# On the stability of the physically reasonable solution to the two-dimensional Navier-Stokes equations

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# 1. Stability of two-dimensional exterior flows

$\Omega$: exterior domain in $\mathbb{R}^2$ with smooth boundary

$U = (U_1(t,x), U_2(t,x))$: velocity field of fluid, $(t,x) \in [0,\infty) \times \Omega$

- $\mathrm{div}\, U = 0$
- No-slip boundary condition

**Basic interest**

Assuming that $U$ is a global solution to the Navier-Stokes equations, we are interested in the stability property of $U$.

# 1. Stability of two-dimensional exterior flows

The stability property of $U$ is closely related to the decay property (and its size) of $U$ in time and space.

It is heuristically known that the invariant spaces about the scaling

$$U_\lambda(t,x) = \lambda U(\lambda^2 t, \lambda x), \qquad \lambda > 0$$

play a fundamental role. The Banach space $Y$ is called a scale-critical space for NS if

$$\|f_\lambda\|_Y = \|f\|_Y, \qquad \lambda > 0.$$

Typical examples for 2D Navier-Stokes equations are

$$Y = L^\infty(0,\infty; L^2(\mathbb{R}^2)) \qquad \text{or} \qquad Y = L^\infty(0,\infty; L^{2,\infty}(\mathbb{R}^2)),$$

where $L^{2,\infty}(\mathbb{R}^2)$ is the weak $L^2$ space, which contains a function such as $f(x) = |x|^{-1}$.

# 1. Stability of two-dimensional exterior flows

We expect that if $Y$ is a scale-critical space and

$$\|U\|_Y \ll 1\,,$$

then $U$ is stable (in a suitable sense) about disturbances.

However, for the $2$D NS in $\Omega$, such a general result has not been established yet, though $3$D case is well understood; Heywood (1970), Borchers and Miyakawa (1995), Kozono-Yamazaki (1998), Yamazaki (2000), and so on.

**Several difficulties in $2$D case:**

- absence of the scale-critical Hardy inequality:

$$\|\frac{f}{|x|}\|_{L^d(\mathbb{R}^d)} \leq C\|\nabla f\|_{L^d(\mathbb{R}^d)} \quad \text{for } d \geq 3,\ d \neq 2.$$

- decay of Newton potential:

$$E_{\mathbb{R}^2}(x) = c_2 \log|x|\,, \qquad E_{\mathbb{R}^3}(x) = c_3|x|^{-1}$$

# 2. Physically reasonable solutions to 2D Navier-Stokes equations

$$
(\mathrm{NS}_\alpha)\begin{cases}
\partial_t u + \alpha u\cdot\nabla u - \Delta u + \alpha\partial_1 u + \nabla p = 0\,, & t>0\,,\quad x\in\Omega\,,\\
\operatorname{div} u = 0\,, & t\geq 0\,,\quad x\in\Omega\,,\\
u|_{\partial\Omega} = -\mathbf{e}_1\,,\quad \lim_{|x|\to\infty} u = 0\,, & u|_{t=0} = u_0\,.
\end{cases}
$$

- The origin is interior of $\mathbb{R}^2 \setminus \Omega$.
- $\operatorname{diam}(\mathbb{R}^2 \setminus \Omega) = 1$
- The number $\alpha > 0$ represents the Reynolds number.

# 2. Physically reasonable solutions to 2D Navier-Stokes equations

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u|_{\partial\Omega} = -\mathbf{e}_1\,,\quad \lim_{|x|\to\infty} u = 0\,,\quad u|_{t=0} = u_0\,. &
\end{cases}
$$

After pioneering work by Leray (1933), the following result was proved.

**Thm (Finn-Smith, 1967; Galdi, 1994): Existence for small $\alpha$**

There exists $\alpha_0 > 0$ such that if $0 < \alpha \le \alpha_0$ then there exists a unique stationary solution $U$ to (NS) such that

$$
|U(x)| \le \frac{C}{|\log\alpha|}\,\frac{1}{|\alpha x|^{\frac{1}{2}}}\,,\qquad x\in\Omega\,.
$$

# 2. Physically reasonable solutions to 2D Navier-Stokes equations

$$\begin{cases} \alpha U\cdot\nabla U - \Delta U + \alpha\partial_1 U + \nabla P = 0\,, & x\in\Omega\,,\\ \operatorname{div} U = 0\,, & x\in\Omega\,,\\ U|_{\partial\Omega} = -\mathbf{e}_1\,, \quad \lim_{|x|\to\infty} U = 0\,. \end{cases}$$

$$|U(x)| \le \frac{C}{|\log\alpha|}\,\frac{1}{|\alpha x|^{\frac{1}{2}}}\,, \qquad x\in\Omega\,.$$

**Remark.** (1) The decay order $O(|\alpha x|^{-\frac{1}{2}})$ is much slower than the scale-critical oder $O(|\alpha x|^{-1})$.

