From Newton's law to the linear Boltzmann equation without cut-off

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Organisation of the talk

1- Physical and historical context of the result

2- Difficulty in our framework

3- Presentation of the usual tools for the proof

4- Adaptation of the tools + new strategy in our context

Context

Kinetic theory of gases = Describe a gas as a physical system constituted of a large number of small particles.

Statistical point of view : we are interested in the evolution of *the density of particles* $f(t, x, v)$ where

t = time
x = position
v = velocity

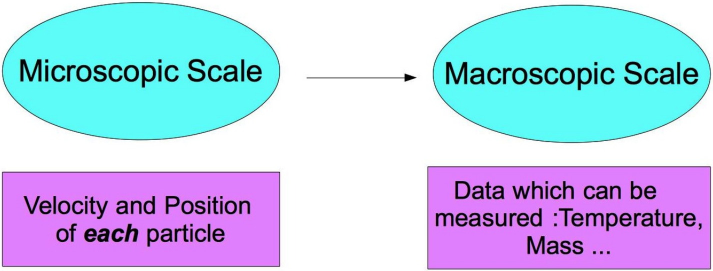
For all infinitesimal volume $dxdv$ **around the point** (x, v)**:**
$f(t, x, v)dxdv =$ *number of particles* which have position x and velocity v at time t.

Historical results:

- A fundamental example, **the Boltzmann equation** (1872) = the evolution equation for the density of particles of a sufficiently rarefied gas.

$$\underbrace{\partial_t f + v.\nabla_x f}_{free\ transport} = \underbrace{Q(f,f)}_{localized\ binary\ collisions}$$

- In **the sixth problem of Hilbert** (1900), **idea** = Boltzmann equation as **an intermediate step** in the transition between atomistic and contiuous model for gas dynamics.



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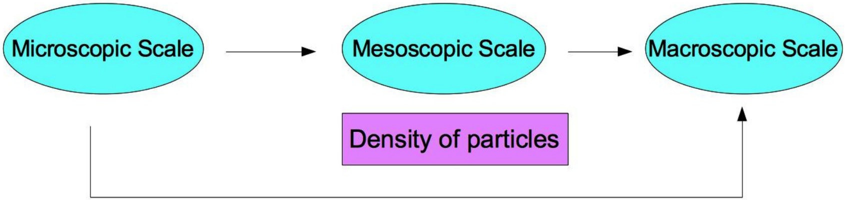


The Boltzmann-Grad scaling

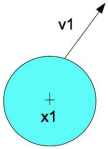
Change of scale = passage to the limit on one *precise parameter* of the system

Boltzmann-Grad scaling = $N \to \infty$, $N\varepsilon^{d-1} = 1$.

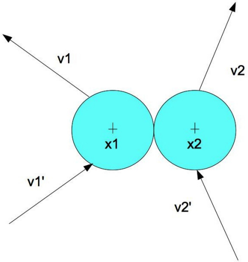
Rarefied gas : $N\varepsilon^d \ll 1$, *not too much* collisions.

Historical Panorama

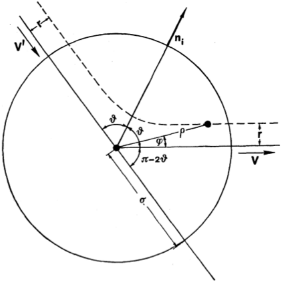
Lanford proved the derivation of **the Boltzmann equation** from systems of particles in the context of **hard-spheres** (1975).



Particles bounces off according to the laws of **elastic reflection**.



Proof recently improved by *Gallagher, Saint-Raymond and Texier*, *Pulvirenti, Saffirio and Simonella* in the context of **hard-spheres** and **short range potentials**.



(Figure: *The Boltzmann equation and its application, Cercignani*.)

Our context: **infinite-range potentials**.

