Estimating and using deformation constraints

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INTRODUCTION
Studying populations of shapes
Idea: characterizing the difference between two shapes thanks to the "best" diffeomorphism transforming one into the other.

_D’Arcy Thompson (On Growth and Form, 1917)_
Definition (S. Arguillère)

A **shape space** on $\mathbb{R}^d$ ($d \in \mathbb{N}^*$) is a manifold $\mathcal{O}$ such that:

- The group of diffeomorphisms of $\mathbb{R}^d$ continuously acts on $\mathcal{O}$
- This action can be differentiated at $id$, giving the infinitesimal action $\xi$

*Diffeomorphometry and geodesic positioning systems for human anatomy*, Miller et al, Technology 2014. ($\mathcal{M} = \mathcal{O}$)
Definition (Large deformation)
For $v \in L^1([0, 1], V)$, we set $\varphi^v$ the flow of $v$:

$$\begin{align*}
\dot{\varphi}^v(t) &= v(t) \circ \varphi^v(t) \\
\varphi^v(0) &= Id
\end{align*}$$

$\rightarrow$ Large deformation: $\varphi^v_t$.
Estimating and using deformation constraints

Introduction

Metric on $V \rightarrow$ metric on $\mathcal{O}$


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Incorporating a structure in large deformations:


- **Higher order momentum** [S. Sommer M. Nielsen, F. Lauze, and X. Pennec. Higher-order momentum distributions and locally affine lddmm registration. SIAM Journal on Imaging Sciences, 2013]


1. Defining easily complex generators
2. Evolution of generators during integration of the flow
3. Ensuring mathematical properties
Introduction

Deformation modules
STRUCTURED VECTOR FIELDS USING A DEFORMATION PRIOR
\[ M = (\mathcal{O}, H, \zeta, \xi, c) \]
Estimating and using deformation constraints

- Structured vector field
- Deformation module: definition and first examples
Estimating and using deformation constraints

Structured vector field

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Structured vector field

Deformation module: definition and first examples

\[ H \]

Controls

\[ \mathcal{O} \times H \]

Geometrical descriptor

Infinitesimal action

\[ TO \]

Cost

\[ C \]

Field generator

\[ \zeta \]

\[ \mathcal{C}_0^\ell(\mathbb{R}^d) \]
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ESTIMATING AND USING DEFORMATION CONSTRAINTS

Modular large deformations

STRUCTURED LARGE DEFORMATION
Estimating and using deformation constraints

- Modular large deformations
- From a deformation module to a deformation model

\[ M = (O, H, \zeta, \xi, c) \]

Field generator

Infinitesimal action

Cost

Controls

Geometrical descriptors

\[ C^\ell_0(\mathbb{R}^d) \]
Definition (Finite energy controlled paths on $\mathcal{O}$)

We denote by $\Omega$ the set of measurable curves $t \mapsto (q_t, h_t) \in \mathcal{O} \times H$ such that:

- $\dot{q}_t = \xi_{q_t}(v_t)$ where $v_t = \zeta_{q_t}(h_t) \in \zeta_{q_t}(H)$
Estimating and using deformation constraints

Modular large deformations

From a deformation module to a deformation model

\[ M = (O, H, \zeta, \xi, c) \]

\( \mathbb{R}^+ \)

\( H \)

\( O \times H \)

\( C^\ell_0(\mathbb{R}^d) \)

Field generator

Infinitesimal action

Geometrical descriptors

Controls

Cost

\[ \zeta \]

\( T \)
Definition (Finite energy controlled paths on \( \mathcal{O} \))

We denote \( \Omega \) the set of measurable curves \( t \mapsto (q_t, h_t) \in \mathcal{O} \times H \) such that:

\[ \dot{q}_t = \xi_{q_t}(v_t) \quad \text{where} \quad v_t = \zeta_{q_t}(h_t) \in \zeta_{q_t}(H) \]

\[ \text{Energy} \ E(q, h) = \int_0^1 c_{q_t}(h_t) dt < \infty \]
Definition

Let $M = (\mathcal{O}, H, \zeta, \xi, c)$ be a $C^k$-deformation module of order $\ell$. We say that $M$ satisfies the **Uniform Embedding Condition (UEC)** if there exists a Hilbert space of vector fields $V$ continuously embedded in $C_0^{\ell+k}(\mathbb{R}^d)$ and a constant $C > 0$ such that for all $o \in \mathcal{O}$ and for all $h \in H$, $\zeta_o(h) \in V$ and

$$|\zeta_o(h)|_V^2 \leq Cc_o(h)$$

Proposition

If $M^l, l = 1 \cdots L$, are $C^k$-deformation modules of order $\ell$ that satisfy UEC, then $\mathcal{C}(M^l, l = 1 \cdots L)$ satisfies UEC.
Definition (Finite energy controlled paths on $\mathcal{O}$)

We denote $\Omega$ the set of measurable curves $t \mapsto (q_t, h_t) \in \mathcal{O} \times H$ such that:

- $\dot{q}_t = \xi_{q_t}(v_t)$ where $v_t = \zeta_{q_t}(h_t) \in \zeta_{q_t}(H)$
- Energy $E(q, h) = \int_0^1 c_{q_t}(h_t) dt < \infty$

$\varphi_{t=1}^{\zeta_q(h)}$ is a modular large deformation.

