On the stability of the physically reasonable solution to the two-dimensional Navier-Stokes equations

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1. Stability of two-dimensional exterior flows

\( \Omega \): exterior domain in \( \mathbb{R}^2 \) with smooth boundary

\( U = (U_1(t, x), U_2(t, x)) \): velocity field of fluid, \( (t, x) \in [0, \infty) \times \Omega \)

- \( \text{div} \ U = 0 \)
- No-slip boundary condition

**Basic interest**

Assuming that \( U \) is a global solution to the Navier-Stokes equations, we are interested in the stability property of \( U \).
The stability property of $U$ is closely related to the decay property (and its size) of $U$ in time and space.

It is heuristically known that the invariant spaces about the scaling

$$U_\lambda(t, x) = \lambda U(\lambda^2 t, \lambda x), \quad \lambda > 0$$

play a fundamental role. The Banach space $Y$ is called a scale-critical space for NS if

$$\|f_\lambda\|_Y = \|f\|_Y, \quad \lambda > 0.$$ 

Typical examples for 2D Navier-Stokes equations are

$$Y = L^\infty(0, \infty; L^2(\mathbb{R}^2)) \quad \text{or} \quad Y = L^\infty(0, \infty; L^{2,\infty}(\mathbb{R}^2)),$$ 

where $L^{2,\infty}(\mathbb{R}^2)$ is the weak $L^2$ space, which contains a function such as $f(x) = |x|^{-1}.$
We expect that if $Y$ is a scale-critical space and

$$\|U\|_Y \ll 1,$$

then $U$ is stable (in a suitable sense) about disturbances.

However, for the 2D NS in $\Omega$, such a general result has not been established yet, though 3D case is well understood; Heywood (1970), Borchers and Miyakawa (1995), Kozono-Yamazaki (1998), Yamazaki (2000), and so on.

**Several difficulties in 2D case:**
- absence of the scale-critical Hardy inequality:

  $$\left\| \frac{f}{|x|} \right\|_{L^d(\mathbb{R}^d)} \leq C\|\nabla f\|_{L^d(\mathbb{R}^d)} \quad \text{for } d \geq 3, \ d \neq 2.$$

- decay of Newton potential:

  $$E_{\mathbb{R}^2}(x) = c_2 \log |x|, \quad E_{\mathbb{R}^3}(x) = c_3 |x|^{-1}$$
2. Physically reasonable solutions to 2D Navier-Stokes equations

\[
\begin{align*}
\partial_t u + \alpha u \cdot \nabla u - \Delta u + \alpha \partial_1 u + \nabla p &= 0, \quad t > 0, \quad x \in \Omega, \\
\text{div } u &= 0, \quad t \geq 0, \quad x \in \Omega, \\
\left. u \right|_{\partial \Omega} &= -e_1, \quad \lim_{|x| \to \infty} u = 0, \quad \left. u \right|_{t=0} = u_0.
\end{align*}
\]

- The origin is interior of \( \mathbb{R}^2 \setminus \Omega \).
- \( \text{diam} \left( \mathbb{R}^2 \setminus \Omega \right) = 1 \)
- The number \( \alpha > 0 \) represents the Reynolds number.
2. Physically reasonable solutions to 2D Navier-Stokes equations

\[\frac{\partial}{\partial t} u + \alpha u \cdot \nabla u - \Delta u + \alpha \partial_1 u + \nabla p = 0, \quad t > 0, \quad x \in \Omega,\]

\[\text{div} \ u = 0, \quad t \geq 0, \quad x \in \Omega,\]

\[u|_{\partial \Omega} = -e_1, \quad \lim_{|x| \to \infty} u = 0, \quad u|_{t=0} = u_0.\]

After pioneering work by Leray (1933), the following result was proved.

Thm (Finn-Smith, 1967; Galdi, 1994): Existence for small \(\alpha\)

There exists \(\alpha_0 > 0\) such that if \(0 < \alpha \leq \alpha_0\) then there exists a unique stationary solution \(U\) to (NS) such that

\[|U(x)| \leq \frac{C}{|\log \alpha|} \frac{1}{|\alpha x|^{\frac{1}{2}}}, \quad x \in \Omega.\]
2. Physically reasonable solutions to 2D Navier-Stokes equations

\[
\begin{aligned}
\alpha U \cdot \nabla U - \Delta U + \alpha \partial_1 U + \nabla P &= 0, \quad x \in \Omega, \\
\text{div} \, U &= 0, \quad x \in \Omega, \\
U|_{\partial \Omega} &= -e_1, \quad \lim_{|x| \to \infty} U = 0.
\end{aligned}
\]

\[
|U(x)| \leq \frac{C}{|\log \alpha|^{|1/2|}} \cdot \frac{1}{|\alpha x|^{-1/2}} , \quad x \in \Omega.
\]

**Remark.** (1) The decay order \(O(|\alpha x|^{-1/2})\) is much slower than the scale-critical order \(O(|\alpha x|^{-1})\).