The Boltzmann equation without cut-off

The Boltzmann equation

$$\partial_t f + v.\nabla_x f = \int_{\mathbf{R}^d}\int_{\mathbf{S}^{d-1}} (f' f_1' - f f_1) b(v - v_1, \nu) d\nu dv_1$$

$b(v - v_1, \nu)$ = **cross-section**, θ = deviation angle

$$b(v - v_1, \nu) = b(|v - v_1|, cos(\theta))$$

integrability with respect to θ $\Rightarrow$ Boltzmann with cut-off
non-integrability with respect to θ $\Rightarrow$ **Boltzmann without cut-off**

Example

Inverse-power law potentials: $\Phi(r) = \frac{1}{r^{s-1}},\ s > 2$. The **cross-section** satisfies

$$b(|v - v_1|, \cos\theta) = q(cos\theta)|v - v_1|^{\gamma},$$

$$sin^{d-2}\theta q(\cos\theta) \sim C\theta^{-1-\alpha},\ \alpha > 0$$

Hard-spheres, Short range potentials $\Rightarrow$ **Boltzmann with cut-off**

Difficulty = infinite range potential $\Rightarrow$ **singularity** due to **grazing collisions** $\Rightarrow$ **Boltzmann without cut-off**

Intuitive idea= use of **compensation** between **gain** and **loss** terms

$$f(v)f(v_1) \approx f(v')f(v_1')$$

The Boltzmann equation

$$\partial_t f + v.\nabla_x f = \int_{\mathbf{R}^d} \int_{\mathbf{S}^{d-1}} (f' f_1' - f f_1) b(v - v_1, \nu) d\nu dv_1$$

The Hard-spheres case

Microscopic Model: for $i \in \{1, \ldots, N\}$,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, & x_i, v_i \in \mathbf{R}^d \\ \dfrac{dv_i}{dt} = 0, & |x_i(t) - x_j(t)| > \varepsilon. \end{cases}$$

If $|x_i - x_j| = \varepsilon$,

$$v_i' = v_i + ((v_j - v_i) \cdot \nu)\nu$$
$$v_j' = v_j - ((v_j - v_i) \cdot \nu)\nu$$

Appropriate quantities:

$f_N(t, Z_N)$ = **distribution function** of N particles

with $z_i = (x_i, v_i)$, $Z_N = (z_1, \ldots, z_N)$. **Marginal** of order one :

$$f_N^{(1)}(t, x_1, v_1) = \int f_N dx_2 \ldots dx_N dv_2 \ldots dv_N.$$

Formal justification of the limit

Liouville equation + Green formula $\Rightarrow$

$$\partial_t f_N^{(1)} + v_1 \cdot \nabla_{x_1} f_N^{(1)} = (N-1)\varepsilon^{d-1} \int_{\mathbf{S}^{d-1}\times\mathbf{R}^d} \Big[f_N^{(2)}(t, x_1, v_1', x_1 + \varepsilon\nu, v_2') - f_N^{(2)}(t, x_1, v_1, x_1 - \varepsilon\nu, v_2) \Big] ((v_2 - v_1)\cdot\nu)_+ d\nu dv_2.$$

$N \to \infty$,

$$\partial_t f + v_1 \cdot \nabla_{x_1} f = \int_{\mathbf{S}^{d-1}\times\mathbf{R}^d} \Big[f^{(2)}(t, x_1, v_1', x_1, v_2') - f^{(2)}(t, x_1, v_1, x_1, v_2) \Big] ((v_2 - v_1)\cdot\nu)_+ d\nu dv_2$$

If $f^{(2)}(Z_2) = f(z_1)f(z_2) \Rightarrow$ the **Boltzmann equation**.

Key notion = **Propagation of chaos.**

Technical proof

The **BBGKY hierarchy**: for $s < N$,

$$\partial_t f_N^{(s)} + V_s \cdot \nabla_{X_s} f_N^{(s)} = \mathcal{C}_{s,s+1} f_N^{(s+1)}$$

Iterated Duhamel formula $\Rightarrow$ expression of the marginals as series.

The **Boltzmann hierarchy :** $g_s(t, Z_s) = g(t, z_1)g(t, z_2) \dots g(t, z_s)$
with g **solution** of the **Boltzmann equation** is a solution of

$$\partial_t g_s + V_s \cdot \nabla_{X_s} g_s = \mathcal{C}^0_{s,s+1} g_{s+1}$$

Two steps to prove the result :

- bounds for the series,
- the termwise convergence of each term of the series.

Key idea = **Geometrical interpretation** of the terms

Iterated Duhamel's formula.

