Ondes périodiques dans un contexte hamiltonien

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Periodic waves
Perturbations of periodic waves

Localized perturbation

Coperiodic perturbation

Subharmonic perturbation

Modulation
Notions of stability


Orbital stability

Modulational stability
Stability criteria in Hamiltonian framework

Not known for:

Based on action integral and basic conservation laws for:

Open in general - require additional conservation laws - for:

Criterion also encoded by action integral for:
Hamiltonian framework

Dispersive perturbations of hyperbolic PDEs

- Generalized Korteweg–de Vries equation

\[ \partial_t v = \partial_x(\delta e[v]), \quad e = f(v) + \frac{1}{2} \kappa(v)v_x^2, \quad \mathcal{H} = e(v,v_x). \]

\[ \delta e[v] := \partial_v e(v,v_x) - (\partial_{v_x} e(v,v_x))_x = f'(v) - \frac{1}{2} \kappa'(v)v_x^2 - \kappa(v)v_{xx} \]

- Euler–Korteweg system

\[
\begin{align*}
\partial_t v &= \partial_x w, \\
\partial_t w &= \partial_x(\delta e[v]), \\
\partial_t \rho + \partial_x(\rho u) &= 0, \\
\partial_t u + u\partial_x u + \partial_x(\delta E[\rho]) &= 0,
\end{align*}
\]

\[ \mathcal{E} = F(v) + \frac{1}{2} K(\rho)\rho_x^2, \quad \mathcal{H} = \frac{1}{2} \rho u^2 + \mathcal{E}(\rho, \rho_x). \]

- Ex: shallow water with capillarity, superfluids, quantum fluids, fluid formulation of nonlinear Schrödinger equation.

... endowed with Hamiltonian structure
Travelling wave equations

Impulse $Q(U) := \frac{1}{2} U \cdot B^{-1} U$ is such that $U = B \nabla Q(U)$.

Hence $U = U(x - ct)$ solves Hamiltonian system $\partial_t U = B \partial_x (\delta H[U])$

iff $B \partial_x (\delta (H + cQ)[U]) = 0$, or equivalently $\delta (H + cQ)[U] \equiv \lambda$.

Constrained energy (Lagrangian)

$$\mathcal{L}[U] := H[U] + cQ(U) - \lambda \cdot U.$$ 

Integrated profile equation

$$U_x \cdot \nabla U_x \mathcal{L}(U, U_x) - \mathcal{L}(U, U_x) = \mu$$

Speed $c$, Lagrange multiplier $\lambda \in \mathbb{R}^N$ and energy level $\mu$ (locally) determine periodic waves.
For a periodic wave of speed $c$, Lagrange multiplier $\lambda$, energy level $\mu$, spatial period $\Xi = \Xi(c, \lambda, \mu)$, and profile $U = U(\xi; c, \lambda, \mu)$,

$$\Theta(c, \lambda, \mu) := \int_0^\Xi (\mathcal{H}[U] + cQ(U) - \lambda \cdot U + \mu) \, d\xi = \int_0^\Xi U_x \cdot \nabla U_x \mathcal{L}(U, U_x) \, d\xi$$

**Remark** it goes to zero in small amplitude limit, and to Boussinesq’s moment of instability $\mathcal{M}(c)$ in soliton limit.

**Crucial fact** - because $\Theta$ is an action integral

$$\Theta_\mu = \int_0^\Xi d\xi \ , \ \nabla_\lambda \Theta = -\int_0^\Xi U \, d\xi \ , \ \Theta_c = \int_0^\Xi Q(U) \, d\xi.$$  

**Message # 1** Our stability criteria are actually encoded by the Hessian matrix $\nabla^2 \Theta$. 

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Action integral

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Stability w.r.t. coperiodic perturbations

**Theorem** [B.–Mietka–Rodrigues’16] If $\Theta_{\mu\mu} \neq 0$ and $\det(\nabla^2 \Theta) \neq 0$,
- if $n(\nabla^2 \Theta) - N$ is odd then wave is spectrally unstable;
- if $n(\nabla^2 \Theta) = N$ then wave is - conditionally - orbitally stable.

Case $N = 1$ subsumes earlier results by Johnson et al on gKdV.
Also see [Deconinck–Kapitula’10], [Natali–Neves’13],...

Case $N = 2$ is newer, and applies to EK.
Also see [Gallay–Haragus’07], [Hakkaev–Stanislavova–Stefanov’14], [Bronski–Johnson–Kapitula’14].

Analogous to Grillakis–Shatah–Strauss alternative for solitary waves
- if $M''(c) < 0$ then the solitary wave is unstable;
- if $M''(c) > 0$ then the solitary wave is stable.