(2) The smallness factor $\frac{1}{|\log\alpha|}$ reflects the Stokes paradox for $2$D exterior flows (Chang and Finn (1961)): there exist no solutions for the Stokes problem $-\Delta u + \nabla p = 0$ in $\Omega$ and $u|_{\partial\Omega} = -\mathbf{e}_1$ satisfying $\lim_{|x|\to\infty} u = 0$.

# 3. Wake estimate and scale-critical decay

$$\begin{cases} \alpha U \cdot \nabla U - \Delta U + \alpha \partial_1 U + \nabla P = 0\,, & x \in \Omega\,, \\ \operatorname{div} U = 0\,, & x \in \Omega\,, \\ U|_{\partial\Omega} = -\mathbf{e}_1\,, \quad \lim_{|x|\to\infty} U = 0\,. \end{cases}$$

It is well known that $U$ possesses a wake structure about the decay at $|x| \to \infty$; Smith (1965); Babenko (1970); Galdi (1994)

**Decay estimate of physically reasonable solutions**

There exists an extension of $U$ to $\mathbb{R}^2$ such that $\operatorname{div} U = 0$ in $\mathbb{R}^2$ and $V(X) = U(\frac{X}{\alpha})$ satisfies

$$|V(X)| \le \frac{C}{|\log \alpha|}\Big(\frac{1}{|X|^{\frac{1}{2}}(1 + |X| - X_1)^{\frac{3}{4}}} + \frac{1}{1 + |X|}\Big), \qquad X \in \mathbb{R}^2$$

and $\|\nabla U\|_{L^4} \le C$. Here $C$ is independent of small $\alpha$.

**Lemma**

There exist $\kappa, C > 0$ such that

$$|X| - X_1 \geq \begin{cases} C|X| & \text{if} \quad |X_2| \geq \kappa |X_1|, \\ \dfrac{X_2^2}{4|X_1|} & \text{if} \quad |X_2| \leq \kappa |X_1|. \end{cases}$$

*Proof.* Use $|X| = |X_1|\left(1 + |\frac{X_2}{X_1}|^2\right)^{\frac{1}{2}}$ for $|X_2| \leq \kappa |X_1|$ with small $\kappa > 0$.

$$|V(X)| \le \frac{C}{|\log \alpha|}\left(\frac{1}{|X|^{\frac{1}{2}}(1+|X|-X_1)^{\frac{3}{4}}} + \frac{1}{1+|X|}\right), \qquad X \in \mathbb{R}^2$$

$$\frac{1}{|X|^{\frac{1}{2}}(1+|X|-X_1)^{\frac{3}{4}}} \le \frac{C}{|X|^{\frac{1}{2}}+|X_2|}\left(\min\left\{1, \frac{|X_1|^{\frac{1}{4}}}{|X_2|^{\frac{1}{2}}}\right\} + \frac{1}{1+|X|^{\frac{1}{2}}}\right)$$

**Corollary: scale-criticality of physically reasonable solutions**

$X_2 V(X) \in L^\infty(\mathbb{R}^2)$. Moreover, $V \in L^\infty_{X_1} L^1_{X_2} + L^{2,\infty}(\mathbb{R}^2)$.

Note that $f_\lambda(X) = \lambda f(\lambda X)$ satisfies

$$\|X_2 f_\lambda\|_{L^\infty} = \|X_2 f\|_{L^\infty} \qquad \text{for } \lambda > 0\,.$$

# 4. Stability of physically reasonable solutions

$$D(A_{\alpha,\Omega}) = W^{2,3}(\Omega) \cap W_0^{1,3}(\Omega), \qquad A_{\alpha,\Omega} f = -\Delta f + \alpha \partial_1 f$$

$$L^3_\sigma(\Omega) = \overline{\{f \in C_0^\infty(\Omega)^2 \mid \text{div } f = 0\}}^{\|f\|_{L^3}} \qquad \mathbb{P}_\Omega : L^3(\Omega)^2 \to L^3_\sigma(\Omega)$$

Pure Oseen operatpr

$$D(\mathbb{A}_{\alpha,\Omega}) = D(A_{\alpha,\Omega})^2 \cap L^3_\sigma(\Omega), \qquad \mathbb{A}_{\alpha,\Omega} f = \mathbb{P}_\Omega A_{\alpha,\Omega} f$$

**Thm (Hishida, 2016): $L^p - L^q$ Estimates for Oseen semigroup $e^{-t\mathbb{A}_{\alpha,\Omega}}$**

$$\|e^{-t\mathbb{A}_{\alpha,\Omega}} f\|_{L^p} \le C\,(1 + \alpha^{-\beta_{p,q}})\, t^{-\frac{1}{q}+\frac{1}{p}} \|f\|_{L^q}, \qquad t > 0,$$

for $1 < q \le p < \infty$. Here $C$ depends only on $\Omega$, $\alpha$, $p$, and $q$, and $\beta_{p,q} > 0$ depends on $p$ and $q$.