Duhamel's formula:

$$f_N^{(s)}(t) = \mathcal{T}_s(t) f_N^{(s)}(0) + \int_0^t \mathcal{T}_s(t-t_1)\mathcal{C}_{s,s+1} f_N^{(s+1)}(t_1)dt_1.$$

The BBGKY series:

$$f_N^{(s)}(t) = \sum_{n=0}^{N-s} \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \mathcal{T}_s(t-t_1)\mathcal{C}_{s,s+1}\mathcal{T}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots\mathcal{T}_{s+n}(t_n) f_N^{(s+n)}(0)dt_n\dots dt_1.$$

The Boltzmann series:

$$g^{(s)}(t) = \sum_{n\geq 0} \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \mathcal{T}_s^0(t-t_1)\mathcal{C}_{s,s+1}^0\mathcal{T}_{s+1}^0(t_1-t_2)\mathcal{C}_{s+1,s+2}^0\dots\mathcal{T}_{s+n}^0(t_n) g^{(s+n)}(0)dt_n\dots dt_1$$

Strategy: Notion of **pseudo-trajectories.**

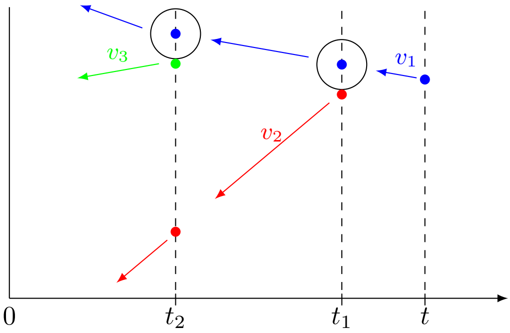


FIGURE : Representation of **a pseudo-trajectory** associated to the term $\int_0^t \int_0^{t_1} \mathcal{T}_1(t-t_1)\mathcal{C}_{1,2}\mathcal{T}_2(t_1-t_2)\mathcal{C}_{2,3}\mathcal{T}_3(t_2)f_N^{(3)}(0)dt_2dt_1$ for the BBGKY hierarchy.

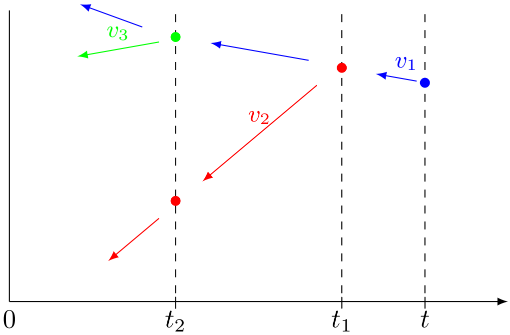


FIGURE : Representation of **a pseudo-trajectory** associated to the term $\int_0^t \int_0^{t_1} \mathcal{T}_1^0(t-t_1)\mathcal{C}_{1,2}^0 \mathcal{T}_2^0(t_1-t_2)\mathcal{C}_{2,3}^0 \mathcal{T}_3^0(t_2) f_N^{(3)}(0) dt_2 dt_1$ for the Boltzmann hierarchy.

Strategy: **Coupling** of the pseudo-trajectories.

Recollision

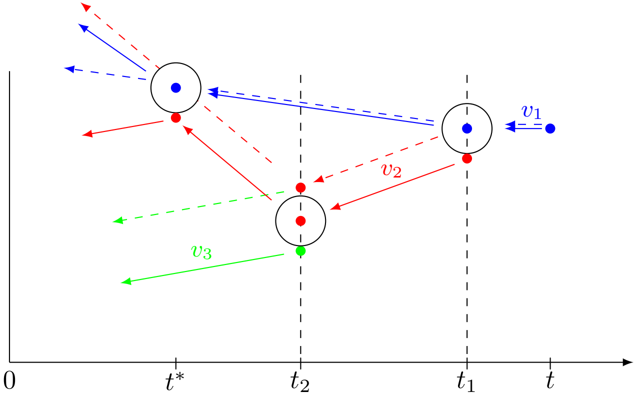


Figure: An example of a recollision between particles 1 and 2 at time t^*.

Strategy: **Geometrical control** of the recollisions

Strategy of the proof and limit

Main term with no pathological situations **converges** to **Boltzmann**.

Remainders associated to **recollisions** **vanishes** when passing to the limit.

Limit : **Short time validity** of the results for the nonlinear Boltzmann equation due to the fact that **compensation between gain and loss terms** are **ignored** for the bounds of the series.

