : \mathbb{R}^+ \to \mathbb{R}^d \) are the controls (\( i \in \{1, \ldots, N\} \)).

Sparse control:

\[
u \in U_M := \left\{ u : \mathbb{R}^+ \to (\mathbb{R}^d)^N \mid u \text{ measurable, } \sum_{i=1}^{N} \|u_i(t)\| \leq M \text{ for a.e. } t \in \mathbb{R}^+ \right\}
\]

Aim: control the system away from consensus and clustering.
Looking for a functional to characterize dispersion: the variance

**Variance**

\[
V(x) = \frac{1}{2N^2} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \|x_i - x_j\|^2
\]

**Lemma**

The system \((x_i)_{i \in \{1, \ldots, N\}}\) is in the state of consensus if and only if \(V(x) = 0\).

By extension, \(V(x) = 0 \Rightarrow (x_i)_{i \in \{1, \ldots, N\}}\) is clustered.

**However**, \((x_i)_{i \in \{1, \ldots, N\}}\) is clustered \(\nRightarrow V(x) = 0\)
Looking for a functional to characterize dispersion: the entropy

**Entropy**

\[ W(x) = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \ln \|x_i - x_j\|. \]

**Lemma**

\( \forall (i, j), \|x_i - x_j\| \geq \epsilon \Rightarrow W(x) > K \epsilon. \)

However, the converse does not hold.
Looking for a functional to characterize dispersion: a modified entropy

**Modified entropy**

Let $g \in C^1(\mathbb{R}^+) \) be strictly increasing such that $\lim_{s \to 0} g(s) = -\infty$ and $\lim_{s \to +\infty} g(s) < \infty$.

$$W_g(x) = \frac{1}{2N^2} \sum_{i=1}^{N} \sum_{j=i+1}^{N} g(\|x_i - x_j\|).$$

**Theorem**

$$\forall (i,j), \|x_i - x_j\| \geq \epsilon \iff W_g(x) > K'_\epsilon.$$
Two control strategies

Theorem

Let $R_i := \frac{1}{N} \sum_{j \neq i} (x_i - x_j)$.
Let $i_V := \arg\max_{i \in \{1, \ldots, N\}} \|R_i\|$.
The control $u^V$ defined by
$$u^V_i = \begin{cases} M \frac{R_i}{\|R_i\|} & \text{for } i = i_V \\ 0 & \text{for all } i \neq i_V \end{cases}$$
maximizes $\dot{V}$ instantaneously.

Theorem

Let $S_i := \frac{1}{N} \sum_{j \neq i} g'(\|x_i - x_j\|^2)(x_i - x_j)$.
Let $i_W := \arg\max_{i \in \{1, \ldots, N\}} \|S_i\|$.
The control $u^W$ defined by
$$u^W_i = \begin{cases} M \frac{S_i}{\|S_i\|} & \text{for } i = i_W \\ 0 & \text{for all } i \neq i_W \end{cases}$$
maximizes $\dot{W}_g$ instantaneously.
**Black hole region**

**Definition**

*Black hole region:*

\[ \mathcal{R}_{BH}^M = \{ x_0 \in (\mathbb{R}^d)^N \mid \forall u \in U_M, \exists T > 0, V(x(T)) = 0 \} \].
Condition for the Black Hole

**Theorem**

Suppose that \( \lim_{s \to 0} s a(s) = +\infty \). Then for all \( M > 0 \), there exists \( \epsilon > 0 \) such that if for all \( (i, j) \in \{1, \ldots, N\}^2 \), \( \|x_i(0) - x_j(0)\| < \epsilon \), then given any control \( u \in U_M \), the system converges to consensus in finite time.

**Proof.**

\[
\dot{V} = -\frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=i+1}^{N} a(\|x_i - x_j\|) \|x_i - x_j\|^2 + \frac{1}{2N^2} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \langle x_i - x_j, u_i - u_j \rangle.
\]

Let \( \epsilon > 0 \) such that for all \( s < \epsilon \), \( a(s) \geq \frac{2M}{s} \).