$$D(A_{\alpha,\Omega}) = W^{2,3}(\Omega) \cap W_0^{1,3}(\Omega), \qquad A_{\alpha,\Omega} f = -\Delta f + \alpha \partial_1 f,$$

$$D(L_{\alpha,\Omega}) = D(A_{\alpha,\Omega})^2, \qquad L_{\alpha,\Omega} f = A_{\alpha,\Omega} f + \alpha(U \cdot \nabla f + f \cdot \nabla U).$$

Pure Oseen operatpr

$$D(\mathbb{A}_{\alpha,\Omega}) = D(A_{\alpha,\Omega})^2 \cap L^3_\sigma(\Omega), \qquad \mathbb{A}_{\alpha,\Omega} f = \mathbb{P}_\Omega A_{\alpha,\Omega} f,$$

Full Oseen operatpr

$$D(\mathbb{L}_{\alpha,\Omega}) = D(\mathbb{A}_{\alpha,\Omega}), \qquad \mathbb{L}_{\alpha,\Omega} f = \mathbb{P}_\Omega L_{\alpha,\Omega} f.$$

$$v(t) = e^{-t\mathbb{L}_{\alpha,\Omega}} v_0 - \alpha \int_0^t e^{-(t-s)\mathbb{L}_{\alpha,\Omega}} \mathbb{P}_\Omega \nabla \cdot (v \otimes v)(s)\, \mathrm{d}s \qquad (\mathrm{INS}_\alpha)$$

**Thm (M.): stability of physically reasonable solutions**

Set $b_\alpha(x) = |\alpha x|^{\frac{1}{2}} + |\alpha x_2|$. For any $\delta \in (0,1)$ there exists $\alpha_0 = \alpha_0(\delta) > 0$ such that the following statement holds for all $\alpha \in (0, \alpha_0]$. There is $\epsilon = \epsilon(\delta, \alpha) > 0$ such that for any $v_0 \in L^3_\sigma(\Omega)$ satisfying

$$\|(1 + b_\alpha)^{1+\delta} v_0\|_{L^\infty} \le \epsilon\,,$$

equation $(\mathrm{INS}_\alpha)$ admits a unique solution $v \in C([0,\infty); L^3_\sigma(\Omega)) \cap C((0,\infty); W^{1,3}_0(\Omega)^2)$ satisfying

$$\lim_{t\to\infty} \|v(t)\|_{L^3} = \lim_{t\to\infty} t^{\frac{1}{2}} \|v(t)\|_{L^\infty} = 0\,.$$

# 4. Stability of physically reasonable solutions

For $3$D case, the nonlinear stability of physically reasonable solutions is proved by

Heywood (1970): for $L^2$ disturbances
Shibata (1999): for $L^3$ disturbances based on the linear analysis of Kobayashi-Shibata (1998)

In $3$D case, the stationary solution $U$ satisfies the estimate

$$|U(x)| \le \frac{C}{|x|}, \qquad x \in \Omega.$$

**cf)** In $2$D case: $$|U(x)| \le \frac{C}{|\log \alpha|\, |\alpha x|^{\frac{1}{2}}}.$$

# 4. Stability of physically reasonable solutions

The core part of the proof is the analysis of $e^{-t\mathbb{L}_{\alpha,\Omega}}$.

## Basic strategy for the linear analysis of $e^{-t\mathbb{L}_{\alpha,\Omega}}$

**1. Estimate of flows far from the boundary:** Introduce a suitable weighted function space by taking into account the wake structure and the transport term $\alpha\partial_1$. The key step is to analyze $u(t) = e^{-t\mathbb{L}_{\alpha,\mathbb{R}^2}} f$ with

$$u(t) = e^{-t\mathbb{A}_{\alpha,\mathbb{R}^2}} f - \alpha \int_0^t e^{-(t-s)\mathbb{A}_{\alpha,\mathbb{R}^2}} \mathbb{P}_{\mathbb{R}^2} \nabla \cdot (U \otimes u + u \otimes U) \, \mathrm{d}s$$

**2. Estimate of flows near the boundary:** ] Resolvent analysis for small time frequency and use the formula

$$e^{-t\mathbb{L}_{\alpha,\Omega}} \mathbb{P}_\Omega f = \frac{1}{2\pi i} \int_\Gamma e^{t\lambda} (\lambda + \mathbb{L}_{\alpha,\Omega})^{-1} \mathbb{P}_\Omega f \, \mathrm{d}\lambda$$

with a compactly supported $f$.