(2) The smallness factor \(\frac{1}{|\log \alpha|}\) reflects the Stokes paradox for 2D exterior flows (Chang and Finn (1961)): there exist no solutions for the Stokes problem \(-\Delta u + \nabla p = 0\) in \(\Omega\) and \(u|_{\partial \Omega} = -e_1\) satisfying \(\lim_{|x| \to \infty} u = 0\).
3. Wake estimate and scale-critical decay

\[
\begin{cases}
\alpha U \cdot \nabla U - \Delta U + \alpha \partial_1 U + \nabla P = 0, & x \in \Omega, \\
\text{div } U = 0, & x \in \Omega, \\
U|_{\partial \Omega} = -e_1, & \lim_{|x| \to \infty} U = 0.
\end{cases}
\]

It is well known that \( U \) possesses a \textit{wake structure} about the decay at \(|x| \to \infty\); Smith (1965); Babenko (1970); Galdi (1994).

**Decay estimate of physically reasonable solutions**

There exists an extension of \( U \) to \( \mathbb{R}^2 \) such that \( \text{div } U = 0 \) in \( \mathbb{R}^2 \) and \( V(X) = U(\frac{X}{\alpha}) \) satisfies

\[
|V(X)| \leq \frac{C}{|\log \alpha|} \left( \frac{1}{|X|^{1/2}(1 + |X| - X_1)^{3/4}} + \frac{1}{1 + |X|} \right), \quad X \in \mathbb{R}^2.
\]

and \( \|\nabla U\|_{L^4} \leq C \). Here \( C \) is independent of small \( \alpha \).
Lemma

There exist $\kappa, C > 0$ such that

$$|X| - X_1 \geq \begin{cases} 
  C|X| & \text{if } |X_2| \geq \kappa|X_1|, \\
  \frac{X_2^2}{4|X_1|} & \text{if } |X_2| \leq \kappa|X_1|. 
\end{cases}$$

Proof. Use $|X| = |X_1| \left(1 + \left|\frac{X_2}{X_1}\right|^2\right)^{\frac{1}{2}}$ for $|X_2| \leq \kappa|X_1|$ with small $\kappa > 0$. 

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3. Wake estimate and scale-critical decay

\[ |V(X)| \leq \frac{C}{|\log \alpha|} \left( \frac{1}{|X|^\frac{1}{2}(1 + |X| - X_1)^\frac{3}{4}} + \frac{1}{1 + |X|} \right), \quad X \in \mathbb{R}^2 \]

\[ \frac{1}{|X|^\frac{1}{2}(1 + |X| - X_1)^\frac{3}{4}} \leq \frac{C}{|X|^\frac{1}{2} + |X_2|} \left( \min \{1, \frac{|X_1|^\frac{1}{4}}{|X_2|^\frac{1}{2}} \} + \frac{1}{1 + |X|^\frac{1}{2}} \right) \]

Corollary: scale-criticality of physically reasonable solutions

\[ X_2 V(X) \in L^{\infty}(\mathbb{R}^2). \text{ Moreover, } V \in L^{\infty}_{X_1} L^{1}_{X_2} + L^{2,\infty}(\mathbb{R}^2). \]

Note that \( f_{\lambda}(X) = \lambda f(\lambda X) \) satisfies

\[ \|X_2 f_{\lambda}\|_{L^{\infty}} = \|X_2 f\|_{L^{\infty}} \quad \text{for } \lambda > 0. \]
4. Stability of physically reasonable solutions

\[ D(A_{\alpha,\Omega}) = W^{2,3}(\Omega) \cap W^{1,3}_0(\Omega), \quad A_{\alpha,\Omega}f = -\Delta f + \alpha \partial_1 f \]

\[ L^3_\sigma(\Omega) = \{ f \in C^\infty_0(\Omega)^2 \mid \text{div } f = 0 \} \]

Pure Oseen operator

\[ D(A_{\alpha,\Omega}) = D(A_{\alpha,\Omega})^2 \cap L^3_\sigma(\Omega), \quad A_{\alpha,\Omega}f = \mathbb{P}_\Omega A_{\alpha,\Omega} f \]