Then while \( \|x_i - x_j\| \leq \epsilon \),

\[
\dot{V} \leq -\frac{M}{N^2} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \|x_i - x_j\| \leq -\frac{\sqrt{2M}}{N} \sqrt{V}
\]

so \( V \to 0 \) in finite time.
Condition for the Black Hole

**Theorem**

Suppose that \( \lim_{s \to 0} sa(s) = +\infty \). Then for all \( M > 0 \), there exists \( \epsilon > 0 \) such that if for all \( (i, j) \in \{1, \ldots, N\}^2 \), \( ||x_i(0) - x_j(0)|| < \epsilon \), then given any control \( u \in U_M \), the system converges to consensus in finite time.
Safety region

**Definition**

\[ \mathcal{R}_S^M = \{ x_0 \in (\mathbb{R}^d)^N | \exists u \in U_M, \exists K \in \mathbb{R}, \forall t \geq 0, W_g(x(t)) \geq K \}. \]
Theorem

Suppose that $\lim_{s \to +\infty} sa(s) = 0$. Then for all bound $M > 0$ on the control, there exists a safety region $\mathcal{R}^M_S \neq \emptyset$. Furthermore, confinement to the safety region can be obtained with the sparse control $u^W \in U_M$.

Proof.

$$\max_u \dot{W}_g = \frac{1}{N} \sum_{i=1}^N \langle S_i, \frac{1}{N} \sum_{k=1}^N a(\|x_i - x_k\|)(x_i - x_k) \rangle + \frac{M}{N} \|S_{iW}\|.$$ 

Since $\lim_{s \to +\infty} sa(s) = 0$, we can suppose $\frac{1}{N} \sum_{k=1}^N a(\|x_i - x_k\|) \|x_i - x_k\| \leq \epsilon$.

Then $\max_u \dot{W}_g \geq \|S_{iW}\| (\frac{M}{N} - \epsilon) \geq 0$. 

$\square$
Theorem

Suppose that \( \lim_{s \to +\infty} sa(s) = 0 \). Then for all bound \( M > 0 \) on the control, there exists a safety region \( \mathcal{R}_s^M \neq \emptyset \). Furthermore, confinement to the safety region can be obtained with the sparse control \( u^W \in U_M \).
Black hole horizon

**Definition**

We define the *black hole horizon* $\mathcal{H}_{\text{BH}}^M$ as the subset of $(\mathbb{R}^d)^N$ given by:

$$\mathcal{H}_{\text{BH}}^M := (\mathbb{R}^d)^N \setminus (\mathcal{R}_{\text{BH}}^M \cup \mathcal{R}_{S}^M).$$

If there is no *safety region* and $\mathcal{R}_{\text{BH}}^M = (\mathbb{R}^d)^N$, the black hole horizon is *infinite*. If $\mathcal{H}_{\text{BH}}^M = \emptyset$ while $\mathcal{R}_{\text{BH}}^M \neq \emptyset$ and $\mathcal{R}_{S}^M \neq \emptyset$, then the state space $(\mathbb{R}^d)^N$ is divided between the black hole and the safety region, and the black hole horizon is *sharp*. 
Consider the interaction function \( a : s \mapsto \frac{1}{s^2} \). Then \( sa(s) = \frac{1}{s} \), and we have

- \( \lim_{s \to 0} sa(s) = +\infty \Rightarrow \text{Existence of a black hole} \)
- \( \lim_{s \to +\infty} sa(s) = 0 \Rightarrow \text{Existence of a safety region} \)

We can show that they satisfy:

\[
\{ x \in \mathbb{R}^{dN} \mid \sum_{i<j} \| x_i - x_j \|^2 < \frac{1}{M^2} \} \subseteq \mathcal{R}^M_{\text{BH}}
\]

and

\[
\{ x \in \mathbb{R}^{dN} \mid \sum_{i<j} \frac{1}{\| x_i - x_j \|^2} < \frac{M^2}{N^2} \} \subseteq \mathcal{R}^M_{\text{S}}.
\]
Partition of the state space into the black hole region and the safety region for $M = 1$, $a : s \mapsto \frac{1}{s^2}$ and $g : s \mapsto -\frac{1}{s}$. With $(N, d) = (2, 1)$, the region enclosed by the red lines is a subset of $\mathcal{R}_{BH}^M$ and the region located outside the blue lines is a subset of $\mathcal{R}_S^M$. The dotted line represents the consensus manifold.
Partition of the state space into the black hole region and the safety region for $M = 1$, $a : s \mapsto \frac{1}{s^2}$ and $g : s \mapsto -\frac{1}{s}$. With $(N, d) = (3, 1)$, the region inside the central cylinder is a subset of $\mathcal{R}^M_{BH}$ and the region outside the hyperbola branches are a subset of $\mathcal{R}^M_{S}$. The grey planes represent the clustering set, and their intersection is the consensus manifold.
Microscopic model (Finite dimension)

Attraction Region

Theorem

Let $M > 0$. If $\lim_{s \to +\infty} sa(s) = +\infty$, then there exists a real number $\mu > 0$ and a set

$$B := \{(x_i)_{i \in \{1, \ldots, N\}} \mid \min_{i \neq j} \|x_i - x_j\| \leq \mu\}$$

such that for any $x(0)$, there exists $T < \infty$ such that $x(T) \in B$, for any $u \in U_M$.

