Discontinuous Galerkin finite element methods for Hamilton–Jacobi–Bellman equations with Cordes coefficients

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joint work with

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Overview

Talk outline

1. *Introduction:* Hamilton–Jacobi–Bellman (HJB) equations.
2. *Analysis:* Analysis of HJB equations with Cordes coefficients.
3. *Numerical methods:* High-order discontinuous Galerkin methods for HJB equations with Cordes coefficients.

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1. Stochastic optimal control

Stochastic differential equation

$$\mathrm{d}X_t = b(X_t, \alpha_t)\,\mathrm{d}t + \sigma(X_t, \alpha_t)\,\mathrm{d}B_t, \quad X_0 = x,$$

Find a control $\alpha(\cdot)\colon t \mapsto \alpha_t \in \Lambda$ that minimises

$$J(x, \alpha(\cdot)) = \mathbb{E}\left[\int_0^{\tau_{\mathrm{exit}}} f(X_t, \alpha_t)\; \mathrm{e}^{-\int_0^t c(X_s, \alpha_s)\,\mathrm{d}s} \mathrm{d}t\right]$$

- $b(x, \alpha) \in \mathbb{R}^d$ drift, $\sigma(x, \alpha) \in \mathbb{R}^{d \times m}$ volatility
- scalar f and c : running cost and discount
- α_t control variable
- Λ controls set (assumed to be a compact metric space).
- stopping time τ_{exit}: first exit from bounded domain $\Omega \subset \mathbb{R}^d$

Example applications: energy, engineering, finance ...

1. Dynamic programming principle

The dynamic programming principle (DPP) is a solution process for a stochastic control problem.

Overview: stages of DPP

1. Define the *value function* of the optimal control problem.
2. DPP: the value function is the solution of an HJB equation.
3. The optimal controls can be computed once the value function is available.

Richard Bellman (1920–1984)

Feedback control map $\alpha_{\text{feedback}} : \Omega \to \Lambda \implies \alpha(t) := \alpha_{\text{feedback}}(X_t)$.

1. Dynamic programming principle

Details in Fleming & Soner 2006

- Define the *value function* V defined by

$$V(x) := \inf\{J(x,\alpha(\cdot)) \mid \alpha(\cdot)\colon t \in [0,\infty) \mapsto \alpha_t \in \Lambda,\ \alpha(\cdot) \in \mathcal{A}\}.$$

$\mathcal{A}$ = set of *admissible controls*: progressively measurable w.r.t. filtration.

- The function $u := -V$ solves the HJB equation

$$\begin{aligned} \sup_{\alpha\in\Lambda}[L^{\alpha}u - f^{\alpha}] &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{HJB}$$

where $L^{\alpha}u := a^{\alpha}(x) : D^2u + b^{\alpha}(x)\cdot\nabla u - c^{\alpha}(x)\,u$, with

$$a^{\alpha}(x) := \tfrac{1}{2}\sigma(x,\alpha)\,\sigma^{\top}(x,\alpha) \in \mathbb{R}^{d\times d},$$

$$\textit{Notation:}\quad a^{\alpha} : D^2u = \sum_{i,j=1}^{d} a^{\alpha}_{ij}(x)u_{x_ix_j}.$$

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$$\sup_{\alpha\in\Lambda}[L^\alpha u - f^\alpha] = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{HJB}$$

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1. Hamilton–Jacobi–Bellman Equation

Elliptic Dirichlet problem:

$$\begin{aligned} \sup_{\alpha\in\Lambda}[L^\alpha u - f^\alpha] &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \qquad \text{(Elliptic HJB)}$$

where $L^\alpha u := a^\alpha(x) : D^2 u + b^\alpha(x)\cdot\nabla u - c^\alpha(x)\, u$.

Notation: $a^\alpha(x) : D^2 u = \sum_{i,j=1}^{d} a^\alpha_{ij}(x) u_{x_i x_j}, \quad b^\alpha(x)\cdot\nabla u = \sum_{i=1}^{d} b^\alpha_i(x) u_{x_i}.$

Assumptions in this talk:

- $\Omega\subset\mathbb{R}^d$ is bounded and convex, Λ a compact metric space.
- a, b, c and f are continuous functions in $x\in\overline{\Omega}$, $\alpha\in\Lambda$.
- a^α are symmetric positive definite, uniformly on $\overline{\Omega}\times\Lambda$, and $c^\alpha\geq 0$.
- Cordes coefficients: the coefficient functions a, b, c satisfy *the Cordes condition* (coming soon!)