Bodineau, Gallagher and Saint-Raymond (2014): **overcome the difficulty of the short time validity** in the case of a **fluctuation around the equilibrium** $\Rightarrow$ derivation of the **linear** **Boltzmann equation**.

The infinite range potential case

Difficulty = The **singularity** prevents the single-use of Lanford's approach.

Ideas =

- framework of perturbation around the equilibrium,
- $\nabla \Phi = \nabla \phi_{>R} + \nabla \Phi_{<R}$,
- new terms with presence of derivatives $\Rightarrow$ weak approach.

Result

Microscopic Model: for $i \in \{1, \ldots, N\}$, $x_i \in \mathbf{T}^d$, $v_i \in \mathbf{R}^d$,

$$\begin{cases} \dfrac{dx_i}{dt} & = & v_i, \\ \dfrac{dv_i}{dt} & = & -\dfrac{1}{\varepsilon} \displaystyle\sum_{j \neq i} \nabla\Phi(\dfrac{x_i - x_j}{\varepsilon}). \end{cases}$$

Tagged particle in a gas at equilibrium:

$$f_N^0(Z_N) := M_{N,\beta}(Z_N)\rho^0(x_1)$$

ρ^0 **density of probability** on $\mathbf{T}^d$, $M_{N,\beta}$ **Gibbs measure**: for $\beta > 0$

$$M_{N,\beta}(Z_N) := \frac{1}{\mathcal{Z}_N}\left(\frac{\beta}{2\pi}\right)^{dN/2} \exp(-\beta H_N(Z_N))$$

with $H_N(Z_N) := \displaystyle\sum_{1 \leq i \leq N} \frac{1}{2}|v_i|^2 + \sum_{1 \leq i < j \leq N} \Phi(\frac{(x_i - x_j}{\varepsilon}).$

Theorem (A., 2016)

Consider ***the initial distribution*** f_N^0 *describing the density of a tagged particle in a gas at equilibrium, then the* ***distribution*** $f_N^{(1)}(t,x,v)$ ***of the tagged particle*** *converges in* $\mathcal{D}'(\mathbf{T}^d \times \mathbf{R}^d)$ *when* N *goes to* ∞ ***under the Boltzmann-Grad scaling*** $N\varepsilon^{d-1} = 1$ *to* $M_\beta(v)h(t,x,v)$ *where* $h(t,x,v)$ *is the solution of* ***the linear Boltzmann equation without cut-off***

$$\partial_t h + v.\nabla_x h = -\int\int [h(v) - h(v_1)] M_\beta(v_1) b(v - v_1) dv_1 d\nu$$

with initial data $\rho^0(x_1)$ *and where* $M_\beta(v) := \left(\frac{\beta}{2\pi}\right)^{d/2} \exp\left(-\frac{\beta}{2}|v|^2\right)$*,* $\beta > 0$*.*

First partial result: Desvillettes and Pulvirenti (1999).

The infinite range potential case

Truncated marginals:

$$\tilde{f}^{(s)}_{N,R}(t, Z_s) := \int_{\mathbf{T}^{d(N-s)}\times\mathbf{R}^{d(N-s)}} f_N(t, Z_s, z_{s+1}, \ldots, z_N) \prod_{\substack{1\le i\le s\\ s+1\le j\le N}} \mathbf{1}_{\{|x_i-x_j|>R\varepsilon\}} dz_{s+1}$$

The **BBGKY hierachy**: for $s < N$,

$$\partial_t \tilde{f}^{(s)}_{N,R} + \sum_{i=1}^{s} v_i.\nabla_{x_i}\tilde{f}^{(s)}_{N,R} - \frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ i\neq j}}^{s} \nabla\Phi^{<}(\frac{x_i - x_j}{\varepsilon}).\nabla_{v_i}\tilde{f}^{(s)}_{N,R}$$

$$= \mathcal{C}_{s,s+1}\tilde{f}^{(s+1)}_{N,R} + \mathcal{C}_{s,s+1}\overline{f}^{(s+1)}_{N,R} + \frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ i\neq j}}^{s} \nabla\Phi^{>}(\frac{x_i - x_j}{\varepsilon}).\nabla_{v_i}\tilde{f}^{(s)}_{N,R}$$

$$+ \frac{(N-s)}{\varepsilon}\sum_{i=1}^{s}\int_{\mathbf{T}^{d(N-s)}\times\mathbf{R}^{d(N-s)}} \nabla\Phi(\frac{x_i - x_{s+1}}{\varepsilon}).\nabla_{v_i} f_N(t, Z_N)$$

$$\prod_{\substack{1\le l\le s\\ s+1\le k\le N}} \mathbf{1}_{\{|x_l-x_k|>R\varepsilon\}} dZ_{(s+1,N)}$$

