Adiabatic control of quantum mechanical systems

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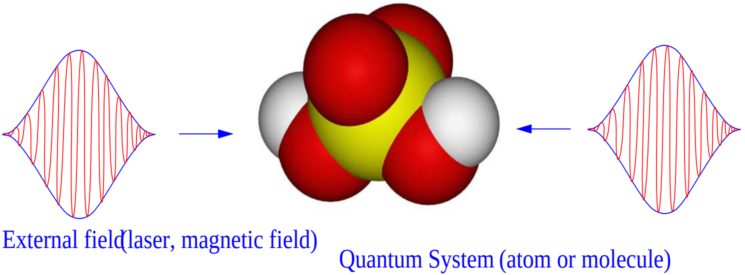
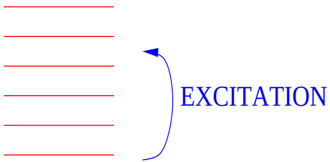
The controllability problem





–) to induce chemical reactions
(material science)

–) to measure the decay --> images
(NMR)

–) to use quantum states as a memory
(quantum computation)

See the European Flagship on quantum technologies

Controllability of the Schroedinger equation

$$i\frac{d\psi}{dt} = (H_0 + \sum_{k=1}^{m} u_k(t)H_k)\psi$$

ψ belongs to the Hilbert sphere of an Hilbert space $\mathcal{H}$
$H_0, H_1 \ldots$ are self-adjoint operators
$u_1(\cdot), \ldots, u_m(\cdot) : \mathbf{R} \to [-M, M]$ are the controls

In this talk $\mathcal{H}$ can be

- finite dimensional (e.g., spin systems). In this case $H_0, H_1, \ldots H_m$ are Hermitian matrices
- infinite dimensional (e.g., dynamics of molecules). For instance

$$i\frac{\partial\psi}{\partial t}(t,x) = (-\Delta + V_0 + \sum_{k=1}^{m} u_k(t)V_k)\psi(t,x), \quad V_k \text{ real potentials}$$

If H_0 has discrete and nondegenerate spectrum $E_0 < E_1 < E_2 < \ldots$
and we write $\psi(t) = \sum c_j(t)\phi_j$ on the base of eigenvectors of H_0
then $|c_j(t)|^2$ is the probability that if we make a measure of energy at time t we get E_j.

finite dim. case for $i\frac{d\psi}{dt} = (H_0 + \sum_{k=1}^{m} u_k(t)H_k)\psi(t)$

as a consequence of the fact that we are working with an analytic system with a recurrent drift, exact controllability is equivalent to the Lie bracket generating condition

$$\dim(\mathrm{Lie}_{\psi}\{-i(H_0 + \sum_{k=1}^{m} u_k H_k)\psi, \quad u_k \in [-M, M]\}) = 2n - 1$$

Remarks

- Why the Lie algebra is important? because for a dynamical system where one can use either X or Y, the bracket $[X, Y]$ is the direction that one can approximate by making quick switching between X and Y.
- In general this condition is not easy to check. Many people worked to look for easy verifyable conditions. Typical conditions read:
  - the spectrum of H_0 is non-resonant (e.g. all gaps different)
  - the control matrices couple all eigenstates of H_0.

infinite dim. case for $i\frac{d\psi}{dt} = (H_0 + \sum_{k=1}^{m} u_k(t) H_k)\psi(t)$

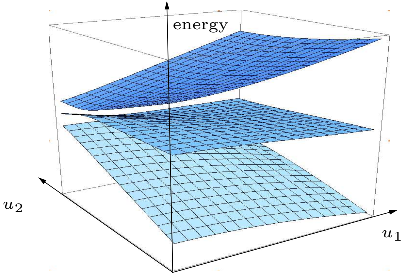
- well posedness of the problem is not trivial
- one cannot expect exact controllability in the natural functional space where the problem is formulated. For instance $H^2 \cap H_0^1$ for

$$i\frac{\partial \psi}{\partial t}(t,x) = (-\Delta + V_0 + \sum_{k=1}^{m} u_k(t) V_k)\psi(t,x), \quad V_k \text{ real potentials}$$

- exact controllability results in a proper subspace of $H^2 \cap H_0^1$ by Coron Beauchard, Laurent, Morancey in 1D [2006,—-]
- approximate controllability results by B. Chambrion, Sigalotti, Mason Boussaid, Caponigro, Nersesyan etc... [2009,—-]

The problem

Consider $\Sigma(\mathbf{u})$=spec$(H_0 + \sum_{k=1}^{m} u_k H_k)$ as function of $\mathbf{u} = (u_1, \ldots u_m)$
Assume that this spectrum is good enough (discrete, bounded from below etc...)



Is it possible to get controllability results from the knowledge of these surfaces ?

→often in experimental situations one knows precisely these surfaces, but not the $(H_0 + \sum_{k=1}^{m} u_k H_k)$.
→in the finite dimensional case we would obtain controllability results without computing any Lie brackets.

The problem

→it seems not obvious, since

- the $\Sigma(u)$ contains information on where you can go by using slow varying controls (by adiabatic theory)
- the brackets contains information on where you can go by using fast controls

However these surfaces encode information on:

- non resonance of eigenvalues of H_0,
- how the controls are coupling the eigenstates the eigenstates of H_0.