Proof.

Let $A > M$. Let $\mu > 0$ such that $s > \mu \Rightarrow sa(s) > A$. Suppose that for all time $t > 0$, $\|x_i - x_j\| > \mu$. Then

$$\dot{V} = -\frac{1}{N^2} \sum_{i < j} a(\|x_i - x_j\|) \|x_i - x_j\|^2 + \frac{1}{N^2} \sum_{i < j} \langle x_i - x_j, u_i - u_j \rangle \leq \frac{M - A}{N^2} \sum_{i < j} \|x_i - x_j\|$$

$$\leq \frac{\sqrt{2}(M - A)}{N} \sqrt{V}.$$

Then $V \to 0$ in finite time which contradicts $\|x_i - x_j\| > \mu$. 

\[\square\]
Attraction Region

Theorem

Let $M > 0$. If $\lim_{s \to +\infty} sa(s) = +\infty$, then there exists a real number $\mu > 0$ and a set

$$B := \{(x_i)_{i\in\{1,\ldots,N\}} \mid \min_{i \neq j} \|x_i - x_j\| \leq \mu\}$$

such that for any $x(0)$, there exists $T < \infty$ such that $x(T) \in B$, for any $u \in U_M$. 

\[\text{Graphical representation of the theorem.}\]
Collapse prevention

Prevention of consensus:

**Theorem**

Suppose that \( \lim_{s \to 0} sa(s) = 0 \). Then the sparse control strategy \( u^V \) prevents consensus, i.e. there exists \( \delta > 0 \) such that if \( V(0) \leq \delta \), then \( V(t) > V(0) \) for all \( t > 0 \).

Prevention of clustering:

**Theorem**

Suppose that \( \lim_{s \to 0} sa(s) = 0 \). Then the sparse control strategy \( u^W \) prevents clustering. More specifically, there exists \( \kappa > 0 \) such that \( \|x_i(t) - x_j(t)\| > \kappa \) for all \( i, j \) and for all \( t \).
Black Hole, Safety Region, Attraction Region and Collapse Prevention

Four different configurations determined by $s \mapsto sa(s)$

<table>
<thead>
<tr>
<th>$s \to 0$</th>
<th>$s \to \infty$</th>
</tr>
</thead>
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<tr>
<td>$sa(s) \to +\infty$</td>
<td><strong>Black Hole</strong></td>
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<td>$sa(s) \to 0$</td>
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Microscopic model (Finite dimension)
For every $t \geq 0$, let $\mu(t, \cdot) \in \mathcal{P}(\mathbb{R}^d)$ be a probability measure representing the density of agents at time $t$. The **mean-field limit** of the HK model is

$$\partial_t \mu_t + \nabla \cdot (\xi[\mu_t] \mu_t) = 0,$$

where the interaction kernel is defined by

$$\xi[\mu_t](x) = \int_{\mathbb{R}^d} a(\|x - y\|)(y - x) d\mu_t(y).$$

The solution is considered in the weak sense, *i.e.*

$$\forall \phi \in C_c^\infty(\mathbb{R}^d), \quad \frac{d}{dt} \int_{\mathbb{R}^d} \phi(x) d\mu_t(x) - \int_{\mathbb{R}^d} \nabla \phi(x) \cdot \xi[\mu_t](x) d\mu_t(x) = 0.$$
Mean-field limit

Let \( \mu_N := \frac{1}{N} \sum_{i=1}^N \delta_{x_i} \) denote the empirical measure associated with the solution to:

\[
\dot{x}_i = \frac{1}{N} \sum_{j=1}^N a(\|x_i - x_j\|)(x_j - x_i), \quad x_i(0) = x_i^0 \in \mathbb{R}^d, \quad i \in \{1, \ldots, N\}. \quad (1)
\]

**Definition: Mean-field limit**

The PDE

\[
\partial_t \mu + \nabla \cdot (\xi[\mu] \mu) = 0, \quad \mu(0) = \mu^0 \quad (2)
\]

is said to be the **mean-field limit** of (1) if

\[
\mu_N(0) \rightharpoonup_{N \to \infty} \mu(0) \quad \Rightarrow \quad \mu_N(t) \rightharpoonup_{N \to \infty} \mu(t) \quad \text{for all } t \geq 0.
\]

**Sufficient conditions:**

(i) When \( \mu^0 \) is an empirical measure associated with an initial data \( x^0 \in \mathbb{R}^{dN} \), then the dynamics (2) can be rewritten as the system (1).