Duhamel's formula:

$$\begin{aligned}\tilde{f}^{(s)}_{N,R}(t,Z_s) = & \mathcal{S}_s(t)\tilde{f}^{(s)}_{N,R}(0,Z_s)\\ & + \int_0^t \mathcal{S}_s(t-t_1)\mathcal{C}_{s,s+1}\tilde{f}^{(s+1)}_{N,R}(t_1,Z_s)dt_1\\ & + \int_0^t \mathcal{S}_s(t-t_1)\mathcal{C}_{s,s+1}\overline{f}^{(s+1)}_{N,R}(t_1,Z_s)dt_1\\ & + \frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ i\neq j}}^{s}\int_0^t \mathcal{S}_s(t-t_1)\left[\nabla\Phi^{>}(\frac{x_i-x_j}{\varepsilon}).\nabla_{v_i}\tilde{f}^{(s)}_{N,R}\right](t_1,Z_s)dt_1\\ & + \frac{(N-s)}{\varepsilon}\sum_{i=1}^{s}\int_0^t \mathcal{S}_s(t-t_1)\left[\int_{\mathbf{T}^{d(N-s)}\times\mathbf{R}^{d(N-s)}}\nabla\Phi(\frac{x_i-x_{s+1}}{\varepsilon}).\nabla_{v_i}f_N\right.\\ & \left.\prod_{\substack{1\leq l\leq s\\ s+1\leq k\leq N}}\mathbf{1}_{\{|x_l-x_k|>R\varepsilon\}}dZ_{(s+1,N)}\right](t_1,Z_s)dt_1\end{aligned}$$

Obstacles to the convergence

Four **possible obstacles** to the convergence:

- the **very long-range** interactions,
- **clusters** (or multiple simultaneous interactions),
- the presence of **recollisions**,
- **super-exponential** collision process.

The pruning process

Definition

Let $\tau > 0$ and denote $t := K\tau$ for some large integer K. We split $[0, t]$ into $\cup_{1\leq k\leq K}[(k-1)\tau, k\tau]$. We call **a collision tree "of controlled size"** a collision tree such that it has **less than** $n_k = 2^k$ **branch points** on the interval $[t - k\tau, t - (k-1)\tau]$.
We call **a collision tree with super-exponential growth** a collision tree which does not satisfy the above property.

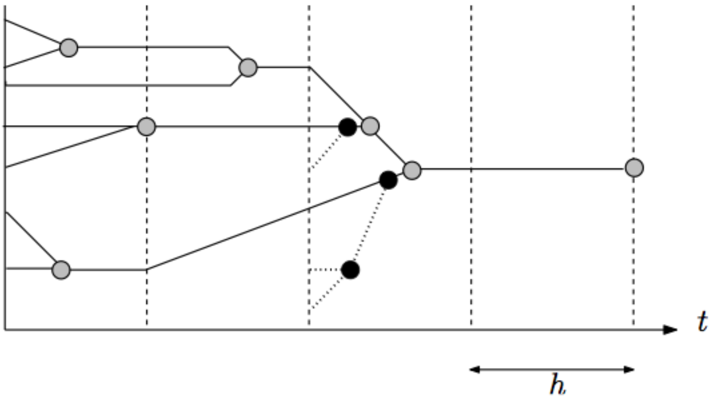


Figure: A **super-exponential tree** (Source: *The Brownian Motion as the limit of a deterministic system of hard-spheres*, Bodineau et al).

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Iteration on the term:

$$\int_0^t \mathcal{S}_s(t-t_1)\mathcal{C}_{s,s+1}\tilde{f}_{N,R}^{(s+1)}(t_1, Z_s)dt_1.$$

New strategy = truncations at each iteration step.

Truncations to avoid pathological situations

Elimination of **recollisions**: notion of **good configurations**.