→I am going to describe this problem in finite dimension. But essentially all results hold in infinite dimension

Answer to this question for a class of systems

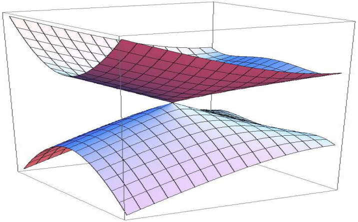
I will consider the following class of systems

- $m = 2$ i.e.
$$i\frac{d\psi}{dt} = (H_0 + u_1(t)H_1 + u_2(t)H_2)\psi(t).$$
- there exists a basis of $\mathbf{C}^n$ where H_0, H_1, H_2 are real (symmetric)
- $(u_1(.), u_2(.)) : [0, T] \to \mathbf{U}$ connected and containing an an open set

$\rightarrow$the hypothesis of having at least 2 controls is crucial.
$\rightarrow$the hypothesys that H_0, H_1, H_2 are real can be relaxed by taking $m > 2$ (see the recent paper by Francesca Chittaro and Paolo Mason)

Special features of this class of systems

Eigenvalue intersection are generically conical:



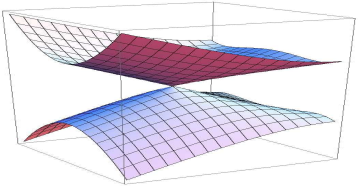
Definition

We say that $\bar{\mathbf{u}} \in \mathbf{R}^2$ is a *conical intersection* between the eigenvalues λ_j and λ_{j+1} if $\lambda_j(\bar{\mathbf{u}}) = \lambda_{j+1}(\bar{\mathbf{u}})$ has multiplicity two and there exists a constant $c > 0$ such that for any unit vector $\mathbf{v} \in \mathbf{R}^2$ and $t > 0$ small enough we have that

$$\lambda_{j+1}(\bar{\mathbf{u}} + t\mathbf{v}) - \lambda_j(\bar{\mathbf{u}} + t\mathbf{v}) > ct\,. \tag{1}$$

(the presence of eigenvalues intersection will be crucial to get controllability results)

Conical singularities are generic



- there exists a generic (i.e., open and dense) set of systems for which all eigenvalue intersections are conical
- in particular each conical intersections is "structurally stable" by perturbation of the system

→this is due to the fact that the condition for a symmetric matrix to have a double eigenvalue is of codimension 2.

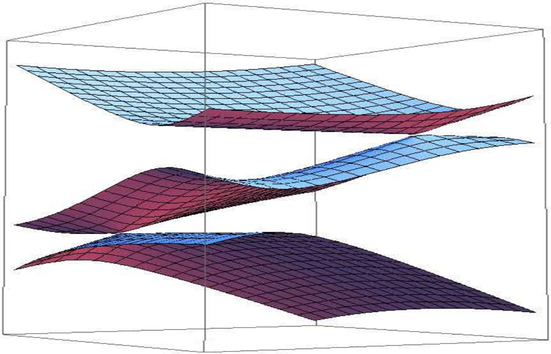
Eigenvalues of $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ are $\frac{1}{2}\left(a + c \pm \sqrt{4b^2 + (a-c)^2}\right)$.

→this result extends naturally in infinite dimension. It was formalized in [Boscain, F. Chittaro, P. Mason, M. Sigalotti, IEEE TAC, 2012] (for ∞-dim systems), but was essentially known from long time

→for Hermitian matrices, one need 3 controls

Definition

We say that the spectrum Σ of $H_0 + u_1 H_1 + u_2 H_2$ is *conically connected* if **all eigenvalue intersections are conical** and for every $j = 1, \ldots, n-1$, there exists a conical intersection $\bar{\mathbf{u}}_j \in \mathbf{U}$ between the eigenvalues λ_j, λ_{j+1}, with $\lambda_l(\bar{\mathbf{u}}_j)$ simple if $l \neq j, j+1$.



The main result (B, Gauthier, Sigalotti, Rossi CMP 2015)

Theorem (finite dimensional case)

Assume that the spectrum Σ is conically connected. Then system is exactly controllable (and hence Lie bracket generating).

→This result is not trivial: It is known how to climb energy levels through eigenvalue intersections to go from one eigenstate to another one, but:

- one arrives to the final state only approximately (because of the adiabatic theorem);
- controllability among eigenstates is much less than controllability on the full space (all superpositions, with all possible phases, of eigenstates);

→we get the Lie-bracket-generating condition without computing any bracket, but just looking to the spectrum.

In the infinite dimensional case we have

Theorem (infinite dimensional case)

Assume that the spectrum Σ is conically connected. Then system is approximately controllable.

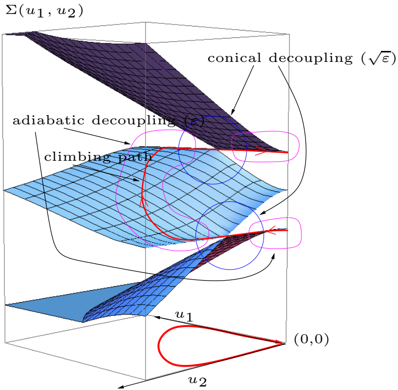
→technical hypotheses will be specified later on

A constructive (not the shortest) proof in 4 steps

- some of the steps are constructive and interesting by themself
- these ideas can be used to obtain results of "ensemble" controllability (controlling a continuous set of systems all with the same control)

STEP 1: approximate controllability among eigenstates

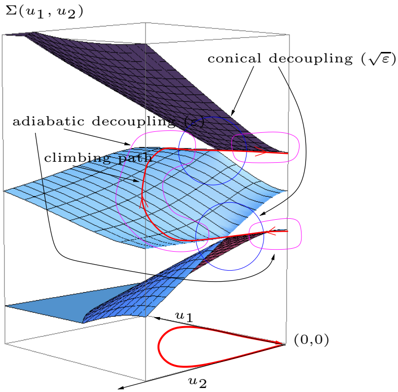
one can cross the eigenvalue intersections and move between eigenstates at the order $\sqrt{\varepsilon}$.



