Concentration analysis of brittle damage

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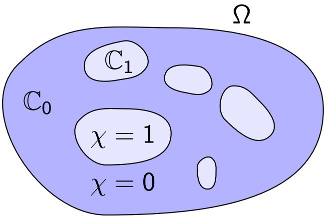
Brittle damage

Let $\Omega \subset \mathbb{R}^n$ be the composition of a "strong" and a "weak" material (= two different, well-ordered, elasticity tensors $\mathbb{C}_0$ and $\mathbb{C}_1$)

$u : \Omega \to \mathbb{R}^n$ elastic displacement
$\chi : \Omega \to \{0,1\}$ damage variable

$\{\chi = 0\} \rightsquigarrow \mathbb{C}_0 \rightsquigarrow$ strong material
$\{\chi = 1\} \rightsquigarrow \mathbb{C}_1 \rightsquigarrow$ weak material



Static energy of the damage process:

$$E(u,\chi) := \int_\Omega \left(\frac{1}{2}\Big((1-\chi)\mathbb{C}_0 + \chi\mathbb{C}_1\Big)\nabla^{sym}u : \nabla^{sym}u + \kappa\chi\right)\mathrm{d}x,$$

where $u \in H^1(\Omega)$, $\chi \in X := L^\infty(\Omega,\{0,1\})$, and $\kappa > 0$.

In the literature

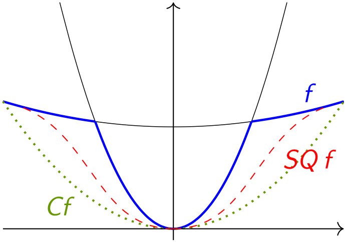
Assume $\mathbb{C}_i\xi := \lambda_i(\operatorname{tr}\xi)\mathbb{I} + 2\mu_i\xi$ (isotropic homogeneous materials), $0 \le \lambda_0 \le \lambda_1$, $0 < \mu_0 \le \mu_1$ (one weaker than the other).

Connection with optimal design: fix $\xi \in \mathbb{R}^{n\times n}_{sym}$,

$$\inf_{\chi\in X}\ \inf_{u=\xi x\ \partial Q} \int_Q \left(\frac{1}{2}\Big((1-\chi)\mathbb{C}_0 + \chi\mathbb{C}_1\Big)\nabla^{sym}u : \nabla^{sym}u + \kappa\chi\right)\mathrm{d}x = \\ = SQ\,f(\xi),$$

for
$f(\xi) := \min\{\frac{1}{2}\mathbb{C}_0\xi : \xi, \frac{1}{2}\mathbb{C}_1\xi : \xi + \kappa\}$

Symmetric quasiconvex envelope:
$$SQ\,f(\xi) := \inf_{\varphi\in C^1_c(Q)} \int_Q f(\xi + \nabla^{sym}\varphi)\,\mathrm{d}x$$



In the literature

First explored in the full gradient case:

$$f(\xi) := \min\{\beta_0|\xi|^2, \beta_1|\xi|^2 + \kappa\}, \qquad \xi \in \mathbb{R}^{n\times n}, \text{ with } \beta_i > 0$$

1. [Kohn–Strang '86] (explicit computation of Qf in 2d when $\beta_0 = +\infty$, $Qf = Rf$ where Rf is the rank-1 convex envelope)
2. [Allaire–Kohn '93] (optimal bounds for the effective behavior of a mixture of two elastic materials via Hashin-Shtrikman principle)
3. [Francfort–Marigo '93] (global stability criterion for evolution)
4. [Dacorogna–Marcellini '95] (necessary and sufficient conditions for the existence of minimizers when $\beta_0 = +\infty$)
5. [Allaire–Francfort '98] (generalization of 1. and 4. to the dimension n)
6. [Allaire–Lods '99] (generalization of 5. to the case $\beta_0 < +\infty$ (double-well) and to the symmetric case)
7. [Francfort–Garroni '06] (relaxed variational evolution)
8. [Allaire-Jouve-Van Goethem '11] (numerical implementations)