Definition

The set of good configuration $\mathcal{G}_k(\varepsilon_0)$ is defined as follows:

$$\mathcal{G}_k(\varepsilon_0) := \{Z_k \in \mathbf{T}^{dk} \times \mathbf{R}^{dk} | \forall u \in [0,t] \; \forall i \neq j \; d(x_i - uv_i, x_j - uv_j) \geq \varepsilon_0\}.$$

Bad sets to remove **geom(s+k)** :
$s+k-1$ particles in a good configuration $\longrightarrow$ after a delay δ, the $s+k$ particles are in a good configuration.

Truncations associated to the **elimination of recollisions**:

- cutting off the **energy of the system** via a smooth function such that

$$\chi^{E^2}(x) = \begin{cases} 1 & if\ |x| \leq E^2 \\ 0 & if\ |x| \geq E^2 + 1. \end{cases}$$

$\chi^{E^2}(H_k(Z_k)) := \chi_{\{H_k(Z_k) \leq E^2\}}$ for all integer k,

- separation of the **collision times** by δ.

Additional truncation:

- **small relative velocities** via a smooth function such that

$$\chi^{\eta}(x) = \begin{cases} 0 & if\ |x| \leq \eta/2 \\ 1 & if\ |x| \geq \eta \end{cases}$$

$$\chi_{\{\forall i \in \{1,\ldots,k\} |v_i - v_{k+1}| \geq \eta\}} := \prod_{i=1}^{k} \chi^{\eta}(v_i - v_{k+1}).$$

One iteration:

$$
\begin{aligned}
&\int_0^t \mathcal{S}_s(t-t_1)\mathcal{C}_{s,s+1}\tilde{f}_{N,R}^{(s+1)}(t_1,Z_s)dt_1 \\
&= \int_{t-\delta}^{t} \mathcal{S}_s(t-t_1)\mathcal{C}_{s,s+1}\tilde{f}_{N,R}^{(s+1)}(t_1,Z_s)dt_1 \\
&+ \int_0^{t-\delta} \mathcal{S}_s(t-t_1)\mathcal{C}_{s,s+1}\left(1-\chi_{\{H_{s+1}(Z_{s+1})\leq E^2\}}\right)\tilde{f}_{N,R}^{(s+1)}(t_1,Z_s)dt_1 \\
&+ \int_0^{t-\delta} \mathcal{S}_s(t-t_1)\mathcal{C}_{s,s+1}\chi_{\{H_{s+1}(Z_{s+1})\leq E^2\}}\chi_{geom(s+1)}\tilde{f}_{N,R}^{(s+1)}(t_1,Z_s)dt_1 \\
&+ \int_0^{t-\delta} \mathcal{S}_s(t-t_1)\mathcal{C}_{s,s+1}\chi_{\{H_{s+1}(Z_{s+1})\leq E^2\}}\left(1-\chi_{geom(s+1)}\right)\left(1-\chi_{\{\forall i\in\{1,\dots,s\}|v_i-v_{s+1}|\geq\eta\}}\right) \\
&\qquad \tilde{f}_{N,R}^{(s+1)}(t_1,Z_s)dt_1 \\
&+ \int_0^{t-\delta} \mathcal{S}_s(t-t_1)\mathcal{C}_{s,s+1}\chi_{\{H_{s+1}(Z_{s+1})\leq E^2\}}\left(1-\chi_{geom(s+1)}\right)\chi_{\{\forall i\in\{1,\dots,s\}|v_i-v_{s+1}|\geq\eta\}} \\
&\qquad \tilde{f}_{N,R}^{(s+1)}(t_1,Z_s)dt_1.
\end{aligned}
$$

Definition of the **operators**:

$$Q_{s,s}(t) := \mathcal{S}_s(t)$$

$$Q_{s,s+n}(t) := \int_0^{t-\delta}\int_0^{t_1-\delta}\dots\int_0^{t_{n-1}-\delta} \mathcal{S}_s(t-t_1)\mathcal{C}_{s,s+1}\chi_{H_{s+1}}\left(1-\chi_{geom(s+1)}\right)\chi_{\eta_s+1}\dots$$
$$\dots\mathcal{S}_{s+n-1}(t_{n-1}-t_n)\mathcal{C}_{s+n-1,s+n}\chi_{H_{s+n}}\left(1-\chi_{geom(s+n)}\right)\chi_{\eta_{s+n}}\mathcal{S}_{s+n}(t_n)dt_n\dots dt_1.$$