→"at order ε" means that for a transfer with an error ε, one needs $T = C/\varepsilon$.

→"at order $\sqrt{\varepsilon}$" means that for a transfer with an error ε, one needs $T = C/\varepsilon^2$.

→this step cannot be realized with only one control



there are two decouplings:

- the adiabatic decoupling far from the eigenvalue intersections (order ε)
- the "conical decoupling" close to the eigenvalue intersections (order $\sqrt{\varepsilon}$)

A1: the adiabatic Theorem (rougher form)

- $\lambda(u_1, u_2)$ be an eigenvalue of $H(u_1, u_2)$ depending $\mathcal{C}^2$ on (u_1, u_2)
- for every $u_1, u_2 \in K$ (K compact subset of $\mathbf{R}^2$), $\lambda(u_1, u_2)$ is simple.

Let $\phi(u_1, u_2)$ be the corresponding eigenvector (defined up to a phase). Consider a $\mathcal{C}^2$ path $(u_1, u_2) : [0, 1] \to K$ and its reparametrization $(u_1^\varepsilon(t), u_2^\varepsilon(t)) = (u_1(\varepsilon t), u_2(\varepsilon t))$, defined on $[0, 1/\varepsilon]$.

Then the solution $\psi_\varepsilon(t)$ of the equation $i\frac{d\psi_\varepsilon}{dt} = (H_0 + u_1^\varepsilon(t)H_1 + u_2^\varepsilon(t)H_2)\psi_\varepsilon(t)$ with initial condition $\psi_\varepsilon(0) = \phi(u_1(0), u_2(0))$ satisfies

$$\left\|\psi_\varepsilon\left(1/\varepsilon\right) - e^{i\vartheta}\phi\left(u_1^\varepsilon\left(1/\varepsilon\right), u_2^\varepsilon\left(1/\varepsilon\right)\right)\right\| \leq C\varepsilon \tag{2}$$

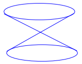
for some $\vartheta = \vartheta(\varepsilon) \in \mathbf{R}$.

- This means that, if the controls are slow enough, then, up to phases, the state of the system follows the evolution of the eigenstates of the time-dependent Hamiltonian.
- The constant C depends on the **gap** between the eigenvalue λ and the other eigenvalues.

The Conical decoupling: why a trajectory passing through a conical singularity induce a transition?

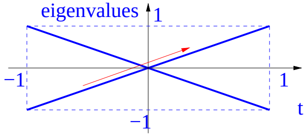
Two level systems:

$$i\begin{pmatrix}\dot\psi_1\\ \dot\psi_2\end{pmatrix}=\begin{pmatrix}u_1 & u_2\\ u_2 & -u_1\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\quad \lambda_\pm=\sqrt{u_1^2+u_2^2}$$



Let us take $u_1=t$, $t\in[-1,1]$, $u_2=0$

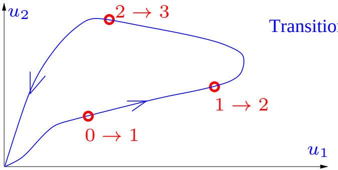
$$i\begin{pmatrix}\dot\psi_1\\ \dot\psi_2\end{pmatrix}=\begin{pmatrix}t & 0\\ 0 & -t\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\qquad \begin{array}{l}\psi_1(-1)=1\\ \psi_2(-1)=0\\ \Rightarrow\lambda=-1\end{array}\ \Rightarrow\ \begin{array}{l}|\psi_1(1)|=1\\ |\psi_2(1)|=0\\ \Rightarrow\lambda=+1\end{array}$$



→For generic two level systems there is an exact climb (on special curves)

→For generic n-level systems there is a climb at order ε (on special curves) [Boscain, F. Chittaro, P. Mason, M. Sigalotti, IEEE TAC, 2012]

→On generic smooth curves the transition is of order $\sqrt{\varepsilon}$



Transition between level 0 and 3

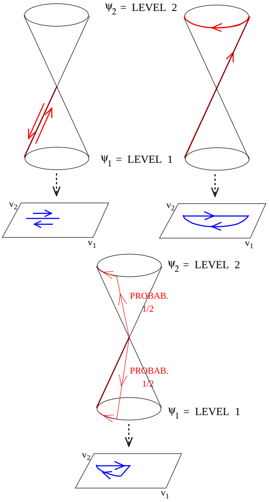
this idea is very old

- Born, Fock 1928,
- Dijon school: Jauslin, Guerin, Yatsenko, 2002,
- Teufel, 2003.

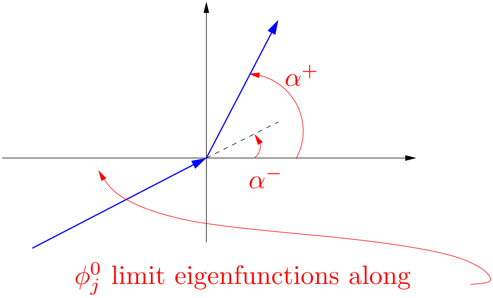
It was formalized in the language of control theory in [Boscain, F. Chittaro, P. Mason, M. Sigalotti, IEEE TAC, 2012]

STEP 2: spread controllability (without phases)

By using the adiabatic theory, is it possible to reach some other state than eigenstates?