Thm (Hishida, 2016): \( L^p - L^q \) Estimates for Oseen semigroup \( e^{-tA_{\alpha,\Omega}} \)

\[ \| e^{-tA_{\alpha,\Omega}} f \|_{L^p} \leq C \left( 1 + \alpha^{-\beta_{p,q}} \right) t^{-\frac{1}{q} + \frac{1}{p}} \| f \|_{L^q}, \quad t > 0, \]

for \( 1 < q \leq p < \infty \). Here \( C \) depends only on \( \Omega, \alpha, p, \) and \( q \), and \( \beta_{p,q} > 0 \) depends on \( p \) and \( q \).
4. Stability of physically reasonable solutions

\[ D(A_{\alpha,\Omega}) = W^{2,3}(\Omega) \cap W_0^{1,3}(\Omega), \quad A_{\alpha,\Omega}f = -\Delta f + \alpha \partial_1 f, \]
\[ D(L_{\alpha,\Omega}) = D(A_{\alpha,\Omega})^2, \quad L_{\alpha,\Omega}f = A_{\alpha,\Omega}f + \alpha(U \cdot \nabla f + f \cdot \nabla U). \]

**Pure Oseen operator**

\[ D(\mathbb{A}_{\alpha,\Omega}) = D(A_{\alpha,\Omega})^2 \cap L^3_\sigma(\Omega), \quad \mathbb{A}_{\alpha,\Omega}f = \mathbb{P}_\Omega A_{\alpha,\Omega}f, \]

**Full Oseen operator**

\[ D(\mathbb{L}_{\alpha,\Omega}) = D(\mathbb{A}_{\alpha,\Omega}), \quad \mathbb{L}_{\alpha,\Omega}f = \mathbb{P}_\Omega L_{\alpha,\Omega}f. \]
4. Stability of physically reasonable solutions

\[ v(t) = e^{-t|\mathbb{L}_{\alpha,\Omega}}v_0 - \alpha \int_0^t e^{-(t-s)|\mathbb{L}_{\alpha,\Omega}}P_{\Omega} \nabla \cdot (v \otimes v)(s) \, ds \quad \text{(INS}_{\alpha}) \]

**Thm (M.):** stability of physically reasonable solutions

Set \( b_{\alpha}(x) = |\alpha x|^{1/2} + |\alpha x_2| \). For any \( \delta \in (0, 1) \) there exists \( \alpha_0 = \alpha_0(\delta) > 0 \) such that the following statement holds for all \( \alpha \in (0, \alpha_0] \). There is \( \epsilon = \epsilon(\delta, \alpha) > 0 \) such that for any \( v_0 \in L^3_{\sigma}(\Omega) \) satisfying

\[ \|(1 + b_{\alpha})^{1+\delta} v_0\|_{L^\infty} \leq \epsilon, \]

equation (INS_{\alpha}) admits a unique solution \( v \in C([0, \infty); L^3_{\sigma}(\Omega)) \cap C((0, \infty); W^{1,3}_0(\Omega)^2) \) satisfying

\[ \lim_{t \to \infty} \|v(t)\|_{L^3} = \lim_{t \to \infty} t^{\frac{1}{2}} \|v(t)\|_{L^\infty} = 0. \]
For 3D case, the nonlinear stability of physically reasonable solutions is proved by

Heywood (1970): for $L^2$ disturbances

In 3D case, the stationary solution $U$ satisfies the estimate

$$|U(x)| \leq \frac{C}{|x|}, \quad x \in \Omega.$$  

cf) In 2D case:

$$|U(x)| \leq \frac{C}{|\log \alpha||\alpha x|^{1/2}}.$$
4. Stability of physically reasonable solutions

The core part of the proof is the analysis of $e^{-t\mathbb{L}_\alpha,\Omega}$.

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<th>Basic strategy for the linear analysis of $e^{-t\mathbb{L}_\alpha,\Omega}$</th>
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| **1. Estimate of flows far from the boundary:** Introduce a suitable weighted function space by taking into account the *wake structure* and the transport term $\alpha \partial_1$. The key step is to analyze $u(t) = e^{-t\mathbb{L}_\alpha,\mathbb{R}^2} f$ with

$$u(t) = e^{-tA_{\alpha,\mathbb{R}^2}} f - \alpha \int_0^t e^{-(t-s)A_{\alpha,\mathbb{R}^2}} \nabla \cdot (U \otimes u + u \otimes U) \, ds$$

**2. Estimate of flows near the boundary:** Resolvent analysis for small time frequency and use the formula

$$e^{-t\mathbb{L}_{\alpha,\Omega} \mathbb{P}_\Omega} f = \frac{1}{2\pi i} \int_{\Gamma} e^{t\lambda} (\lambda + \mathbb{L}_{\alpha,\Omega})^{-1} \mathbb{P}_\Omega f \, d\lambda$$

with a compactly supported $f$. |
4. Stability of physically reasonable solutions