Remainders associated to the **long-range part**:

$$r^{Pot,a}_{s,m+1}(0,t,Z_s) := \sum_{n=0}^{m}\int_0^{t-n\delta} Q_{s,s+n}(t-t_{n+1})$$

$$\frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ i\neq j}}^{s+n}\left[\nabla\Phi^{>}\left(\frac{x_i-x_j}{\varepsilon}\right).\nabla_{v_i}\tilde{f}^{(s+n)}_{N,R}\right](t_{n+1},Z_s)dt_{n+1}$$

$$r^{Pot,b}_{s,m+1}(0,t,Z_s) := \sum_{n=0}^{m} \int_0^{t-n\delta} Q_{s,s+n}(t-t_{n+1})$$

$$\frac{(N-(s+n))}{\varepsilon} \sum_{i=1}^{s+n} \Bigg[\int_{\mathbf{T}^{d(N-(s+n))}\times\mathbf{R}^{d(N-(s+n))}} \nabla\Phi(\frac{x_i - x_{s+n+1}}{\varepsilon}).\nabla_{v_i} f_N$$

$$\prod_{\substack{1\leq l\leq s+n \\ s+n+1\leq k\leq N}} \mathbf{1}_{\{|x_l-x_k|>R\varepsilon\}} dZ_{(s+n,N)} \Bigg] (t_{n+1}, Z_s) dt_{n+1}.$$

Control of the new terms

Maximum Principle for **the Liouville equation** $\Rightarrow$ **A priori estimates** on **the truncated marginals**.

Problem: Control on the marginals but not on the derivatives of the marginals $\Rightarrow$ **Weak approach**.

Advantage of the iteration method

no pathological situations $\Rightarrow$ easy to pass from $\mathbf{Z_m}(\mathbf{t_m})$ to $\mathbf{z}$ (state of particle 1 at time t) via **changes of variables** such as

$$\begin{array}{rcl} \mathbf{T}^d \times \mathbf{R}^d \times [0, t-\delta] \times \mathbf{S}^{d-1} \times \mathbf{R}^d \times \cdots \times [0, t_{m-1}-\delta] \times \mathbf{S}^{d-1} \times \mathbf{R}^d & \to & \mathbf{T}^{(m+1)d} \times \mathbf{R}^{(m+1)d} \\ (z, t_1, \nu_2, v_2, \ldots, t_m, \nu_{m+1}, v_{m+1}) & \mapsto & \tilde{Z}_{m+1} = Z_{m+1}(t_m). \end{array}$$

Key point: z Lipschitz function of $(\tilde{x}_1, \tilde{v}_1, \ldots, \tilde{x}_{m+1}, \tilde{v}_{m+1})$

Tools to prove it: study of the reduced dynamics

- **bound on the microscopic time of interaction** thanks to the **lower bound on relative velocites**,
- use of **the Cauchy-Lipschitz theorem**, $\nabla\Phi$ being Lipschitz,
- **Lipschitz character** of the collision times.

Proposition

The function $(\tilde{x}_1, \tilde{v}_1, \ldots, \tilde{x}_{k+1}, \tilde{v}_{k+1}) \mapsto z = z(\tilde{x}_1, \tilde{v}_1, \ldots, \tilde{x}_{k+1}, \tilde{v}_{k+1})$*, where* $(\tilde{x}_1, \tilde{v}_1, \ldots, \tilde{x}_{k+1}, \tilde{v}_{k+1})$ *is* ***the state of the*** $k+1$ ***particles after*** k ***collisions at time*** t^+_{k+1} *(i.e. before the* $k+1$*-th collision) on a* ***pseudo-trajectory*** *is a* $(C_{R,\eta,\varepsilon})^k$*-Lipschitz function with* $C_{R,\eta,\varepsilon} = \dfrac{CRe^{CR^3/\eta}}{\eta\,|\cos\left(\frac{\pi}{2}-\varepsilon\right)|\varepsilon}$ *and* C *is a constant which can only depend on* $\nabla\Phi$*.*

Lipschitz control associated to the pseudo-trajectories $\Longrightarrow$ bound of the remainder associated to the long-range part controlled by

$$\left(e^{C\frac{R^3}{\eta}}\right)^{2^{K+1}} \|\nabla\Phi^>\|_\infty.$$

Limit of the approach: Very decreasing potentials.

Thank you for your attention.