A4: how to compute angles



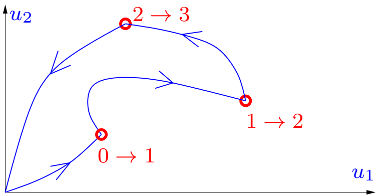
$$p_1 = |\cos(\theta(\alpha_-) - \theta(\alpha_+))| \quad p_2 = |\sin(\theta(\alpha_-) - \theta(\alpha_+))|,$$

where $\theta(\alpha)$ is the solution to:

$$\left(\cos\alpha, \sin\alpha\right) \mathcal{M}(\phi_i^0, \phi_{i+1}^0) \begin{pmatrix} \cos 2\theta(\alpha) \\ \sin 2\theta(\alpha) \end{pmatrix} = 0.$$

and by definition

$$\mathcal{M}(\phi_i, \phi_{i+1}) = \begin{pmatrix} \langle \phi_i, H_1 \phi_{i+1} \rangle & \frac{1}{2}\left(\langle \phi_{i+1}, H_1 \phi_{i+1} \rangle - \langle \phi_i, H_1 \phi_i \rangle\right) \\ \langle \phi_i, H_2 \phi_{i+1} \rangle & \frac{1}{2}\left(\langle \phi_{i+1}, H_2 \phi_{i+1} \rangle - \langle \phi_i, H_2 \phi_i \rangle\right) \end{pmatrix}.$$

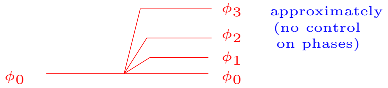


Transition between level 0 and a superposition of levels 0,1,2,3

by making angles at the eigenvalues intersections one can "spread the probability"

→this can be done at order $\sqrt{\varepsilon}$ or at order ε on special curves

→this step is constructive but we cannot control the phases



→this step extends to ∞-dimension

STEP 3: spread controllability (with phases)

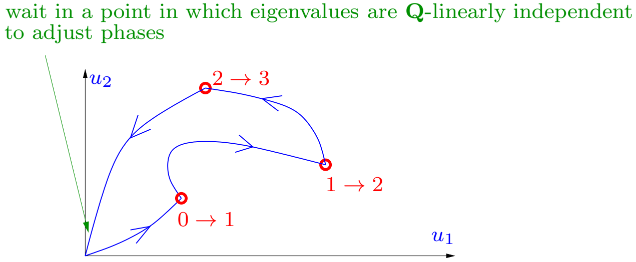
One can control the phases by using the following result:

Lemma

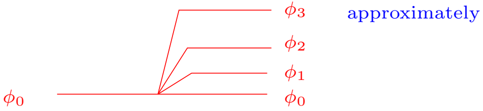
Let Σ be conically connected. Then there exists $\bar{\mathbf{U}} \subset \mathbf{U}$ which is dense and with zero-measure complement in $\mathbf{U}$ such that $\sum_{j=1}^{n} \alpha_j \lambda_j(\bar{\mathbf{u}}) = 0$ with $(\alpha_1, \ldots, \alpha_n) \in \mathbf{Q}^n$ and $\mathbf{u} \in \bar{\mathbf{U}}$ implies $\alpha_1 = \alpha_2 = \cdots = \alpha_n$.

This means that in the space of controls, close to every point there is a value of control for which the eigenvalues are $\mathbf{Q}$-linearly independent (except for the trace).

Hence one can modify a little the path by passing through a point in which the eigenvalues are $\mathbf{Q}$-linearly independent and wait in such a way that the phases take the corrected values (approximately).



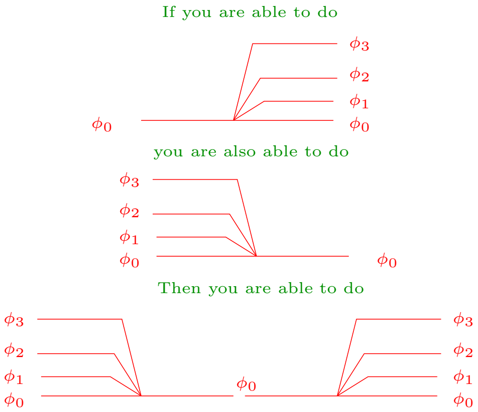
$\rightarrow$this step is not really constructive since it is hard to take track of relative phases after an adiabatic path



$\rightarrow$this step extends to ∞-dimension

STEP 4: approximate controllability

Since if $u(t)$ send ψ_0 in ψ_1 in time T then $u(T-t)$ send $\bar{\psi}_1$ in $\bar{\psi}_0$



We have approximate controllability

→this step extends to ∞-dimension

The approximate controllability result

Both for finite and infinite dimension we get an approximate controllability result in L^2

$\rightarrow$recall that in the infinite dimensional case one cannot get exact controllability on the domain where $\Delta + V_0$ is essentially self adjoint (usually $H^2 \cap H_0^1$), (Ball Marsden Slemrod, '82).

$\rightarrow$in dimension 1, Beauchard, Coron, Laurent proved exact controllability in a suitable subset of $H^2 \cap H_0^1$ (i.e. they have a complete characterization of the reachable set).

Precise statement in the infinity dimensional case

($\mathbf{A}^\infty$) Let $\mathbf{U}$ be an open and connected subset of $\mathbf{R}^m$. Assume that the Hamiltonian $H(\cdot)$ has the form

$$H(\mathbf{u}) = H_0 + u_1 H_1 + u_2 H_2, \quad \mathbf{u} = (u_1, u_2) \in \mathbf{U},$$

where $H_0, \ldots, H_2$ are self-adjoint operators on $\mathcal{H}$, with H_0 bounded from below and H_1, H_2 bounded.

A typical case for which $(\mathbf{A}^\infty)$ is satisfied is when $H_0 = -\Delta + V$, where Δ is the Laplacian on $\mathbf{R}^d$ and V is a continuous real-valued confining potential, i.e., $\lim_{|x|\to\infty} V(x) = +\infty$, and $H_1, \ldots, H_m$ are multiplication operators by continuous and bounded functions.