Let $\delta, \delta' \in [0, 1]$. Let $\rho_{\delta,\delta'}(\tau, X)$ be the function defined by

$$
\rho_{\delta,\delta'}(\tau, X) = 1 + |X_1|^{\frac{1+\delta}{2}} + |X_1 - \tau|^{\frac{1+\delta}{2}} (1 + |X_1|^{\frac{\delta'}{2}}) + |X_2|^{1+\delta}.
$$

$$
\|f\|_{L^\infty_{\rho_{\delta,\delta'}(\alpha^2 t, \alpha \cdot)}} = \|\rho_{\delta,\delta'}(\alpha^2 t, \alpha \cdot) f\|_{L^\infty} = \sup_{x \in \Omega} |\rho_{\delta,\delta'}(\alpha^2 t, \alpha x) f(x)|.
$$

Note: $\|f\|_{L^\infty} \leq C(1 + \alpha^2 t)^{-\frac{1+\delta}{2}} \|f\|_{L^\infty_{\rho_{\delta,\delta'}(\alpha^2 t, \alpha \cdot)}}$
4. Stability of physically reasonable solutions

\[ \rho_{\delta,\delta'}(\tau, X) = 1 + |X_1|^{\frac{1+\delta}{2}} + |X_1 - \tau|^{\frac{1+\delta}{2}} (1 + |X_1|^{\frac{\delta'}{2}}) + |X_2|^{1+\delta}. \]

**Thm (M.): Linear estimate for full Oseen semigroup** \( e^{-t\|L_\alpha,\Omega} \)

Let \( \delta \in (0, 1) \), \( 0 < \delta' \ll 1 \), and \( \tilde{\delta} \in (\delta + \delta', 1) \). Then there exists \( \alpha_0 = \alpha_0(\delta, \delta', \tilde{\delta}) > 0 \) such that if \( \alpha \in (0, \alpha_0] \) then

\[ \|e^{-t\|L_\alpha,\Omega} f\|_{L^\infty} \leq C \|f\|_{L^\infty} \rho_{2\tilde{\delta},\delta'}^{\rho_{\alpha^2 t,\alpha'}}(0,\alpha'), \quad t > 0. \]
4. Stability of physically reasonable solutions

\[ \| e^{-tI_{\alpha,\Omega}} f \|_{L^\infty_{\rho_\delta,\delta'}(\alpha^2 t, \alpha t)} \leq C\| f \|_{L^\infty_{\rho_{2\delta,\delta'}(0, \alpha t)}} , \quad t > 0. \]

The main problem is the estimate for large time. The proof is based on:

(i) Estimate of \( e^{-tI_{\alpha,\mathbb{R}^2}} \) (whole space problem)

(ii) Estimate of local energy decay, which is roughly speaking the estimate of \( e^{-tI_{\alpha,\Omega}} f \) for a compactly supported \( f \)

The argument of local energy decay is known in the analysis in the exterior domains;
Dan-Shibata (1999) for the Stokes semigroup \( e^{-tA_{\Omega}} \)
Hishida (2016) for the pure Oseen semigroup \( e^{-tA_{\alpha,\Omega}} \)
5. Estimate of $e^{-t\mathbb{L}_{\alpha,\mathbb{R}^2}}$ (the whole space problem)

Thm (M.): Estimate of $e^{-t\mathbb{L}_{\alpha,\mathbb{R}^2}}$

Let $\delta \in (0, 1)$ and $\delta' \in (0, \frac{1+\delta}{2}]$. Then for sufficiently small $\alpha > 0$ it follows that

$$\|e^{-t\mathbb{L}_{\alpha,\mathbb{R}^2}} f\|_{L^\infty_{\rho_{\delta,\delta'}(\alpha^2 t, \alpha^2)}} \leq C \|f\|_{L^\infty_{\rho_{2\delta,\delta'}(0, \alpha^2)}} , \quad t > 0 .$$

Here $C$ depends only on $\delta$ and $\delta'$. 
5. Estimate of $e^{-t\mathbb{L}_{\alpha,\mathbb{R}^2}}$ (the whole space problem)

Set $\tau = \alpha^2 t$ and $X = \alpha x$, and introduce the rescaling

$$\nu(\tau, X) = e^{-t\mathbb{L}_{\alpha,\mathbb{R}^2}} f \quad \quad \alpha q(\tau, X) = p(t, x).$$

Then $\nu$ solves

$$\begin{cases}
\partial_\tau \nu - \Delta \nu + \partial_1 \nu + V \cdot \nabla \nu + \nu \cdot \nabla V + \nabla q = 0, & \tau > 0, \quad X \in \mathbb{R}^2, \\
\text{div} \, \nu = 0, & \tau \geq 0, \quad X \in \mathbb{R}^2, \\
\nu|_{\tau=0} = \nu_0.
\end{cases}$$

That is,

$$\nu(\tau) = e^{-\tau\mathbb{A}_1} \nu_0 - \int_0^\tau e^{-(\tau-s)\mathbb{A}_1} \text{P} \nabla \cdot (V \otimes \nu + \nu \otimes V) \, ds.$$  