Also see [Angulo Pava’09], [De Bièvre–Genoud–Rota Nodari’14].
Modulational stability

As observed in [B.–Noble–Rodrigues’13], modulated equations read

\[
\begin{align*}
\partial_T (\partial_\mu \Theta) &+ c \partial_X (\partial_\mu \Theta) - (\partial_\mu \Theta) \partial_X c = 0, \\
\partial_T (\nabla_\lambda \Theta) &+ c \partial_X (\nabla_\lambda \Theta) + (\partial_\mu \Theta) B \partial_X \lambda = 0, \\
\partial_T (\partial_c \Theta) &+ c \partial_X (\partial_c \Theta) - (\partial_\mu \Theta) \partial_X \mu = 0.
\end{align*}
\]

Therefore

- they are evolutionary iff \( \det(\nabla^2 \Theta) \neq 0 \),
- if this is the case and \( \Theta_{\mu\mu} \neq 0 \), they are hyperbolic iff \( S \nabla^2 \Theta \) is \( \mathbb{R} \)-diagonalizable, where

\[
S := \begin{pmatrix}
0 & 0 & -1 \\
0 & B^{-1} & 0 \\
-1 & 0 & 0
\end{pmatrix}.
\]
KdV case

Principal minors of $\nabla^2 \Theta$ (log scale)

Discrepancies in computed eigenvalues of Whitham eqs

Signs $(+, -, -) \Rightarrow n(\nabla^2 \Theta) = 1$

Coperiodic stability & modulational stability.

[Deconinck–Kapitula’10] [Whitham’1967]
Cubic NLS / shallow water EK case

Principal minors of $\nabla^2 \Theta$ (log scale)

Discrepancies in computed eigenvalues of Whitham eqs

Signs $(+, - , +, +) \Rightarrow n(\nabla^2 \Theta) = 2$

Coperiodic stability

Message # 2 Large wave-lengths are hard to capture.

& modulational stability.

[Gallay–Haragus’07]
Principal minors of $\nabla^2 \Theta$ (log scale)

Signs $(+, -, -) \Rightarrow n(\nabla^2 \Theta) = 1$

Coperiodic stability.

[Natali–Neves'13]

Real part of eigenvalues of Whitham eqs

Modulational instability of small amplitude waves.

[Haragus–Kapitula’08], [Bronski–Johnson’10].
EK–Boussinesq case

Principal minors of $\nabla^2 \Theta$ (log scale)

Real part of eigenvalues of Whitham eqs

Signs $(+, -, +, +) \Rightarrow n(\nabla^2 \Theta) = 2$

Coperiodic stability & modulational instability of small amplitude waves.
EK–Boussinesq case with coperiodic stability transition

Principal minors of $\nabla^2 \Theta$ (log scale)

Many more cases in Mietka’s thesis

Signs $(+, -, +, +) \Rightarrow n(\nabla^2 \Theta) = 2$ hence stability for small amplitude, signs $(+, -, +, -) \Rightarrow n(\nabla^2 \Theta) = 3$ hence instability for wavelength larger than transition point. [Hakkaev–Stanislavova–Stefanov'14]
**Aim** Investigate analytically **small amplitude** and **large wavelength** limits, thanks to reduced profile equation
\[
\frac{1}{2} \kappa(v) v_x^2 + W(v; \lambda, c) = \mu.
\]

When \( \mu \) goes to \( \mu_s \), almost solitary wave.
When \( \mu \) goes to \( \mu_0 \), almost harmonic wave.

**Action reads**
\[
\Theta = 2 \int_{v_2(\mu,\lambda,c)}^{v_3(\mu,\lambda,c)} \sqrt{2\kappa(v)} \left( \mu - W(v; c, \lambda) \right) dv
\]
By crucial fact, \( \nabla \Theta = 2 \int_{v_2(\mu,\lambda,c)}^{v_3(\mu,\lambda,c)} \sqrt{\frac{2\kappa(v)}{Z(v; \mu, \lambda, c)}} \nabla Z(v; \mu, \lambda, c) \, dv, \)

\( Z(v; \mu, \lambda, c) := \mu - W(v; \lambda, c). \)

Factoring out singularity by Taylor formula and change of variable

- \( v = \sigma v_2 + (1 - \sigma)v_3, \quad \nabla \Theta = \int_0^1 Y \nabla Z \frac{d\sigma}{\sqrt{\sigma(\rho + \sigma)}}, \quad \rho := \frac{v_2 - v_1}{v_3 - v_2}, \)

- \( v = v_2 + (1 + \sin \theta)\delta, \quad \nabla \Theta = \int_{-\pi/2}^{\pi/2} Y \nabla Z d\theta, \quad \delta := \frac{v_3 - v_2}{2}. \)

**Solitary** wave limit is when \( \rho \) goes to zero.

**Harmonic** wave limit is when \( \delta \) goes to zero.
Harmonic limit

**Theorem** [B.–Mietka–Rodrigues'17] There are vectors $V, W, Z, T$, positive numbers $\gamma, \varepsilon$, real numbers $a, b$, such that

$$
\nabla^2 \Theta = a \, V \otimes V + b \left( V \otimes W + W \otimes V \right) + 2 \gamma \, W \otimes W \\
+ \gamma \left( Z \otimes V + V \otimes Z \right) - \varepsilon \, T \otimes T + O(\delta^2).
$$

Furthermore, vectors $V, W, Z, T$ satisfy orthogonality properties involving matrix $S$

$$(V \cdot S \, V = 0, V \cdot S \, W = 0, V \cdot S \, T = 0, T \cdot S \, T = 0, T \cdot S \, Z = 0)$$

and we can eventually compute $n(\nabla^2 \Theta) = n(S \, \nabla^2 \Theta \, S)$ by looking at quadratic form in suitable basis, according to Sylvester’s rule.