# 4. Stability of physically reasonable solutions

Let $\delta, \delta' \in [0,1]$. Let $\rho_{\delta,\delta'}(\tau, X)$ be the function defined by

$$\rho_{\delta,\delta'}(\tau, X) = 1 + |X_1|^{\frac{1+\delta}{2}} + |X_1 - \tau|^{\frac{1+\delta}{2}}(1 + |X_1|^{\frac{\delta'}{2}}) + |X_2|^{1+\delta}.$$

$$\|f\|_{L^\infty_{\rho_{\delta,\delta'}(\alpha^2 t,\alpha\cdot)}} = \|\rho_{\delta,\delta'}(\alpha^2 t, \alpha\cdot) f\|_{L^\infty} = \sup_{x\in\Omega} \left|\rho_{\delta,\delta'}(\alpha^2 t, \alpha x) f(x)\right|.$$

**Note:** $\|f\|_{L^\infty} \le C(1+\alpha^2 t)^{-\frac{1+\delta}{2}} \|f\|_{L^\infty_{\rho_{\delta,\delta'}(\alpha^2 t,\alpha\cdot)}}$

$$\rho_{\delta,\delta'}(\tau, X) = 1 + |X_1|^{\frac{1+\delta}{2}} + |X_1 - \tau|^{\frac{1+\delta}{2}}(1 + |X_1|^{\frac{\delta'}{2}}) + |X_2|^{1+\delta}.$$

**Thm (M.): Linear estimate for full Oseen semigroup $e^{-t\mathbb{L}_{\alpha,\Omega}}$**

Let $\delta \in (0,1)$, $0 < \delta' \ll 1$, and $\tilde{\delta} \in (\delta + \delta', 1)$. Then there exists $\alpha_0 = \alpha_0(\delta, \delta', \tilde{\delta}) > 0$ such that if $\alpha \in (0, \alpha_0]$ then

$$\|e^{-t\mathbb{L}_{\alpha,\Omega}} f\|_{L^\infty_{\rho_{\delta,\delta'}(\alpha^2 t, \alpha\cdot)}} \le C\|f\|_{L^\infty_{\rho_{2\tilde{\delta},\delta'}(0,\alpha\cdot)}}, \qquad t > 0.$$

$$\|e^{-t\mathbb{L}_{\alpha,\Omega}} f\|_{L^\infty_{\rho_{\delta,\delta'}(\alpha^2 t,\alpha\cdot)}} \le C\|f\|_{L^\infty_{\rho_{2\tilde{\delta},\delta'}(0,\alpha\cdot)}}\,, \qquad t > 0\,.$$

The main problem is the estimate for large time. The proof is based on:

(i) Estimate of $e^{-t\mathbb{L}_{\alpha,\mathbb{R}^2}}$ (whole space problem)

(ii) Estimate of local energy decay, which is roughly speaking the estimate of $e^{-t\mathbb{L}_{\alpha,\Omega}} f$ for a compactly supported $f$

The argument of local energy decay is known in the analysis in the exterior domains;
Dan-Shibata (1999) for the Stokes semigroup $e^{-t\mathbb{A}_{\Omega}}$
Hishida (2016) for the pure Oseen semigroup $e^{-t\mathbb{A}_{\alpha,\Omega}}$

**Thm (M.): Estimate of $e^{-t\mathbb{L}_{\alpha,\mathbb{R}^2}}$**

Let $\delta \in (0,1)$ and $\delta' \in (0, \frac{1+\delta}{2}]$. Then for sufficiently small $\alpha > 0$ it follows that

$$\|e^{-t\mathbb{L}_{\alpha,\mathbb{R}^2}} f\|_{L^\infty_{\rho_{\delta,\delta'}(\alpha^2 t,\alpha\cdot)}} \le C\|f\|_{L^\infty_{\rho_{2\delta,\delta'}(0,\alpha\cdot)}}, \qquad t > 0.$$

Here $C$ depends only on $\delta$ and $\delta'$.

# 5. Estimate of $e^{-t\mathbb{L}_{\alpha,\mathbb{R}^2}}$ (the whole space problem)

Set $\tau = \alpha^2 t$ and $X = \alpha x$, and introduce the rescaling

$$v(\tau, X) = e^{-t\mathbb{L}_{\alpha,\mathbb{R}^2}} f \qquad \alpha q(\tau, X) = p(t, x) .$$

Then $v$ solves

$$\begin{cases} \partial_\tau v - \Delta v + \partial_1 v + V \cdot \nabla v + v \cdot \nabla V + \nabla q = 0 , & \tau > 0 , \quad X \in \mathbb{R}^2 , \\ \operatorname{div} v = 0 , & \tau \geq 0 , \quad X \in \mathbb{R}^2 , \\ v|_{\tau=0} = v_0 . & \end{cases}$$

That is,

$$v(\tau) = e^{-\tau\mathbb{A}_1} v_0 - \int_0^\tau e^{-(\tau-s)\mathbb{A}_1} \mathbb{P} \nabla \cdot (V \otimes v + v \otimes V) \, \mathrm{d}s .$$

Here $\mathbb{A}_1 = \mathbb{A}_{1,\mathbb{R}^2} = -\Delta + \partial_1$ (since $\mathbb{P} = \mathbb{P}_{\mathbb{R}^2}$ commutes with derivatives).