1. Examples

How do HJB equations relate to other PDEs?

$$\sup_{\alpha\in\Lambda}[L^\alpha u - f^\alpha] = 0$$

The HJB equation generalises many other equations:

- Linear nondivergence form elliptic equations

$$a : D^2 u + b\cdot\nabla u - cu = f, \quad \text{(assume that } \Lambda \text{ is a singleton set).}$$

- Hamilton–Jacobi: e.g. eikonal equation

$$\sup_{\alpha\in\mathbb{S}^d}[\alpha\cdot\nabla u - 1] = |\nabla u| - 1 = 0.$$

- Monge–Ampère equation [Krylov 1987, Jensen & Feng 2017]

$$\begin{cases}\det D^2 u - f = 0\\ u \text{ convex}\end{cases} \iff \inf_{\substack{a\in\mathbb{R}^{d\times d}_{\mathrm{sym},+}\\ \mathrm{Tr}\, a=1}}\left[a : D^2 u - d\,(f\,\det a)^{1/d}\right] = 0.$$

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1. Approaches

What are the available approaches
to PDE theory and numerical discretization?

- Weak solutions in $H^1(\Omega)$: not applicable!
  - $\rightarrow$ most existing finite element techniques cannot be used!
- **Viscosity solutions**: generally applicable, even to degenerate elliptic problems. Solution space $u \in C(\overline{\Omega})$.
  - This leads to *monotone numerical schemes* (c.f. next few slides)...
- **Strong solutions in $H^2(\Omega)$:** under the Cordes condition ...
- Classical solutions in $C^2(\Omega)$: Evans–Krylov Theorem and its developments guarantee interior regularity estimates for the viscosity solution under uniform ellipticity and data regularity assumptions.

Some references:

- Weak, strong and classical solutions — Gilbarg & Trudinger, 1998.
- Viscosity solutions for fully nonlinear PDE — Crandall, Lions & Ishii, 1992.
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1. Literature

Pre-existing numerical methods:

- Monotone methods: low-order methods with discrete maximum principles:
  - Convergence theory to the viscosity solution: Barles & Souganidis 1991.
  - mostly finite difference methods: Motzkin & Wasow, Kuo & Trudinger, Kocan, Camilli & Falcone, Barles & Jakobsen, Jakobsen & Debrabant, Fleming & Soner, Bonnans & Zidani,. . .
  - a few on monotone FEM: Jensen & S., SINUM 2013 and Nochetto & Zhang FOCM 2016.
  - limitations for anisotropic problems: Motzkin & Wasow 1953, Kocan 1995, Bonnans & Zidani 2003, Crandall & Lions 2012.
- Other pre-existing non-monotone methods without discrete maximum principles, without convergence theory: Feng, Neilan, Glowinski, Brenner, Lakkis, Pryer, . . . [Feng et al. SIAM Rev. 2013].

Is it possible to have stable, consistent and convergent methods for fully nonlinear PDEs without discrete maximum principles?

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2. *Analysis:* Analysis of HJB equations with Cordes coefficients.
   - Motivation of Cordes coefficients
   - Existence, Uniqueness, Well-posedness
3. *Numerical methods:* High-order discontinuous Galerkin methods for HJB equations with Cordes coefficients.

2. PDE Theory: motivation

Cordes introduced his condition in the context of nondivergence form equations with discontinuous coefficients.

There is a link between HJB equations and nondivergence form equations with discontinuous coefficients:

Classical solution algorithm: policy iteration, due to Howard and Bellman:

1. Choose an initial guess u^0.
2. For $k \in \mathbb{N}$, given a current guess u^k, choose $\alpha_k : \Omega \to \Lambda$, a Lebesgue-measurable selection

$$\alpha_k(x) \in \operatorname{argmax}_{\alpha\in\Lambda} (L^\alpha u^k - f^\alpha)(x), \qquad \forall\, x \in \Omega.$$

3. Then, find u^{k+1} as a solution of the PDE

$$L^{\alpha_k} u^{k+1} = f^{\alpha_k} \quad \text{in } \Omega, \qquad \text{with } u^{k+1} = 0 \text{ on } \partial\Omega,$$

where $f^{\alpha_k} : x \mapsto f^{\alpha_k(x)}(x)$, etc.

Howard 1960, Puterman & Brumelle 1979, Bokanowski et al. 2009, . . .

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2. PDE Theory: motivation

Linearized equation:

$$L^{\alpha_k} u^{k+1} = f^{\alpha_k} \quad \text{in } \Omega, \qquad \text{with } u^{k+1} = 0 \text{ on } \partial\Omega, \tag{1}$$

Question: is Eq. (1) a well-posed PDE?