(ii) The solution \( \mu(t) \) to (2) is continuous with respect to the initial data \( \mu^0 \).

\[
\xi[\mu](x) = \int a(\|x - y\|)(y - x) d\mu(y)
\]
Macroscopic model: well-posedness

Existence and uniqueness (Piccoli-Rossi ’12)

The nonlinear, nonlocal transport PDE

$$\partial_t \mu + \nabla \cdot (\xi[\mu] \mu) = 0, \quad \mu(0) = \mu^0$$

is well-posed if the function $\xi[\cdot] : \mathcal{P}(\mathbb{R}^d) \to C^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ satisfies

- $\xi[\mu]$ is uniformly Lipschitz and uniformly bounded
- $\xi$ is uniformly Lipschitz with respect to the Wasserstein distance

\[ \|\rho_1 - \rho_2\|_{L^1} = O(1); \quad W_1(\rho_1, \rho_2) = O(\delta). \]
Control of macroscopic models

- $\Gamma$-limit: limit of microscopic control (Fornasier, Solombrino...)
- Control by leaders (Albi, Bongini, Colombo, Cristiani, Fornasier, Kalise, Pareschi, Piccoli, Pogodaev, Rossi...)
- Direct control of PDE (Duprez, Morancey, Piccoli, Rossi, Trélat...)

How do we define in infinite dimension:

- The notion of control sparsity?
- The notion of consensus and the variance?
- The notion of clustering and the modified entropy?
Control sparsity in infinite dimension

\[ \partial_t \mu + \nabla \cdot ((\xi[\mu] + \chi_\omega u)\mu) = 0 \]

- In finite dimension, we sought \textbf{sparse} controls, i.e. \( u_j = 0 \) for all \( j \neq i \).
- How to define sparse controls in infinite dimension?

Infinite dimensional control

We define the control as \( \chi_\omega u \), with \( u : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) representing the control strength. Given \( M, c > 0 \), We set the following constraints on \( \omega \) and \( u \):

\[
\begin{align*}
\{ u \in \mathcal{U}_M & := \{ u : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d \text{ measurable} \mid \| u(t, \cdot) \|_{L^\infty(\mathbb{R}^d)} \leq M \} \\
\omega \in \Omega_c & := \{ \omega \subset \mathbb{R}^d \mid |\omega| \leq c \} 
\end{align*}
\]

(3)

where \( |\omega| = \int_\omega dx \) is the Lebesgue measure of \( \omega \).

Remark: Other possible choice of constraint: \( \Omega'_c := \{ \omega \subset \mathbb{R}^d \mid \int_\omega d\mu(x) \leq c \} \).
Macroscopic consensus and variance

**Definition**

For any $x_0 \in \mathbb{R}^d$, the state $\mu = \delta_{x_0}$ is referred to as *consensus*. We define the macroscopic variance

$$V(\mu(t, \cdot)) = \int \int \|x - y\|^2 d\mu_x(t) d\mu_y(t).$$

**Theorem**

$V(\mu) = 0$ if and only if $\mu = \delta_{x_0}$ for some $x_0 \in \mathbb{R}^d$. 
Macroscopic clustering and generalized entropy

**Definition**

If $\mu \in \mathcal{P}(\mathbb{R}^d)$ contains at least one point mass, we say that $\mu$ is in clustered state.