Here $\mathbb{A}_1 = \mathbb{A}_{1,\mathbb{R}^2} = -\Delta + \partial_1$ (since $\text{P} = \text{P}_{\mathbb{R}^2}$ commutes with derivatives).
5. Estimate of $e^{-tL_{\alpha,\mathbb{R}^2}}$ (the whole space problem)

\[ e^{-\tau A_1} f(X) = \int_{\mathbb{R}^2} G(\tau, X - Y - \tau e_1) f(Y) \, dY, \]
\[ e^{-\tau A_1 \mathbb{P}} f(X) = \int_{\mathbb{R}^2} \Phi(\tau, X - Y - \tau e_1) f(Y) \, dY, \]

where \( G(\tau, X) = \frac{1}{4\pi \tau} e^{-\frac{|X|^2}{4\tau}} \) and

\[ \Phi(\tau, X) = \mathcal{F}^{-1} \left[ e^{-\tau|\xi|^2} \left( I - \frac{\xi \otimes \xi}{|\xi|^2} \right) \right](X). \]

Lemma

\[ |\partial_{\tau}^k \nabla_X^j \Phi(\tau, X)| \leq C \tau^{-1 - \frac{j}{2} - k} \left( 1 + \frac{|X|^2}{\tau} \right)^{-1 - \frac{j}{2} - k}. \]
To focus on the essence we only consider the $L^\infty$ estimate of

$$|X_1|^\frac{\sigma}{2} e^{-\tau L_{1,\mathbb{R}^2}} f \quad 0 < \sigma < 1$$

when $f$ is bounded and has enough decay in $X_1$. Let us recall that $v(\tau) = e^{-\tau L_{1,\mathbb{R}^2}} f$ is the solution to

$$v(\tau) = e^{-\tau A_1} f - \int_0^\tau e^{-(\tau-s)A_1} \mathbb{P} \nabla \cdot (V \otimes v + v \otimes V) \, ds.$$
5-1. Estimate of \( e^{-tA_{1,R^2}} \) (the whole space problem)

From \( |X_1| \leq |X_1 - Y_1 - \tau| + \tau + |Y_1| \) we have

\[
|X_1|^\frac{\sigma}{2} |e^{-tA_{1,R^2}} f(X)| \leq C\tau^{-1+\frac{\sigma}{4}} \int_{\mathbb{R}^2} e^{-\frac{|X-Y-\tau e_1|^2}{8\tau}} |f(Y)| \, dY
\]

\[
+ C\tau^{-1+\frac{\sigma}{2}} \int_{\mathbb{R}^2} e^{-\frac{|X-Y-\tau e_1|^2}{4\tau}} |f(Y)| \, dY
\]

\[
+ C\tau^{-1} \int_{\mathbb{R}^2} e^{-\frac{|X-Y-\tau e_1|^2}{4\tau}} |Y_1|^\frac{\sigma}{2} |f(Y)| \, dY
\]

\[
= I + II + III.
\]

It is easy to see \( ||I(\tau)||_{L^\infty} + ||III(\tau)||_{L^\infty} \leq C||(1 + |X_1|^\frac{\sigma}{2})f||_{L^\infty} \). On the other hand, for \( II \) we can only conclude

\[
||II(\tau)||_{L^\infty} \leq C\tau^\frac{\sigma}{4} \|(1 + |X_1|^\frac{\sigma}{2})f\|_{L^\infty}.
\]
Therefore, to obtain the uniform bound in $\tau$ we need a decay of $f$ such as $(1 + |X_1|^{\sigma})^{-1}$. Indeed, we have

$$
\|II(\tau)\|_{L^\infty} \leq C\|(1 + |X_1|^{\sigma})f\|_{L^\infty}, \quad \sigma \in [0, 1).
$$

That is, due to the transport effect, we have a decay-loss estimate

$$
\| (1 + |X_1|^\frac{\sigma}{2})e^{-\tau A_1}f\|_{L^\infty} \leq C\|(1 + |X_1|^{\sigma})f\|_{L^\infty}, \quad \tau > 0, \quad \sigma \in [0, 1).
$$
A similar problem appears for the inhomogeneous term

\[ W[V, \nu](\tau) = \int_0^\tau e^{-(\tau-s)\bar{A}_1} \mathbb{P} \nabla \cdot (V \otimes \nu + \nu \otimes V) \, ds, \]

and in this case the difficulty is crucial since we are not allowed to impose faster spatial decay of \( \nu \), otherwise the estimate can not be closed.

This discrepancy comes from the parabolic/hyperbolic scales of \( e^{-(\tau-s)\bar{A}_1} \), i.e., to gain the decay \( |X_1|^{-\frac{\sigma}{2}} \) we end up with the growth \( (\tau - s)^{\frac{\sigma}{2}} \), which requires the decay \( |Y_1|^{-\sigma} \) in addition for the sourcing term.