(B) The spectrum of H_0 is discrete without accumulation points and each eigenvalue has finite multiplicity.

Theorem

Let hypotheses $(\mathbf{A}^\infty)$ *and* $(\mathbf{B})$ *be satisfied. If the spectrum* $\Sigma(\cdot)$ *is conically connected then* $i\dot{\psi}(t) = (H_0 + u_1(t)H_1 + u_2(t)H_2)\psi(t), \qquad \psi(t) \in \mathcal{S}$, *is approximately controllable.*

STEP 5: approximate controllability implies exact controllability for finite dimensional systems

Theorem

Consider the system

$$i\dot{\psi}(t) = H(\mathbf{u}(t))\psi(t). \tag{3}$$

where $\psi : [0,T] \to S^{2n-1} \subset \mathbf{C}^n$, $\mathbf{u}(\cdot) : [0,T] \to \mathbf{U} \subset \mathbf{R}^m$, $H(\mathbf{u})$, $\mathbf{u} \in \mathbf{U}$, *are* $n \times n$ *Hermitian matrices. Then it is approximately controllable if and only it is exactly controllable.*

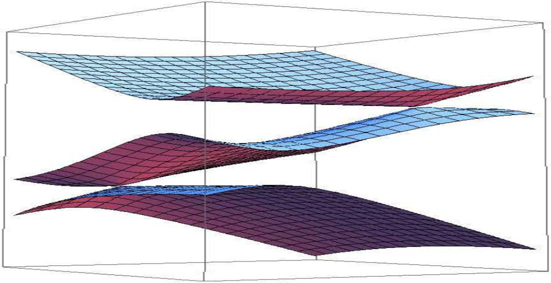
→even for a nonlinear dependence on the control

→here $H(\mathbf{u})$ can be complex (Hermitian)
→this step does not extend to ∞-dimension

- on the group this was "almost not known" (consequence on the "analytic arc" theorem by Smith on Simple Lie groups)
- on the sphere is more complicated (representation theory is necessary) [B. Gauthier, Rossi, Sigalotti CMP 2014]

Conclusion

If you see a spectrum like that:



then

- in finite dimension we get exact controllability i.e. Lie Bracket generated
- in infinite dimension we get approximate controllability

Byproduct:

- in finite dimension approximate controllability = exact controllability

Simultaneous controllability of quantum systems

[Boscain, Sigalotti, 2017, submitted]

In may experimental situations in quantum physics, one is faced to the problem of controlling via a finite number of controls a set of Schroedinger equations with slightly different parameters

$$i\frac{d}{dt}\psi^{\alpha} = (H_0^{\alpha} + \sum_{j=1}^{m} u_j(t)H_j^{\alpha})\psi^{\alpha}(t) \tag{4}$$

Here

- $\alpha \in \mathbf{R}^d$ represents a set of parameters representing
  - either a set of systems with slightly different Hamiltonian all controlled by the same control (as in high power MRI)
  - either a system in which the Hamiltonian is not know precisely (as in many experimental situations)

The simultaneous controllability problem

Let us decompose $\psi^\alpha(t) = \sum_k c_k^\alpha(t)\phi_k^\alpha$ where $\{\phi_k^\alpha\}$ is a base of eigenvectors of H_0^α.

P *Assume that at time zero all systems are in the eigenstate corresponding to an eigenvalue* $E_{k_0}^\alpha$*:*

$$\psi^\alpha(0) = e^{i\theta_0(\alpha)}\phi_{k_0}^\alpha$$

for some $\theta_0(\alpha)$*.*
For every $\varepsilon > 0$*, find a time* T_ε *and a controls* $u_j^\varepsilon(\cdot) \in L^\infty([0, T_\varepsilon], \mathbf{R})$ *(the same for all systems) such that at time* T_ε *all systems are in an eigenstate corresponding to the eigenvalue* $E_{k_1}^\alpha$ *up to errors of order* ε*:*

$$||\psi^\alpha(T_\varepsilon) - e^{i\theta_1(\alpha)}\phi_{k_1}^\alpha|| \leq \varepsilon$$

for some $\theta_1(\alpha)$

Definition

When **P** has a solution for every $E_{k_0}^\alpha$ and $E_{k_1}^\alpha$ we say that the system is simultaneous approximately controllable between eigenstates

When α belongs to a discrete set, the problem is solved [Dirr 2012, Belhadj-Salomon-Turinici 2015].

Otherwise the problem is very difficult:

- even in the finite dimensional case it can be seen as a prototype of system driven by a continuous spectrum operator acting on an infinite dimensional Hilbert space.
- one cannot expect exact controllability (Ball-Marsden-Slemrod, Turinici).

Remark I am considering the problem in which all initial and final conditions are the same. If initial and final conditions depend on α then the problem is much more difficult.

$\rightarrow$See Beauchard, Coron, Rouchon for the general problem for a 2-level quantum system [CMP2010]

What can we adapt our technique to a a one parameter family of systems?