A natural question: concentration analysis

Take $\varepsilon << 1$ and set

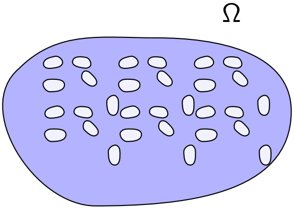
$$E_{\varepsilon}(u,\chi) := \int_{\Omega} \left(\frac{1}{2} \Big((1-\chi)\mathbb{C}_0 + \chi\varepsilon\mathbb{C}_1 \Big) \nabla^{sym} u : \nabla^{sym} u + \frac{\kappa}{\varepsilon}\chi \right) \mathrm{d}x,$$

where $u \in H^1(\Omega)$, $\chi \in X$, and $\kappa > 0$.

One material becomes weaker and weaker in smaller and smaller zones.

In the $L^1 \times L^1$ topology, what about

$$\Gamma\text{-}\lim_{\varepsilon \to 0} E_{\varepsilon}(u,\chi) = ?$$



Remark 1: from quadratic to linear growth

E_ε is quadratic in $\nabla^{sym} u$, but the limit is expected to be linear. Indeed, if $u_\varepsilon \to u$ and $\chi_\varepsilon \to 0$ in $L^1(\Omega) \times L^1(\Omega)$

$$E_\varepsilon(u_\varepsilon, \chi_\varepsilon) := \int_\Omega \Big(\frac{1}{2}\Big((1-\chi_\varepsilon)\mathbb{C}_0 + \chi_\varepsilon \varepsilon \mathbb{C}_1\Big)\nabla^{sym} u_\varepsilon : \nabla^{sym} u_\varepsilon + \frac{\kappa}{\varepsilon}\chi_\varepsilon\Big) dx \geq$$

$$\overset{\text{(Young)}}{\geq} c \int_\Omega \Big((1-\chi_\varepsilon)|\nabla^{sym} u_\varepsilon|^2 + \chi_\varepsilon |\nabla^{sym} u_\varepsilon|\Big) dx.$$

Therefore

$$\|\nabla^{sym} u_\varepsilon\|_1 \leq c$$

Hence u has bounded deformation ($D^{sym} u \in \mathcal{M}_b$), write $u \in BD(\Omega)$, and $\nabla^{sym} u_\varepsilon \mathcal{L}^n \overset{*}{\rightharpoonup} D^{sym} u$ in $\mathcal{M}_b$.

Remark 2: a simple structure in the 1d case

The 1d case is simple since all the envelopes coincide "$Cf = SQf = Rf$".

→ Then, from below:

$$E_\varepsilon(u_\varepsilon, \chi_\varepsilon) := \int_\Omega \left(\frac{1}{2}\Big((1-\chi_\varepsilon) + \chi_\varepsilon \varepsilon\Big)|u_\varepsilon'|^2 + \frac{\kappa}{\varepsilon}\chi_\varepsilon\right)\mathrm{d}x \geq$$

$$\overset{\text{(Young)}}{\geq} \int_\Omega C\Big(\min\{|\cdot|^2/2, \sqrt{2\kappa}|\cdot|\}\Big)(u_\varepsilon')\,\mathrm{d}x.$$

→ From above, the linear slope $\sqrt{2\kappa}|\xi|$ is easy to reach through a piecewise affine recovery sequence.

Hence

$$E_\varepsilon(u,\chi) \xrightarrow{\Gamma} E(u,0) := \int_\Omega Cf(u_\varepsilon')\,\mathrm{d}x + \sqrt{2\kappa}|D^s u|(\Omega),$$

for all $u \in BV(\Omega)$, where $f(\xi) := \min\{|\xi|^2/2, \sqrt{2\kappa}|\xi|\}$.

Remark 3: difficulties in the vector-valued case

The difficulty is to identify the exact linear growth of the volume term.