# 5. Estimate of $e^{-t\mathbb{L}_{\alpha,\mathbb{R}^2}}$ (the whole space problem)

$$e^{-\tau\mathbb{A}_1} f(X) = \int_{\mathbb{R}^2} G(\tau, X - Y - \tau \mathbf{e}_1)\, f(Y)\, \mathrm{d}Y\,,$$

$$e^{-\tau\mathbb{A}_1}\mathbb{P} f(X) = \int_{\mathbb{R}^2} \Phi(\tau, X - Y - \tau \mathbf{e}_1)\, f(Y)\, \mathrm{d}Y\,,$$

where $G(\tau, X) = \dfrac{1}{4\pi\tau} e^{-\frac{|X|^2}{4\tau}}$ and

$$\Phi(\tau, X) = \mathcal{F}^{-1}\Big[e^{-\tau|\xi|^2}\big(\mathbb{I} - \frac{\xi \otimes \xi}{|\xi|^2}\big)\Big](X)\,.$$

## Lemma

$$|\partial_\tau^k \nabla_X^j \Phi(\tau, X)| \le C\tau^{-1-\frac{j}{2}-k}\Big(1 + \frac{|X|^2}{\tau}\Big)^{-1-\frac{j}{2}-k}\,.$$

To focus on the essence we only consider the $L^\infty$ estimate of

$$|X_1|^{\frac{\sigma}{2}} e^{-\tau \mathbb{L}_{1,\mathbb{R}^2}} f \qquad 0 < \sigma < 1$$

when $f$ is bounded and has enough decay in $X_1$. Let us recall that $v(\tau) = e^{-\tau \mathbb{L}_{1,\mathbb{R}^2}} f$ is the solution to

$$v(\tau) = e^{-\tau \mathbb{A}_1} f - \int_0^\tau e^{-(\tau - s)\mathbb{A}_1} \mathbb{P} \nabla \cdot (V \otimes v + v \otimes V)\, ds\,.$$

# 5-1. Estimate of $e^{-t\mathbb{A}_{1,\mathbb{R}^2}}$ (the whole space problem)

From $|X_1| \le |X_1 - Y_1 - \tau| + \tau + |Y_1|$ we have

$$\begin{aligned} |X_1|^{\frac{\sigma}{2}} |e^{-\tau \mathbb{A}_1} f(X)| &\le C\tau^{-1+\frac{\sigma}{4}} \int_{\mathbb{R}^2} e^{-\frac{|X-Y-\tau \mathbf{e}_1|^2}{8\tau}} |f(Y)|\, \mathrm{d}Y \\ &\quad + C\tau^{-1+\frac{\sigma}{2}} \int_{\mathbb{R}^2} e^{-\frac{|X-Y-\tau \mathbf{e}_1|^2}{4\tau}} |f(Y)|\, \mathrm{d}Y \\ &\quad + C\tau^{-1} \int_{\mathbb{R}^2} e^{-\frac{|X-Y-\tau \mathbf{e}_1|^2}{4\tau}} |Y_1|^{\frac{\sigma}{2}} |f(Y)|\, \mathrm{d}Y \\ &= I + II + III\,. \end{aligned}$$

It is easy to see $\|I(\tau)\|_{L^\infty} + \|III(\tau)\|_{L^\infty} \le C\|(1+|X_1|^{\frac{\sigma}{2}}) f\|_{L^\infty}$. On the other hand, for $II$ we can only conclude

$$\|II(\tau)\|_{L^\infty} \le C\tau^{\frac{\sigma}{4}} \|(1+|X_1|^{\frac{\sigma}{2}}) f\|_{L^\infty}\,.$$

Therefore, to obtain the uniform bound in $\tau$ we need a decay of $f$ such as $(1+|X_1|^{\sigma})^{-1}$. Indeed, we have

$$\|II(\tau)\|_{L^\infty} \le C\|(1+|X_1|^{\sigma})f\|_{L^\infty}, \qquad \sigma \in [0,1).$$

That is, due to the transport effect, we have a decay-loss estimate

$$\|(1+|X_1|^{\frac{\sigma}{2}})e^{-\tau\mathbb{A}_1}f\|_{L^\infty} \le C\|(1+|X_1|^{\sigma})f\|_{L^\infty}, \qquad \tau > 0, \quad \sigma \in [0,1).$$

# 5-2. Estimate of the inhomogeneous term

A similar problem appears for the inhomogeneous term

$$W[V, v](\tau) = \int_0^\tau e^{-(\tau-s)\mathbb{A}_1}\mathbb{P}\nabla\cdot(V\otimes v + v\otimes V)\,\mathrm{d}s\,,$$

and in this case the difficulty is crucial since we are not allowed to impose faster spatial decay of $v$, otherwise the estimate can not be closed.

This discrepancy comes from the parabolic/hyperbolic scales of $e^{-(\tau-s)\mathbb{A}_1}$, i.e., to gain the decay $|X_1|^{-\frac{\sigma}{2}}$ we end up with the growth $(\tau - s)^{\frac{\sigma}{2}}$, which requires the decay $|Y_1|^{-\sigma}$ in addition for the sourcing term.

The key idea to keep the same order $|X_1|^{-\frac{\sigma}{2}}$ and $|Y_1|^{-\frac{\sigma}{2}}$ is to make use of the time integral $\int_0^\tau \mathrm{d}s$.