**Corollary** [B.–Mietka–Rodrigues'17] Small amplitude waves are 'generically' stable with respect to coperiodic perturbations.
**Soliton limit**

**Theorem** [B.–Mietka–Rodrigues'17] There are vectors $V, W, Z, T$, positive numbers $\eta, \gamma, \varepsilon$, real numbers $a, b$, and a symmetric matrix $D$ such that

$$\nabla^2 \Theta = \eta \frac{\rho + 1}{\rho^2} V \otimes V + (a V \otimes V + b (V \otimes W + W \otimes V)) \ln \rho$$

$$+ (2 \gamma W \otimes W + \gamma (Z \otimes V + V \otimes Z) + \varepsilon T \otimes T) \ln \rho$$

$$+ D + O(\rho \ln \rho),$$

and $V \cdot SDS V = M''(c)$, plus orthogonality properties.

**Corollary** [B.–Mietka–Rodrigues'17] Close enough to a soliton

- period $\Xi$ is monotonically increasing with $\mu$, i.e. $\Xi_{\mu} = \Theta_{\mu\mu} > 0$;
- if $M''(c) \neq 0$ then $\nabla^2 \Theta$ is nonsingular;
- if in addition $M''(c) > 0$ then $n(\nabla^2 \Theta) = N$. 
Stability of long wavelength periodic waves

It has been known since [Gardner’97] that large wavelength periodic waves are unstable whenever the limiting solitary wave is unstable.

In the other way round, from previous Corollary we have

**Theorem** [B.–Mietka–Rodrigues’17] If $M''(c) > 0$ then nearby cycles inside the homoclinic loop are associated with periodic waves that are orbitally stable under coperiodic perturbations.
Back to modulated equations

Alternative form based on **averaged Hamiltonian** $H := \langle \mathcal{H}[\mathbf{U}] \rangle$

and new variable

$$\alpha := \Xi \left( \langle Q(U) \rangle - Q(\langle U \rangle) \right) = \int_0^\Xi (Q(U) - Q(\langle U \rangle)) \, d\xi.$$ 

**Proposition** $H$ can be viewed as function of $(k, \alpha, M := \langle U \rangle)$,

$$\partial_k H = \Theta - \alpha c, \quad \partial_\alpha H = -k c, \quad \nabla_M H = \langle \delta \mathcal{H}[\mathbf{U}] \rangle.$$ 

Modulated equations equivalently read

$$\begin{align*}
\partial_T k &= \partial_X (\partial_\alpha H), \\
\partial_T \alpha &= \partial_X (\partial_k H), \\
\partial_T M &= B \partial_X (\nabla_M H).
\end{align*}$$

[Gavrilyuk–Serre’95], [B.–Mietka–Rodrigues’17].
### Quasilinear form of modulated equations

\[
\begin{align*}
\partial_T k &= H_{k\alpha} \partial_X k + H_{\alpha\alpha} \partial_X \alpha + H_{\alpha M} \cdot \partial_X M \\
\partial_T \alpha &= H_{k\alpha} \partial_X k + H_{k\alpha} \partial_X \alpha + H_{k M} \cdot \partial_X M \\
\partial_T M &= B H_{k M} \partial_X k + B H_{\alpha M} \partial_X \alpha + B H_{M M} \partial_X M
\end{align*}
\]

- **Small amplitude limit** ($\alpha \to 0$): \(\partial_k H = \Theta - \alpha c \to 0, -H_{\alpha k} \to V_g\), a **double** eigenvalue associated with **Jordan** block.
- **Soliton limit** ($k \to 0$): \(\partial_\alpha H = -k c \to 0, -H_{\alpha k} \to c_s\), again a **double** eigenvalue associated with **Jordan** block.

**Message #3** Modulated equations are at best **weakly hyperbolic** in these limits.

**Benjamin–Feir type criterion** A reduced, modulational stability condition in these limits is, e.g. in case \(N = 1\)

\[
(H_{k\alpha} - H_{MM}) \left( (H_{k\alpha} - H_{MM}) H_{kk} H_{\alpha\alpha} + H_{kk} H_{\alpha M} + H_{\alpha\alpha} H_{k M} \right) > 0.
\]
Gurevich–Pitaevskii problem

Amounts to proving the existence of dispersive shocks

Message # 4 Gurevich-Pitaevskii problem is not solved yet in general!