The key idea to keep the same order \( |X_1|^{-\frac{\sigma}{2}} \) and \( |Y_1|^{-\frac{\sigma}{2}} \) is to make use of the time integral \( \int_0^\tau ds \).
5-2. Estimate of the inhomogeneous term

Setting \( \|v\| = \sup_{\tau>0} \|(1 + |X_1|^\frac{\sigma}{2})v(\tau)\|_{L^\infty} \), we see

\[
\left| W[V, v](\tau, X) \right| \\
\leq C \|v\| \int_0^\tau \int_{\mathbb{R}^2} (\tau - s)^{-\frac{3}{2}} (1 + \frac{|X - Y - (\tau - s)e_1|^2}{\tau - s})^{-\frac{3}{2}} \frac{|V(Y)|}{1 + |Y|^{\frac{\sigma}{2}}} \, dB \, ds \\
\leq C \|v\| \int_{\mathbb{R}^2} \frac{1}{|X - Y|(1 + |X - Y| - (X_1 - Y_1))} \frac{|V(Y)|}{1 + |Y|^{\frac{\sigma}{2}}} \, dY.
\]

Since \( |V(Y)| \leq \frac{C}{|\log \alpha| |Y|^{\frac{1}{2}}} \leq \frac{C}{|\log \alpha| |Y_1|^{\frac{1}{2}}} \) and

\[
\int_{\mathbb{R}} \frac{1}{|X - Y|(1 + |X - Y| - (X_1 - Y_1))} \, dB_2 \leq \frac{C}{|X_1 - Y_1|^{\frac{1}{2}}},
\]
we finally obtain

\[ |W[V, v](\tau, X)| \leq \frac{C||v||}{|\log \alpha|} \int_\mathbb{R} \frac{1}{|X_1 - Y_1|^{\frac{1}{2}}|Y_1|^{\frac{1}{2}}(1 + |Y_1|^{\frac{\sigma}{2}})} \, dY_1 \]

\[ \leq \frac{C||v||}{|\log \alpha|} \frac{1}{|X_1|^{\frac{\sigma}{2}}}, \quad \sigma \in (0, 1). \]

Here \( C \) depends only on \( \sigma \in (0, 1) \).

It is not difficult to estimate \( W[V, v] \) for \( |X_1| \leq 1 \), and we conclude

\[ \|W[V, v]\| \leq \frac{C}{|\log \alpha|} ||v||, \]

which yields by the iteration, if \( \alpha \) is small enough,

\[ \|(1 + |X_1|^{\frac{\sigma}{2}})e^{-\tau L_{1,\mathbb{R}^2}} f\|_{L^\infty} \leq C\|(1 + |X_1|^{\sigma})f\|_{L^\infty}, \quad \tau > 0. \]
6. Local energy decay estimate of $e^{-t\mathbb{L}_\alpha,\Omega}$

Lemma (M.): decay estimate of local energy

For any $\kappa \in (0, 1)$ there exists $\alpha(\kappa) > 0$ such that if $0 < \alpha \leq \alpha(\kappa)$ then

$$
\|e^{-t\mathbb{L}_\alpha,\Omega} \mathbb{P}_\Omega f\|_{W^{1,\infty}(\Omega \cap B_4(0))} \leq C(1 + t)^{-1+\kappa}(1 + t^{-\frac{5}{6}})\|f\|_{L^3(\Omega)}.
$$

Here $f \in L^3_{[5]}(\Omega)^2 = \{f \in L^3(\Omega)^2 \mid f = 0 \text{ for } |x| > 5\}$, and $C$ depends only on $\kappa$ and $\alpha$.

For the pure Oseen operator $\mathbb{P}_\Omega(-\Delta + \alpha \partial_1)$, the decay order is estimated as $O((1 + t)^{-1-\kappa})$ with any $\kappa \in (0, 1)$ by Hishida (2016).

The worse estimate $O((1 + t)^{-1+\kappa})$ is due to the presence of the scale-critical term $\mathbb{P}_\Omega(U \cdot \nabla f + f \cdot \nabla U)$. 

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The analysis of the local energy decay estimate is based on the study of the resolvent problem

\[ \lambda u + L_{\alpha,\Omega} u = P_{\Omega} f, \quad f \in L_{[5]}^3 (\Omega)^2, \]

\[ L_{\alpha,\Omega} u = P_{\Omega} \left( -\Delta u + \alpha \partial_1 + \alpha \nabla \cdot (U \otimes u + u \otimes U) \right). \]

Here \( \lambda \in \mathbb{C} \) is a resolvent parameter, and we consider the behavior of the resolvent operator for

\[ \lambda = i\mu \quad \text{with} \quad \mu \in \mathbb{R} \setminus \{0\}, \quad |\mu| \ll 1. \]
The resolvent \((\lambda + L_{\alpha,\Omega})^{-1}P_{\Omega}\) is constructed as a compact perturbation to the sum of the resolvents in the whole space and in the bounded domain.