# 5-2. Estimate of the inhomogeneous term

Setting $\|v\| = \sup_{\tau>0} \|(1+|X_1|^{\frac{\sigma}{2}})v(\tau)\|_{L^\infty}$, we see

$$\left| W[V,v](\tau,X) \right|$$

$$\le C\|v\| \int_0^\tau \int_{\mathbb{R}^2} (\tau-s)^{-\frac{3}{2}} \Big(1+\frac{|X-Y-(\tau-s)\mathbf{e}_1|^2}{\tau-s}\Big)^{-\frac{3}{2}} \frac{|V(Y)|}{1+|Y_1|^{\frac{\sigma}{2}}}\, \mathrm{d}Y\, \mathrm{d}s$$

$$\le C\|v\| \int_{\mathbb{R}^2} \frac{1}{|X-Y|\,(1+|X-Y|-(X_1-Y_1))} \frac{|V(Y)|}{1+|Y_1|^{\frac{\sigma}{2}}}\, \mathrm{d}Y\,.$$

Since $|V(Y)| \le \dfrac{C}{|\log\alpha|\,|Y|^{\frac{1}{2}}} \le \dfrac{C}{|\log\alpha|\,|Y_1|^{\frac{1}{2}}}$ and

$$\int_{\mathbb{R}} \frac{1}{|X-Y|(1+|X-Y|-(X_1-Y_1))}\, \mathrm{d}Y_2 \le \frac{C}{|X_1-Y_1|^{\frac{1}{2}}}\,,$$

# 5-2. Estimate of the inhomogeneous term

we finally obtain

$$\Big| W[V,v](\tau,X) \Big| \leq \frac{C\|v\|}{|\log\alpha|} \int_{\mathbb{R}} \frac{1}{|X_1 - Y_1|^{\frac{1}{2}} |Y_1|^{\frac{1}{2}} (1 + |Y_1|^{\frac{\sigma}{2}})} \, dY_1$$

$$\leq \frac{C\|v\|}{|\log\alpha|} \frac{1}{|X_1|^{\frac{\sigma}{2}}}, \qquad \sigma \in (0,1).$$

Here $C$ depends only on $\sigma \in (0,1)$.

It is not difficult to estimate $W[V,v]$ for $|X_1| \leq 1$, and we conclude

$$\| W[V,v] \| \leq \frac{C}{|\log\alpha|} \|v\|,$$

which yields by the iteration, if $\alpha$ is small enough,

$$\|(1 + |X_1|^{\frac{\sigma}{2}}) e^{-\tau \mathbb{L}_{1,\mathbb{R}^2}} f\|_{L^\infty} \leq C \|(1 + |X_1|^{\sigma}) f\|_{L^\infty}, \qquad \tau > 0.$$

# 6. Local energy decay estimate of $e^{-t\mathbb{L}_{\alpha,\Omega}}$

**Lemma (M.): decay estimate of local energy**

For any $\kappa \in (0,1)$ there exists $\alpha(\kappa) > 0$ such that if $0 < \alpha \leq \alpha(\kappa)$ then

$$\|e^{-t\mathbb{L}_{\alpha,\Omega}}\mathbb{P}_\Omega f\|_{W^{1,\infty}(\Omega\cap B_4(0))} \leq C(1+t)^{-1+\kappa}(1+t^{-\frac{5}{6}})\|f\|_{L^3_{[5]}(\Omega)}.$$

Here $f \in L^3_{[5]}(\Omega)^2 = \{f \in L^3(\Omega)^2 \mid f = 0 \ \text{ for } |x| > 5\}$, and $C$ depends only on $\kappa$ and $\alpha$.

For the pure Oseen operator $\mathbb{P}_\Omega(-\Delta + \alpha\partial_1)$, the decay order is estimated as $O((1+t)^{-1-\kappa})$ with any $\kappa \in (0,1)$ by Hishida (2016).

The worse estimate $O((1+t)^{-1+\kappa})$ is due to the presence of the scale-critical term $\mathbb{P}_\Omega(U \cdot \nabla f + f \cdot \nabla U)$.

# 6. Local energy decay estimate of $e^{-t\mathbb{L}_{\alpha,\Omega}}$

The analysis of the local energy decay estimate is based on the study of the resolvent problem

$$\lambda u + \mathbb{L}_{\alpha,\Omega} u = \mathbb{P}_\Omega f\,, \qquad f \in L^3_{[5]}(\Omega)^2\,,$$
$$\mathbb{L}_{\alpha,\Omega} u = \mathbb{P}_\Omega \Big( -\Delta u + \alpha\partial_1 + \alpha\nabla\cdot(U\otimes u + u\otimes U)\Big)\,.$$

Here $\lambda \in \mathbb{C}$ is a resolvent parameter, and we consider the behavior of the resolvent operator for

$$\lambda = i\mu \ \text{ with } \ \mu \in \mathbb{R}\setminus\{0\}\,,\ |\mu| \ll 1\,.$$

The resolvent $(\lambda + \mathbb{L}_{\alpha,\Omega})^{-1}\mathbb{P}_\Omega$ is constructed as a compact perturbation to the sum of the resolvents in the whole space and in the bounded domain.