$\varphi_{t=1}^{\zeta_q(h)} \cdot q_0 = q_1$.

$\varphi^{\zeta_q(h)}$ is defined by $(q_{t=0}, h) \in \mathcal{O} \times L^2([0, 1], H)$. 
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Sub-Riemannian structure on $\mathcal{O} = (\mathcal{O}, H, \zeta, \xi, c)$

$M = (\mathcal{O}, H, \zeta, \xi, c)$

$\mathbb{R}^+$

\[ \mathcal{O} \times H \]

Field generator

Cost $c$

Controls

Geometrical descriptors

Infinitesimal action

$C^\ell_0(\mathbb{R}^d)$

$\zeta$
Estimating and using deformation constraints

Sub-Riemannian structure on $\mathcal{O}$

SUB-RIEMANNIAN STRUCTURE ON $\mathcal{O}$
Proposition

Wet set $\rho : (q, h) \in O \times H \mapsto (q, \xi_q \circ \zeta_q(h)) \in T O$. Then $(O \times H, c, \rho)$ defines a sub-Riemannian structure on $O$ and

$$\text{Dist}(a, b)^2 = \inf \{ \int_0^1 c_q(h) \mid h \in L^2([0, 1], H), \dot{q} = \rho_q(h), q_{t=0} = a, q_{t=1} = b \}$$

Theorem

If $\text{Dist}(a, b) < \infty$ the energy $E$, there exists $(q, h) \in \Omega$ such that $q_{t=0} = a, q_{t=1} = b$ and $\text{Dist}(a, b) = \sqrt{\int_0^1 c_q(h)}$.

Proposition

Let $M = (\mathcal{O}, H, \zeta, \xi, c)$ be a deformation modules satisfying the UEC and $\mu : \mathcal{O} \hookrightarrow \mathbb{R}^+ \ C^1$. Let $a \in \mathcal{O}$ and

$$J_a : h \in L^2([0, 1], H) \mapsto \int_0^1 c_{q_t}(h_t) \, dt + \mu(q_{t=1}, b)$$

with $q_{t=0} = a$ and $(q, h)$ horizontal. Minimizers of $J_a$ can be parametrized by an element $\eta \in T_{a}^*\mathcal{O}$. 
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Applications

Image reconstruction

APPLICATIONS
IMAGE RECONSTRUCTION
Estimating and using deformation constraints

- Applications
- Image reconstruction
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Applications

Image reconstruction
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Goal:

- Using $I_0$ as a prior to reconstruct from data $g$
- With a user-defined deformation module $M = (O, H, \zeta, \xi, c)$

Strategy: using geodesics parametrized by $(a, \eta) \in T^*_aO$ to transform $I_0$.

\[
J_{I_0,g}(a, \eta) = \int_0^1 c_{qt}(h_t)dt + \frac{1}{\lambda} D\left( T(\varphi^{\zeta q(h)}_{t=1} \cdot I_0), g \right)
\]

with $(q, h)$ the geodesic parametrized by $(a, \eta)$. 
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Applications

Image reconstruction

[B.G., Incorporation of a deformation prior in image reconstruction (2018)]

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- Image reconstruction
Estimating and using deformation constraints

Applications

Studying a population of genes (joint work with L. Mégret)
Huntington disease:

- Caused by an elongated polyglutamin (PolyQ)
- Worsens with size of PolyQ and age

Data: (Mouse allelic series, HDinHD)

- Expression level of genes in mice
- 3 ages and 6 sizes of PolyQ: 18 data points per gene

Goal:

- Study behavior of genes
- Compensation/decompensation

Strategy:

- Each gene is considered as a surface in dimension 3
- Compare 2 surfaces via modular deformations
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- Applications
- Studying a population of genes (joint work with L. Mégret)
Estimating and using deformation constraints

Applications

Studying a population of genes (joint work with L. Mégret)
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Applications

Studying a population of genes (joint work with L. Mégret)
Define only $\rho = \xi \circ \zeta$ and approximate

$$\varphi^h \cdot S \simeq S + \rho(S, h_0) \doteq h_0 \cdot S$$

For two surfaces (genes) $S_i$ and $S_j$,

$$h_{i,j} = \text{argmin}\{C(h) + D(h \cdot S_i, S_j)\}$$

[McVaillant, J. Glaunes. Surface Matching via Currents]

Clustering from $\left(C(h_{i,j})\right)_{i,j}$
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Conclusion

- Geometric study of data
- Structure coming from shapes
- Controls to manage variability
"Is it possible to mechanize human intuitive understanding of biological pictures that typically exhibit a lot of variability but also possess characteristic structure?"

Ulf Grenander

*Hands : a Pattern Theoric Study of Biological Shapes, 1991*

Questions?