→ Mismatch: from below

$$\int_\Omega \chi_\varepsilon \Big(\frac{\varepsilon}{2}|\nabla^{sym} u_\varepsilon|^2 + \frac{\kappa}{\varepsilon}\Big)\mathrm{d}x \overset{\text{(Young)}}{\geq} \int_\Omega \chi_\varepsilon \sqrt{2\kappa}|\nabla^{sym} u_\varepsilon|\,\mathrm{d}x.$$

→ From above, the $1d$ construction (along good directions) gives the volume slope ($n = 2$)

$$\begin{cases} \sqrt{2\kappa}|\nabla^{sym} u| = \sqrt{2\kappa}\Big(\sum_i |\xi_i|^2\Big)^{1/2} & \text{if } \xi_1\xi_2 \leq 0 \text{ (rank-1-sym)}, \\ \sqrt{2\kappa}\sum_i |\xi_i| & \text{if } \xi_1\xi_2 > 0. \end{cases}$$

where the ξ_i's denote the eigenvalues of $\nabla^{sym} u$.

This is the correct slope ⇔ the sets where ξ_1^ε and $\xi_2^\varepsilon \sim 1/\varepsilon$ (and with the same sign) are always negligible ⇔ "rigidity".

Results in 2d

Theorem (Babadjian–F.I.–Rindler)

Let $n = 2$. Then

$$E_\varepsilon(u,\chi) := \int_\Omega \Big(\frac{1}{2}\Big((1-\chi)\mathbb{C}_0 + \chi\varepsilon\mathbb{C}_1\Big)\nabla^{sym}u : \nabla^{sym}u + \frac{\kappa}{\varepsilon}\chi\Big)dx$$

Γ-converges in $L^1(\Omega)\times L^1(\Omega)$ to

$$E(u,0) := \int_\Omega Cf(\nabla^{sym}u_\varepsilon)\,\mathrm{d}x + \int_\Omega g\Big(\frac{\mathrm{d}D_s^{sym}u}{\mathrm{d}|D_s^{sym}u|}\Big)\,\mathrm{d}|D_s^{sym}u|, \qquad (1)$$

for all $u \in BD(\Omega)$, where $f(\xi) := \min\{\mathbb{C}_0\xi : \xi/2, g(\xi)\}$ and

$$g(\xi) := \sqrt{2\kappa}(\mathbb{C}_1\xi : \xi + 4\mu_1(\det\xi)^+)^{1/2}.$$

Note: it is possible to force $\operatorname{div} u \in L^2(\Omega)$. In this case the Γ-limit is the sum of (1) with $\nabla^{sym}u$ replaced by its deviatoric part and of a term quadratic in $\operatorname{div} u$ (Tresca plasticity model).

Proof and extension to nd

Idea in 2d: it is possible to add a determinant-type correction to $\varepsilon\mathbb{C}_1$:

$$\int_\Omega \varepsilon \det(\nabla^{sym} u_\varepsilon)^+ \, dx \text{ is small.}$$

This comes from Ball '77:

$$\varepsilon^{1/2} u_\varepsilon \rightharpoonup 0 \ w\text{-}H^1 \quad \Rightarrow \quad \varepsilon \det \nabla u_\varepsilon \overset{*}{\rightharpoonup} 0 \text{ in } \mathcal{M}_b.$$

The n-dimensional case is still an ongoing work. One knows that

$$\varepsilon^{1/2} u_\varepsilon \rightharpoonup 0 \ w\text{-}H^1 \quad \Rightarrow \quad \varepsilon \mathrm{adj}_2(\nabla u_\varepsilon) \overset{*}{\rightharpoonup} 0 \text{ in } \mathcal{M}_b. \tag{2}$$

The pointwise limit of the integrands of E_ε is again Cf, but with

$$g(\xi) := \sqrt{2\kappa}\Big(\mathbb{C}_1\xi : \xi + 4\mu_1 \sum_{i<j} (\xi_i \xi_j)^+\Big)^{1/2},$$

ξ_i being an eigenvalue of ξ. Is (2) enough to ensure that

$$\int_\Omega \varepsilon \sum_{i<j} (\xi_i^\varepsilon \xi_j^\varepsilon)^+ \, dx \text{ is small ?}$$