**Bogovskii Lemma**

The Bogovskii operator $\mathbb{B} : L^3(D) \to W_0^{1,3}(D)^2$ in $D = \{4 < |x| < 5\}$ satisfes

$$\nabla \cdot \mathbb{B}g = g \quad \text{in } D$$

for $g \in L^3(D)$ with $\int_D g \, \mathrm{d}x = 0$. Let $\Omega_b = \Omega \cap B_5(0)$.

# 6. Local energy decay estimate of $e^{-t\mathbb{L}_{\alpha,\Omega}}$

Let $\chi \in C_0^\infty(\mathbb{R}^2)$ be a cut-off such that $\chi = 1$ for $|x| \le 4$ and $\chi = 0$ for $|x| \ge 5$. Then we set

$$\begin{aligned}\mathcal{U}_\alpha[\lambda]f \;&=\; (1-\chi)\, u_{\alpha,\mathbb{R}^2}(\lambda) \;+\; \mathbb{B}[\nabla\chi\cdot u_{\alpha,\mathbb{R}^2}(\lambda)] \\ &\quad +\; \chi\, u_{\alpha,\Omega_b}(\lambda) \;-\; \mathbb{B}[\nabla\chi\cdot u_{\alpha,\Omega_b}(\lambda)]\,,\end{aligned}$$

$$\mathcal{P}_\alpha[\lambda]f \;=\; (1-\chi)\, p_{\alpha,\mathbb{R}^2}(\lambda) \;+\; \chi\, p_{\alpha,\Omega_b}(\lambda)\,.$$

Here

$$\begin{aligned}u_{\alpha,\mathbb{R}^2}(\lambda) &= (\lambda + \mathbb{L}_{\alpha,\mathbb{R}^2})^{-1}\mathbb{P}_{\mathbb{R}^2} f\,,\\ u_{\alpha,\Omega_b}(\lambda) &= (\lambda + \mathbb{L}_{\alpha,\Omega_b})^{-1}\mathbb{P}_{\Omega_b} f\,.\end{aligned}$$

**Note.** The construction of $u_{\alpha,\mathbb{R}^2}(\lambda)$ for small $\lambda$ is not trivial, and we need to solve the equations

$$u_\lambda = (\lambda + \mathbb{A}_{\alpha,\mathbb{R}^2})^{-1}\mathbb{P}_{\mathbb{R}^2} f - \alpha(\lambda + \mathbb{A}_{\alpha,\mathbb{R}^2})^{-1}\mathbb{P}_{\mathbb{R}^2}\nabla\cdot\Big(U\otimes u_\lambda + u_\lambda \otimes U\Big).$$

It is easy to see that

$$\mathrm{div}\, \mathcal{U}_\alpha[\lambda] f = 0 \quad \text{in } \Omega, \qquad \mathcal{U}_\alpha[\lambda] f = 0 \quad \text{on } \partial\Omega.$$

Moreover, we have

$$(\lambda + L_{\alpha,\Omega})\mathcal{U}_\alpha[\lambda] f + \nabla \mathcal{P}_\alpha[\lambda] f = (I + S_\alpha[\lambda]) f,$$

where $S_\alpha[\lambda] : L^3_{[5]}(\Omega)^2 \to L^3_{[5]}(\Omega)^2$ is a compact operator.

Although $S_\alpha[\lambda]$ is compact, its operator norm tends to infinity in the limit $\lambda, \alpha \to 0$.

This singularity comes from the behavior of the resolvent for the pure Oseen operator $\mathbb{A}_{\alpha,\mathbb{R}^2} = \mathbb{P}_{\mathbb{R}^2}(-\Delta + \alpha\partial_1)$ in the whole space.

The key is the expansion as follows: for $|x| \leq 5$ and $f \in L^3_{[5]}(\Omega)$,

$$(\lambda + \mathbb{A}_{\alpha,\mathbb{R}^2})^{-1}\mathbb{P}_{\mathbb{R}^2} f = E_0^0 * f + \Big( -\frac{1}{8\pi}\log(4\lambda + \alpha^2)\,\mathbb{I} + \mathbb{J}(\alpha,\lambda)\Big)\int_{\mathbb{R}^2} f \,\mathrm{d}x + \tilde{E}_\lambda^\alpha * f\,.$$

Here

$$E_0^0(x) = \Big( -\frac{1}{4\pi}\log|x|\Big)\mathbb{I} + \frac{x \otimes x}{|x|^2}\,,$$

$$|\mathbb{J}(\alpha,\lambda)| \leq C\,,$$

$$\|\tilde{E}_\lambda^\alpha * f\|_{W^{1,\infty}(\{|x|\leq 5\})} \leq C\,|\sqrt{4\lambda+\alpha^2}|\,|\log(4\lambda+\alpha^2)|\,\|f\|_{L^3_{[5]}(\Omega)}\,.$$