Theorem (B., Sigalotti)

Consider a system with real Hamiltonians of the type

$$H_0 + u_1 H_1 + u_2 H_2,$$

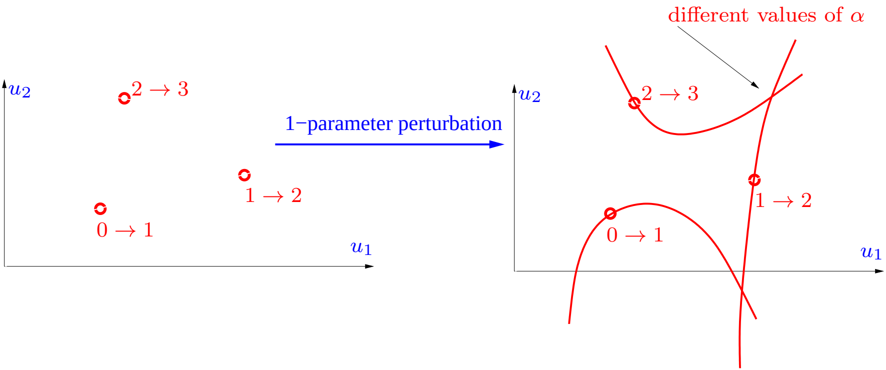
for which the spectrum is conically connected.

Consider a smooth curve of perturbations

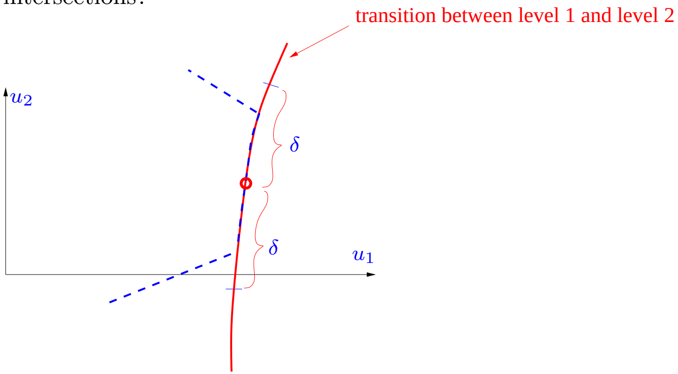
$$H_0^\alpha + u_1 H_1^\alpha + u_2 H_2^\alpha, \quad \alpha \in [-\alpha_0, \alpha_0]$$

such that $H_0^0 = H_0$, $H_1^0 = H_1$, $H_2^0 = H_2$.
Then under generic conditions on the curve of perturbations, there exist $\delta > 0$ *such that*

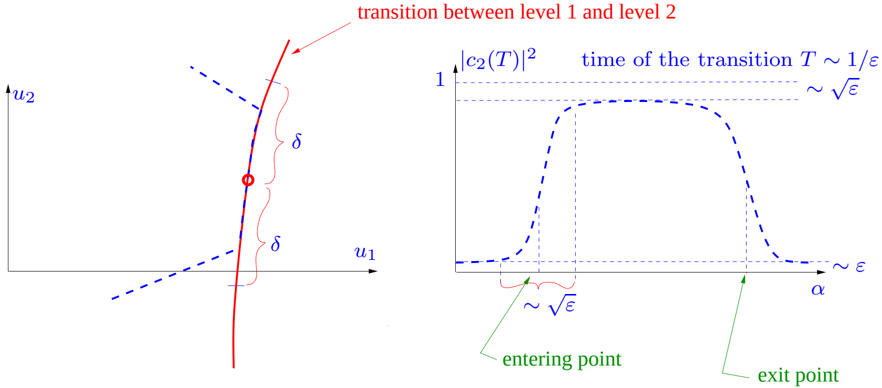
- *for every* $\alpha \in [-\delta, +\delta]$ *the spectrum is conically connected*
- *as* α *varies in* $[-\delta, +\delta]$ *eigenvalue intersections describe a smooth curve in the space of controls (with no intersections for the same value of* α*)*

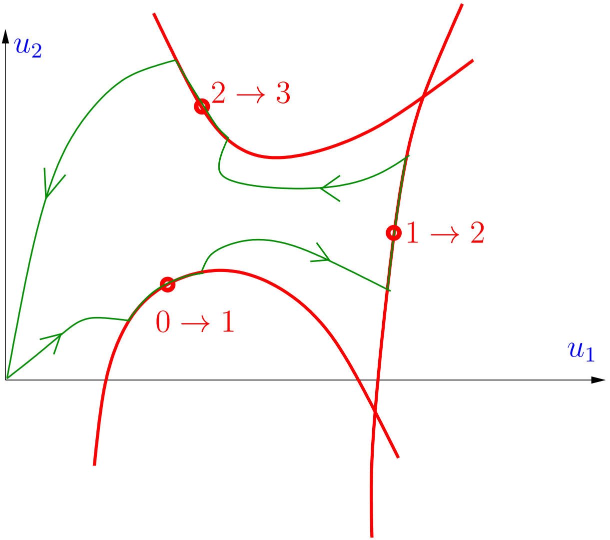

u_2
$2 \to 3$
$1 \to 2$
$0 \to 1$
u_1
1−parameter perturbation
different values of α
u_2
$2 \to 3$
$1 \to 2$
$0 \to 1$
u_1

What happen if we make a path that “enter” inside a curve of conical intersections?



we get a transition for all systems at order $\sqrt{\varepsilon}$



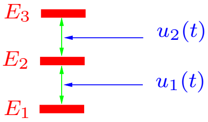

u_2
$2 \to 3$
$1 \to 2$
$0 \to 1$
u_1

Theorem

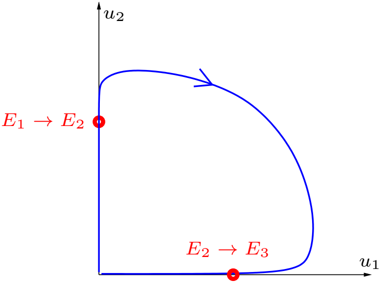
When Theorem 1 applies, then he system is **simultaneous approximately controllable between eigenstates** *in a proper subinterval of* $[-\delta, +\delta]$.