Let $g \in C^1(\mathbb{R}^+) \cap C^0(\mathbb{R}^+)$ be a strictly increasing function such that $\lim_{s \to 0} g(s) = -\infty$ and $\lim_{s \to +\infty} g(s) < \infty$. We define the macroscopic generalized entropy

$$\mathcal{W}_g(\mu(t, \cdot)) = \int\int g(\|x - y\|^2) d\mu_x(t) d\mu_y(t).$$

**Theorem**

Suppose that $\mathcal{W}_g(\mu) \geq K \in \mathbb{R}$ and $\sup_{s > 0} g(s) = m$. Then for all $r$ small enough,

$$\int\int_{\|x - y\| \leq r} d\mu_x d\mu_y \leq \frac{m - K}{-g(r^2)}.$$
Behavior of the kinetic equation without control

Theorem

Let $\mu$ satisfy the kinetic Hegselmann-Krause equation (without control)

$$
\begin{cases}
\partial_t \mu + \nabla \cdot (\xi[\mu] \mu) = 0 \\
\mu(0) = \mu_0
\end{cases}
$$

where the convolution kernel is $\xi[\mu](x) = \int_{\mathbb{R}^d} a(\|x - y\|)(y - x)d\mu(y)$, for a continuous function $a \in C(\mathbb{R}^+, \mathbb{R}^+)$. If $a$ is bounded away from zero, then $\nu(\mu(t, \cdot))$ tends to zero exponentially.
Control of the macroscopic model

\[ \mathcal{W}_g(\mu(t, \cdot)) = \int \int g(\|x - y\|^2) d\mu_x(t) d\mu_y(t). \]

**Proposition**

Consider \( \mu \in \mathcal{P}_c(\mathbb{R}^d) \) and let \( S(x) = \int g'(\|x - y\|^2)(x - y) d\mu(t, y) \). Let \( M, c > 0 \) and let \( u \) and \( \omega \) satisfying the conditions (3). The control \( \chi_{\omega}u \) such that

\[
\omega(t) := \arg\max_{\omega \in \mathcal{\Omega}_c} \int_{\omega} S(x, t) d\mu(t, x); \quad u(x, t) = M \frac{S(x, t)}{\|S(x, t)\|}
\]

maximizes \( \dot{\mathcal{W}}_g \) instantaneously.
**Macroscopic Black Hole and Attraction Region**

**Black hole region:**

\[
\mathcal{R}_M^{BH} = \{ \mu_0 \in \mathcal{P}(\mathbb{R}^d) \mid \forall u, \omega, \exists T > 0, \mathcal{V}(\mu_T) = 0 \}.
\]

**Theorem**

Suppose that \( \lim_{s \to 0} sa(s) = +\infty \). Then for all \( M > 0 \), there exists a black hole region \( \mathcal{R}_M^{BH} \neq \emptyset \). More specifically, for all \( \mu_0 \in \mathcal{R}_M^{BH} \), \( \mathcal{V}(\mu_t) \) tends to 0 in finite time, and this for any control \( u \in \mathcal{U}_M, \omega \in \Omega_c \).

**Attraction region:**

**Theorem**

Suppose that \( \lim_{s \to +\infty} sa(s) = +\infty \). Then there exists a diameter \( d > 0 \), a constant \( \delta \in (0, \frac{1}{2}) \) and a time \( T > 0 \) such that for any \( \mu_0 \in \mathcal{P}(\mathbb{R}^d) \), for any control \( \chi \omega u \),

\[
\int \int_{\|x-y\| \leq d} d\mu_x(T)d\mu_y(T) \geq \delta.
\] (6)
Suppose that $\lim_{s \to 0} sa(s) = 0$. Let $N \in \mathbb{N}$, $N \geq 2$, and $r > 0$ small enough that $N|B(0, r)| \leq c$. Let $R > 4r$ be small enough that for all $s \leq R$, $a(s)(s) \leq \epsilon < M$. Let $\{x_1, \ldots, x_N\} \in (\mathbb{R}^d)^N$ be such that for all $i \neq j$, $2r \leq \|x_i - x_j\| \leq \frac{R}{2}$. If $\mu_0$ satisfies: $\text{supp}(\mu_0) \subset \bigcup_{i=1}^{N} B(x_i, r)$, then for $\omega := \bigcup_{i=1}^{N} B(x_i, r) \subset \Omega_c$ and $u \in U_M$ maximizing $\dot{\mathcal{W}}_g$, $\text{supp}(\mu(t, \cdot)) \subset \bigcup_{i=1}^{N} B(x_i, r)$. 