As in the works by Dan-Shibata (1999, for the Stokes operator) and by Hishida (2016, for the pure Oseen operator), $I + S_\alpha[\lambda]$ is written in the form

$$I + S_\alpha[\lambda] = \Theta_\alpha[\lambda] + N_\alpha[\lambda]$$

$\Theta_\alpha[\lambda]$: invertible for small $\lambda$ and $\alpha$ together with the uniform bound

$N_\alpha[\lambda]$: finite rank operator of the form

$$N_\alpha[\lambda] = \sum_{j=1}^{N} \langle \phi_j(\alpha,\lambda), f\rangle_{L^2(\Omega)}\, w_j\,, \qquad 2 \le \exists N \le 4,.$$

Here $\phi_j(\alpha,\lambda)$ is a constant vector in $\mathbb{R}^2$ (containing $\log(4\lambda+\alpha^2)$), and $w_j$ is a compactly supported function independent of $\lambda$ and $\alpha$.

Then we can write

$$I + S_\alpha[\lambda] = \Theta_\alpha[\lambda]\left(I + \Theta_\alpha[\lambda]^{-1}N_\alpha[\lambda]\right),$$

$$\Theta_\alpha[\lambda]^{-1}N_\alpha[\lambda] = \sum_{j=1}^{N}\langle\phi_j(\alpha,\lambda), f\rangle_{L^2(\Omega)}\,\Theta_\alpha[\lambda]^{-1}w_j\,.$$

Let $M(\alpha,\lambda)$ be the $N \times N$ matrix given by

$$M(\alpha,\lambda) = \mathbb{I} + \left(\langle\phi_j(\alpha,\lambda), \Theta_\alpha[\lambda]^{-1}w_k\rangle_{L^2(\Omega)}\right)_{1\le j,k\le N}\,.$$

# 6. Local energy decay estimate of $e^{-t\mathbb{L}_{\alpha,\Omega}}$

The direct computation shows

$$\det M(\alpha, \lambda) = \sum_{j=0}^{2} C_j(\alpha, \lambda) \left(\frac{1}{2} \log(4\lambda + \alpha^2)\right)^j ,$$

where $C_j(\alpha, \lambda)$ is bounded and continuous on small $\lambda$ and $\alpha$. The constant $C_2(0, 0)$ is already considered in the analysis of $\mathbb{P}_\Omega(-\Delta + \alpha\partial_1)$ by Hishida (2016), and it is shown by Hishida (2016) that $C_2(0, 0) \neq 0$.

Hence, the matrix $M(\alpha, \lambda)$ is invertible for small $\alpha$ and $\lambda$, and with its inverse $M(\alpha, \lambda)^{-1} = (b_{jk}(\alpha, \lambda))_{1 \le j,k \le N}$, the inverse of $I + \Theta_\alpha[\lambda]^{-1} N_\alpha[\lambda]$ is described as

$$\left(I + \Theta_\alpha[\lambda]^{-1} N_\alpha[\lambda]\right)^{-1} f = f - \sum_{j,k} \langle \phi_k(\alpha, \lambda), f \rangle_{L^2(\Omega)} \, b_{jk}(\alpha, \lambda) \, w_j \, .$$

In the end, we arrive at the formula

$$(I + S_\alpha[\lambda])^{-1} = \Big(I + \Theta_\alpha[\lambda]^{-1} N_\alpha[\lambda]\Big)^{-1} \Theta_\alpha[\lambda]^{-1},$$

and this verifies the representation

$$(\lambda + \mathbb{L}_{\alpha,\Omega})^{-1}\mathbb{P}_\Omega f = \mathcal{U}_\alpha[\lambda]\,(I + S_\alpha[\lambda])^{-1} f, \qquad f \in L^3_{[5]}(\Omega).$$

With these formulas one can show

$$\|(\lambda + \mathbb{L}_{\alpha,\Omega})^{-1}\mathbb{P}_\Omega f\|_{W^{1,3}(\Omega\cap B_4(0))} \le C\|f\|_{L^3_{[5]}(\Omega)},$$

$$\begin{aligned}\|\partial_\lambda(\lambda + \mathbb{L}_{\alpha,\Omega})^{-1}\mathbb{P}_\Omega f\|_{W^{1,3}(\Omega\cap B_4(0))} \le \frac{C}{|\lambda|\,|\log(\lambda+\alpha^2)|}\|f\|_{L^3_{[5]}(\Omega)} \\ + \frac{C_\kappa}{|\lambda|\,|\log\alpha|}\Big(1 + \frac{\alpha^{2\kappa}}{|\lambda|^\kappa}\Big)\|f\|_{L^3_{[5]}(\Omega)},\end{aligned}$$

for small $\lambda = i\mu$ and $\alpha$.

# 7. Open question

**1.** Can we show the $L^p$ Boundedness ?

$$\|e^{-t\mathbb{L}_{\alpha,\Omega}} f\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}\,, \qquad t > 0\,. \tag{1}$$

**2.** Can we remove the dependence on $\alpha$ for the size of the initial perturbations in the main theorem ?

**3.** Analysis for nonsmall $\alpha$ ?