**Bogovskii Lemma**

The Bogovskii operator \(B : L^3(D) \to W^{1,3}_0(D)^2\) in \(D = \{4 < |x| < 5\}\) satisfies

\[
\nabla \cdot Bg = g \quad \text{in } D
\]

for \(g \in L^3(D)\) with \(\int_D g \, dx = 0\). Let \(\Omega_b = \Omega \cap B_5(0)\).
Let $\chi \in C^0_0(\mathbb{R}^2)$ be a cut-off such that $\chi = 1$ for $|x| \leq 4$ and $\chi = 0$ for $|x| \geq 5$. Then we set

$$U_\alpha[\lambda]f = (1 - \chi) u_{\alpha,\mathbb{R}^2}(\lambda) + \mathcal{B}[\nabla \chi \cdot u_{\alpha,\mathbb{R}^2}(\lambda)] + \chi u_{\alpha,\Omega_b}(\lambda) - \mathcal{B}[\nabla \chi \cdot u_{\alpha,\Omega_b}(\lambda)],$$

$$P_\alpha[\lambda]f = (1 - \chi) p_{\alpha,\mathbb{R}^2}(\lambda) + \chi p_{\alpha,\Omega_b}(\lambda).$$

Here

$$u_{\alpha,\mathbb{R}^2}(\lambda) = (\lambda + \mathbb{L}_{\alpha,\mathbb{R}^2})^{-1} P_{\mathbb{R}^2} f,$$

$$u_{\alpha,\Omega_b}(\lambda) = (\lambda + \mathbb{L}_{\alpha,\Omega_b})^{-1} P_{\Omega_b} f.$$

**Note.** The construction of $u_{\alpha,\mathbb{R}^2}(\lambda)$ for small $\lambda$ is not trivial, and we need to solve the equations

$$u_\lambda = (\lambda + \mathbb{A}_{\alpha,\mathbb{R}^2})^{-1} P_{\mathbb{R}^2} f - \alpha (\lambda + \mathbb{A}_{\alpha,\mathbb{R}^2})^{-1} P_{\mathbb{R}^2} \nabla \cdot \left( U \otimes u_\lambda + u_\lambda \otimes U \right).$$
It is easy to see that

$$\text{div } \mathcal{U}_\alpha[\lambda] f = 0 \quad \text{in } \Omega, \quad \mathcal{U}_\alpha[\lambda] f = 0 \quad \text{on } \partial \Omega.$$ 

Moreover, we have

$$(\lambda + L_{\alpha,\Omega}) \mathcal{U}_\alpha[\lambda] f + \nabla P_\alpha[\lambda] f = (I + S_\alpha[\lambda]) f,$$

where $S_\alpha[\lambda] : L^3_{[5]}(\Omega)^2 \to L^3_{[5]}(\Omega)^2$ is a compact operator.

Although $S_\alpha[\lambda]$ is compact, its operator norm tends to infinity in the limit $\lambda, \alpha \to 0$.

This singularity comes from the behavior of the resolvent for the pure Oseen operator $A_{\alpha,R^2} = \mathbb{P}_{R^2}(-\Delta + \alpha \partial_1)$ in the whole space.
6. Local energy decay estimate of $e^{-t\mathbb{L}_{\lambda,\Omega}}$

The key is the expansion as follows: for $|x| \leq 5$ and $f \in L^3_{[\Omega]}$, 

$$(\lambda + A_{\alpha,\mathbb{R}^2})^{-1}P_{\mathbb{R}^2}f = E^0_0 \ast f$$

$$+ \left( - \frac{1}{8\pi} \log(4\lambda + \alpha^2) \mathbb{I} + \mathbb{J}(\alpha, \lambda) \right) \int_{\mathbb{R}^2} f \, dx$$

$$+ \tilde{E}^\alpha_\lambda \ast f.$$

Here

$$E^0_0(x) = \left( - \frac{1}{4\pi} \log |x| \right) \mathbb{I} + \frac{x \otimes x}{|x|^2},$$

$$|\mathbb{J}(\alpha, \lambda)| \leq C,$$

$$||\tilde{E}^\alpha_\lambda \ast f||_{W^{1,\infty}(\{|x| \leq 5\})} \leq C \sqrt{4\lambda + \alpha^2} |\log(4\lambda + \alpha^2)| ||f||_{L^3_{[\Omega]}}.$$
6. Local energy decay estimate of $e^{-tL_{\alpha, \Omega}}$