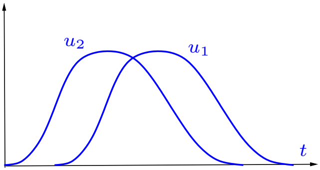
- extension to the case of an infinite dimensional Hilbert space is straightforward (under natural technical hypotheses)

An application: simultaneous controllability of STIRAP processes



$$i\frac{d}{dt}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \begin{pmatrix}E_1 & u_1(t) & 0\\ u_1(t) & E_2 & u_2(t)\\ 0 & u_2(t) & E_3\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}, \ u_1(t), u_2(t) \text{ real controls}$$





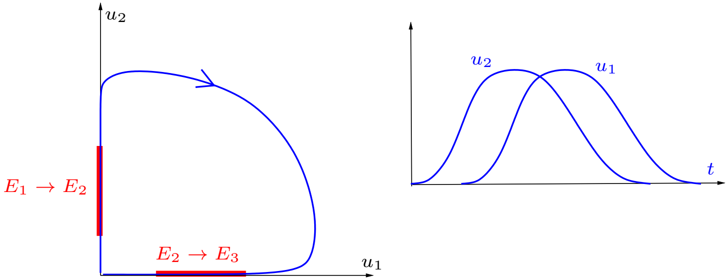
the conunterintuitive strategy is a consequence of the presence of conical intersections

What happens if we have some parameters?

$$i\frac{d}{dt}\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\end{pmatrix}=\begin{pmatrix}\alpha_1E_1&\beta_1u_1&0\\\beta_1u_1&\alpha_2E_2&\beta_2u_2\\0&\beta_2u_2&\alpha_3E_3\end{pmatrix}\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\end{pmatrix}$$

Here $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2>0$ and $\alpha_1E_1<\alpha_2E_2<\alpha_3E_3$

The same strategy with maybe bigger controls, is working for the whole family of systems



→Often only some parameters are responsible for "moving" eigenvalue intersections

→this explain why the counterintuitive strategy is so robust thanks

thanks

Conical intersections are generic (finite dimension)

Let $\mathrm{sym}(n)$ be the set of all $n \times n$ symmetric real matrices. Then, generically with respect to the pair (H_1, H_2) in $\mathrm{sym}(n) \times \mathrm{sym}(n)$ (i.e., for all (H_1, H_2) in an open and dense subset of $\mathrm{sym}(n) \times \mathrm{sym}(n)$), for each $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ and $\lambda \in \mathbf{R}$ such that λ is a multiple eigenvalue of $H_0 + u_1 H_1 + u_2 H_2$, the eigenvalue intersection $\mathbf{u}$ is conical.

Moreover, each conical intersection $\mathbf{u}$ is structurally stable, in the sense that small perturbations of H_0, H_1 and H_2 give rise, in a neighborhood of $\mathbf{u}$, to conical intersections for the perturbed H.

Conical intersections are generic in infinite dimension

Conical intersections are generic in the reference case where
$\mathcal{H} = L^2(\Omega, \mathbf{C})$,
$H_0 = -\Delta + V_0 : D(H_0) = H^2(\Omega, \mathbf{C}) \cap H^1_0(\Omega, \mathbf{C}) \to L^2(\Omega, \mathbf{C})$,
$H_1 = V_1,\ H_2 = V_2$,
Ω a bounded domain of $\mathbf{R}^2$ and $V_j \in \mathcal{C}^0(\Omega, \mathbf{R})$ for $j = 0, 1, 2$.

- V_0 is a continuous real-valued confining potential, i.e., $\lim_{|x|\to\infty} V_0(x) = +\infty$, and V_1, V_2 are continuous and bounded functions.
- The spectrum of H_0 is discrete without accumulation points and each eigenvalue has finite multiplicity.

Indeed, generically with respect to the pair (V_1, V_2) in $\mathcal{C}^0(\Omega, \mathbf{R}) \times \mathcal{C}^0(\Omega, \mathbf{R})$ (i.e., for all (V_1, V_2) in a countable intersection of open and dense subsets of $\mathcal{C}^0(\Omega, \mathbf{R}) \times \mathcal{C}^0(\Omega, \mathbf{R})$), for each $\mathbf{u} \in \Omega$ and $\lambda \in \mathbf{R}$ such that λ is a multiple eigenvalue of $H_0 + u_1 H_1 + u_2 H_2$, the eigenvalue intersection $\mathbf{u}$ is conical.

Moreover, each conical intersection $\mathbf{u}$ is structurally stable, in the sense that small perturbations of V_0, V_1 and V_2 give rise, in a neighbourhood of $\mathbf{u}$, to conical intersections for the perturbed H.

A Consequence of approximate = exact controllability

$$i\dot{\psi}(t) = (H_0 + \sum_{j=1}^{m} u_j(t)H_j)\psi(t), \quad \psi(t) \in S^{2n-1} \subset \mathbf{C}^n, \qquad (5)$$

where the controls $u_j(\cdot) \in L^1(\mathbf{R}, \mathbf{R})$ and $H_0, \ldots, H_m$ are Hermitian matrices.

Theorem

Let $T^* = \inf\{T > 0 \mid$ (5) is controllable in time $T\} \in [0, \infty]$. The following statements are equivalent.

(i) $T^* = 0$,

(ii) *for every $\psi \in S^{2n-1}$, $Lie_\psi(iH_1, \ldots, iH_m) = T_\psi S^{2n-1}$,*

(iii) *for an arbitrary $\psi \in S^{2n-1}$, $Lie_\psi(iH_1, \ldots, iH_m) = T_\psi S^{2n-1}$.*

On the group this was already known (see D'Alessandro Book)

Another quicker, but less constructive way

Both in finite and infinite dimension, the fact that the spectrum is conically connected implies that:

- close to every point in the space of controls, there is a value of control for which the eigenvalues are $\mathbf{Q}$-linearly independent;
- controls couple every pair of eigenspaces.