Diagram: 

- Three points $x_1, x_2, x_3$ arranged in a circle with radius $R$. 
- $x_2$ is at the top, $x_3$ is at the bottom, and $x_1$ is at the left.
- The circle represents $B(0, r)$, with $R > 4r$.
- The set $\omega$ is the union of balls $B(x_i, r)$ for $i = 1, 2, 3$.
Theorem

Suppose that \( \lim_{s \to \infty} sa(s) = 0 \). Let \( N \in \mathbb{N} \), \( N \geq 2 \), and \( r > 0 \) small enough that \( N|B(0, r)| \leq c \). Let \( R > 2r \) be large enough that for all \( s \geq R - 2r \), \( sa(s) \leq \epsilon < M \). Let \( \{x_1, \ldots, x_N\} \in (\mathbb{R}^d)^N \) be such that for all \( i \neq j \), \( 2r \leq \|x_i - x_j\| \geq R \). If \( \mu_0 \) satisfies:

\[
supp(\mu_0) \subset \bigcup_{i=1}^N B(x_i, r),
\]

then for \( \omega := \bigcup_{i=1}^N B(x_i, r) \subset \Omega_c \), there exists \( u \in \mathcal{U}_M \) such that

\[
supp(\mu(t, \cdot)) \subset \bigcup_{i=1}^N B(x_i, r).
\]
Coexistence of the black hole and the safety region

Let $a : s \mapsto \frac{1}{s^2}$. Then $\lim_{s \to 0} s a(s) = +\infty$ and $\lim_{s \to +\infty} s a(s) = 0$, which means that there exist a \textit{black hole} and a \textit{safety region}.

Let us use the generalized entropy $W_g$ defined with $g : s \mapsto -\frac{1}{s}$.

- $x_0 \in \mathcal{R}_{BH}^M$ if $V(0) < \frac{1}{2N^2M^2}$, i.e., if $M < \frac{1}{N\sqrt{2V(0)}}$.
- $x_0 \in \mathcal{R}_S^M$ if $W_g(0) > -\frac{M^2}{2N^4}$, i.e., if $M > N^2\sqrt{-2W_g(0)}$. 

\[
sa(s) \xrightarrow{s \to 0} +\infty
\]
\[
sa(s) \xrightarrow{s \to +\infty} 0
\]
Convergence to the black hole

With $M = 0.16 < \frac{1}{N\sqrt{2V(0)}}$, the system satisfies $x_0 \in \mathcal{R}_{BH}^M$, and converges to the clustering set in finite time. Left: Evolution of the 10 agents’ positions (the controlled agent is in red). Right: Evolution of the generalized entropy.
With $M = 0.16 > N^2 \sqrt{-2W_g(0)}$, the system satisfies $x_0 \in \mathcal{R}_S^M$, and the control $u_W$ manages to steer the system away from the consensus set. Left: Evolution of the 10 agents’ positions (the controlled agent is in red). Right: Evolution of the generalized entropy.
Coexistence of the basin of attraction and collapse prevention

Consider the interaction function \( a : s \mapsto \frac{1}{\sqrt{s}} \). Since \( \lim_{s \to +\infty} sa(s) = +\infty \) and \( \lim_{s \to 0} sa(s) = 0 \), the basin of attraction and collapse prevention are possible.

We fix \( M = 1 \).

- There exists \( T > 0 \) such that \( x(T) \in B := \{(x_i)_{i\in\{1,\ldots,N\}} \mid \min_{i \neq j} \|x_i - x_j\| \leq 1\} \).
- There exists \( \kappa > 0 \) such that \( \forall t \geq 0, \forall (i,j) \in \{1,\ldots,N\}^2, \|x_i(t) - x_j(t)\| \geq \kappa \).
No control can prevent the convergence of the system to the basin of attraction $B = \{(x_i)_{i \in \{1, \ldots, N\}} \mid \min_{i \neq j} \|x_i - x_j\| \leq 1\}$. Left: Evolution of the 10 agents’ positions. Right: Evolution of the generalized entropy.
The control $u_W$ steers the system away from the clustering set. Left: Evolution of the 10 agents’ positions. Right: Evolution of the generalized entropy.
Evolution of 10 agents’ positions in $\mathbb{R}^2$ controlled by $u_W$ with $M = 5$. 

**Time = 3.9**
Numerical simulations

Safety zone

Evolution of 10 agents’ positions in $\mathbb{R}^2$ controlled by $u_W$ with $M = 10$. 
Black hole and safety zone: Entropy

Evolution of $W_g$ (left: $M = 5$; right: $M = 10$).
Future directions

- Sparse kinetic control: control a portion of the population
- Clarify the black hole horizon
- Optimal control
- Second-order clustering: preventing flocking
- Well-posedness of the kinetic aggregation equation