As in the works by Dan-Shibata (1999, for the Stokes operator) and by Hishida (2016, for the pure Oseen operator), $I + S_{\alpha}[\lambda]$ is written in the form

$$I + S_{\alpha}[\lambda] = \Theta_{\alpha}[\lambda] + N_{\alpha}[\lambda]$$

$\Theta_{\alpha}[\lambda]$: invertible for small $\lambda$ and $\alpha$ together with the uniform bound $N_{\alpha}[\lambda]$: finite rank operator of the form

$$N_{\alpha}[\lambda] = \sum_{j=1}^{N} \langle \phi_j(\alpha, \lambda), f \rangle_{L^2(\Omega)} w_j, \quad 2 \leq \exists N \leq 4.$$ 

Here $\phi_j(\alpha, \lambda)$ is a constant vector in $\mathbb{R}^2$ (containing $\log(4\lambda + \alpha^2)$), and $w_j$ is a compactly supported function independent of $\lambda$ and $\alpha$. 
Then we can write

\[ I + S_\alpha[\lambda] = \Theta_\alpha[\lambda] \left( I + \Theta_\alpha[\lambda]^{-1} N_\alpha[\lambda] \right), \]

\[ \Theta_\alpha[\lambda]^{-1} N_\alpha[\lambda] = \sum_{j=1}^{N} \langle \phi_j(\alpha, \lambda), f \rangle_{L^2(\Omega)} \Theta_\alpha[\lambda]^{-1} w_j. \]

Let \( M(\alpha, \lambda) \) be the \( N \times N \) matrix given by

\[ M(\alpha, \lambda) = I + \left( \langle \phi_j(\alpha, \lambda), \Theta_\alpha[\lambda]^{-1} w_k \rangle_{L^2(\Omega)} \right)_{1 \leq j, k \leq N}. \]
The direct computation shows

$$\det M(\alpha, \lambda) = \sum_{j=0}^{2} C_j(\alpha, \lambda) \left( \frac{1}{2} \log(4\lambda + \alpha^2) \right)^j,$$

where $C_j(\alpha, \lambda)$ is bounded and continuous on small $\lambda$ and $\alpha$. The constant $C_2(0, 0)$ is already considered in the analysis of $\mathbb{P}_\Omega(-\Delta + \alpha \partial_1)$ by Hishida (2016), and it is shown by Hishida (2016) that $C_2(0, 0) \neq 0$.

Hence, the matrix $M(\alpha, \lambda)$ is invertible for small $\alpha$ and $\lambda$, and with its inverse $M(\alpha, \lambda)^{-1} = (b_{jk}(\alpha, \lambda))_{1 \leq j, k \leq N}$, the inverse of $I + \Theta_\alpha[\lambda]^{-1}N_\alpha[\lambda]$ is described as

$$\left( I + \Theta_\alpha[\lambda]^{-1}N_\alpha[\lambda] \right)^{-1} f = f - \sum_{j,k} \langle \phi_k(\alpha, \lambda), f \rangle_{L^2(\Omega)} b_{jk}(\alpha, \lambda) w_j.$$
6. Local energy decay estimate of $e^{-tL_{\alpha,\Omega}}$

In the end, we arrive at the formula

$$(I + S_{\alpha}[\lambda])^{-1} = \left(I + \Theta_{\alpha}[\lambda]^{-1}N_{\alpha}[\lambda]\right)^{-1}\Theta_{\alpha}[\lambda]^{-1},$$

and this verifies the representation

$$(\lambda + L_{\alpha,\Omega})^{-1}P_{\Omega}f = U_{\alpha}[\lambda](I + S_{\alpha}[\lambda])^{-1}f, \quad f \in L^3_{[5]}(\Omega).$$

With these formulas one can show

$$\| (\lambda + L_{\alpha,\Omega})^{-1}P_{\Omega}f \|_{W^{1,3}(\Omega \cap B_4(0))} \leq C\|f\|_{L^3_{[5]}(\Omega)},$$

$$\| \partial_{\lambda}(\lambda + L_{\alpha,\Omega})^{-1}P_{\Omega}f \|_{W^{1,3}(\Omega \cap B_4(0))} \leq \frac{C}{|\lambda| \log(\lambda + \alpha^2)}\|f\|_{L^3_{[5]}(\Omega)} + \frac{C\kappa}{|\lambda| \log \alpha}(1 + \frac{\alpha^{2\kappa}}{|\lambda|^\kappa})\|f\|_{L^3_{[5]}(\Omega)},$$

for small $\lambda = i\mu$ and $\alpha$. 
7. Open question

1. Can we show the $L^p$ Boundedness?

\[ \| e^{-tI_{\alpha,\Omega}} f \|_{L^p(\Omega)} \leq C \| f \|_{L^p(\Omega)}, \quad t > 0. \]  

2. Can we remove the dependence on $\alpha$ for the size of the initial perturbations in the main theorem?

3. Analysis for nonsmall $\alpha$?