In infinite dimension

using the following

Theorem (Caponigro, Chambrion, Sigalotti, B., CMP 2012)

Consider the control system

$$i\dot{\psi} = (H_0 + \sum_1^m u_i H_i)\psi, \quad \psi \in S \in \mathcal{H}$$

H_i *self adjoint + technical hypotheses on the domains*

If the differences among eigenvalues of H_0 *are all different and the controls couple every eigenstate of* H_0 *then the system is approximate controllable for the density matrix. Some degeneracies are also admitted under additional hypotheses.*

we get (in a non-constructive way)

conically connected $\Rightarrow$ approximate controllability for the density matrix

in the finite dimensional case we get:

conically connected $\Rightarrow$ exact controllability on the group

Approximate-exact controllability on the group

Approximate and exact controllability on the group $SU(n)$ are equivalent. This is a direct consequence of the following result,

Theorem

If an everywhere dense subgroup H of a simple Lie group G of dimension larger than 1 contains an analytic arc, then $H = G$.

Let us apply this theorem to our system,

$$\begin{cases} \dot{g} = M(u)\, g, \\ y(0) = Id. \end{cases} \tag{6}$$

Let (6) be approximately controllable. Then, the orbit from the identity is an everywhere dense subgroup H of $SU(n)$. Any trajectory of (6) with constant u is an analytic arc, contained in H. Then $H = SU(n)$, i.e., the orbit is the whole group. Since we are in the compact case, we have that the accessible set coincides with the orbit, i.e., that system (6) is exactly controllable.

Approximate controllability on the sphere

Consider the problem for the propagator (on $U(n)$):

$$\begin{cases} \dot{g} = H(\mathbf{u}(t)).g, \\ g(0) = I, \end{cases} \tag{7}$$

Let G be the orbit. It is a subgroup of $U(n)$.

- Step A: Since G is a subgroup of $U(n)$, it is injectable in a compact Lie group. By a theorem of Dixmier we have $G = \mathbf{R}^p \times K$, with K compact.
- STEP B: the inclusion map $i : G \hookrightarrow U(n)$ is a faithful unitary representation of G. It is irreducible as a consequence of approximate controllability.
- STEP C: by a theorem of Weyl we have that the inclusion map is equivalent to $\mathfrak{X}_1 \otimes \mathfrak{X}_2$ where $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are unitary irreducible representations of $\mathbf{R}^p$ and K.
- STEP D: if $\mathbf{R}^p$ admits a irreducible unitary faithful representation, then $p = 0$. Indeed unitary irreducible representations of $\mathbf{R}^p$ are $x \mapsto e^{ix\cdot\xi}$. Hence $G = K$
- STEP E: the reachable set on the sphere S^{2n-1} is $G.\psi_0$. Hence it is compact. Being closed and dense it coincide with S^{2n-1}.

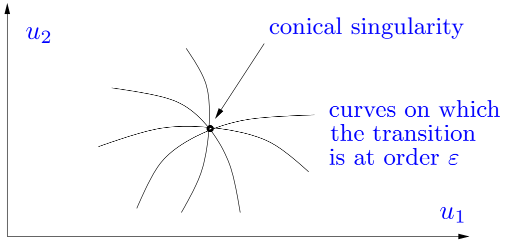
higher dimensional systems: Effective Hamiltonian

By adiabatic theory, at the order ε the dynamics is given by:

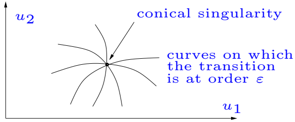
$$H_{eff}(\tau) = \begin{pmatrix} \lambda_\alpha(\tau) & 0 \\ 0 & \lambda_\beta(\tau) \end{pmatrix} + i\varepsilon \begin{pmatrix} 0 & \langle \dot{\phi}_\alpha(\tau), \phi_\beta(\tau) \rangle \\ \langle \dot{\phi}_\alpha(\tau), \phi_\beta(\tau) \rangle & 0 \end{pmatrix}$$

→For a smooth curve passing through a conical intersection the term in $i\varepsilon$ give a contribution of order $\sqrt{\varepsilon}$ [Teufel 2003] (adiabatic theorem gives a decoupling at the order ε, far from singularities)

→on the special curves $\begin{cases} \dot{u}_1 = -\langle \phi_i, V_2 \phi_{i+1} \rangle \\ \dot{u}_2 = \langle \phi_i, V_1 \phi_{i+1} \rangle \end{cases}$ the term in $i\varepsilon$ vanish and hence the climb is of order ε (the same as the adiabatic approximation).



there exists special curves where the conical decoupling is "at order ε"



$\rightarrow$this step extends to ∞-dimension
$\rightarrow$this step is constructive

REMARK In the finite dimensional case as a consequence of the fact that we are working with the projection of a left-invariant control system on $U(n)$, exact controllability is equivalent to (see D'alessandro's book):

$$\mathrm{Lie}\{-i(H_0+\sum_{k=1}^{m} u_k H_k), \quad u_k \in [-M, M]\} \supseteq \begin{cases} \mathrm{su}(n) \text{ if } n \text{ is odd} \\ \mathrm{su}(n) \text{ or } \mathrm{sp}(n/2) \text{ if } n \text{ is even.} \end{cases}$$

(Lie bracket generating condition on S^{2n-1}) however this condition is not easy to verify.