In general, the answer is *no*: the linearization process (esp. the argmax) leads to discontinuous diffusion coefficients: $a^{\alpha_k} \in L^\infty(\Omega)^{d\times d}$ and $a^{\alpha_k} \notin C(\overline{\Omega})^{d\times d}$.

- Calderon–Zygmund: If $a \in C(\overline{\Omega})^{d\times d}$ and $\partial\Omega \in C^{1,1}$, then existence and uniqueness in $W^{2,p}$ [Gilbarg & Trudinger, 1998]
- If $a \in L^\infty(\Omega)^{d\times d}$, $a \notin C(\overline{\Omega})^{d\times d}$, then there are counter-examples showing non-uniqueness in general (Maugeri et al, 2000):

$$\Delta u + \rho \sum_{i,j=1}^{d} \frac{x_i x_j}{|x|^2} u_{x_i x_j} = 0 \text{ in } B \text{ unit ball}, \qquad \rho = -1 + \frac{d-1}{1-\theta}, \quad 0 < \theta < 1,$$

If $d \geq 3$ and $d > 2(2-\theta) > 2$, two solutions in $H^2(B) \cap H^1_0(B)$

$$u_1(x) = 0 \qquad \text{and} \qquad u_2(x) = |x|^\theta - 1$$

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$$u_1(x) = 0 \qquad \text{and} \qquad u_2(x) = |x|^\theta - 1$$

Cordes condition: *Case 1: without advection and reaction*

Assume that there exists $\varepsilon \in (0, 1]$ s. t.

$$\frac{|a(x)|^2}{(\operatorname{Tr} a(x))^2} \leq \frac{1}{d - 1 + \varepsilon} \quad \text{a.e. } x \in \Omega, \tag{Cordes$_0$}$$

Theorem (Cordes, 1956)

If Ω is convex, and if $a \in L^\infty(\Omega)^{d \times d}$ unif. ellipt. satisfies (Cordes$_0$), *then for any $f \in L^2(\Omega)$ there exists a unique $u \in H^2(\Omega) \cap H_0^1(\Omega)$ solving*

$$a : D^2 u = f \quad \text{in } \Omega, \qquad \text{with } u = 0 \text{ on } \partial\Omega,$$

Example

If dimension $d = 2$, (Cordes$_0$) $\iff$ **uniform ellipticity**.

Cordes condition: *Case 1: without advection and reaction*

Assume that there exists $\varepsilon \in (0, 1]$ s. t.

$$\frac{|a^\alpha(x)|^2}{(\mathrm{Tr}\, a^\alpha(x))^2} \le \frac{1}{d - 1 + \varepsilon} \quad \text{a.e. } x \in \Omega, \alpha \in \Lambda \qquad (2)$$

Theorem (Cordes, 1956)

If Ω is convex, and if $a^\alpha \in L^\infty(\Omega)^{d\times d}$ unif. ellipt. satisfies (Cordes$_0$), then for any $f \in L^2(\Omega)$ there exists a unique $u \in H^2(\Omega) \cap H^1_0(\Omega)$ solving

$$a^\alpha : D^2 u = f \quad \text{in } \Omega, \qquad \text{with } u = 0 \text{ on } \partial\Omega,$$

Example

If dimension $d = 2$, (Cordes$_0$) $\iff$ **uniform ellipticity**.

Cordes condition: *Case 2: extension to $b^\alpha \neq 0$ and $c^\alpha \neq 0$*

Assume that there exist $\lambda > 0$ and $\varepsilon \in (0, 1]$ s. t.

$$\frac{|a^\alpha|^2 + |b^\alpha|^2/2\lambda + (c^\alpha/\lambda)^2}{(\mathrm{Tr}\, a^\alpha + c^\alpha/\lambda)^2} \leq \frac{1}{d + \varepsilon} \quad \text{in } \Omega,\ \forall\, \alpha \in \Lambda. \qquad (\text{Cordes}_1)$$

Theorem (Strong solutions of HJB equations with Cordes coefficients)

Let Ω *be a bounded convex open subset of* $\mathbb{R}^d$*, and let* Λ *be a compact metric space.*

Let the data be continuous on $\overline{\Omega} \times \Lambda$*, and satisfy* (Cordes$_1$) *with uniformly elliptic* a^α *and* $c^\alpha \geq 0$ *for all* $\alpha \in \Lambda$*.*

Then, there exists a unique $u \in H^2(\Omega) \cap H^1_0(\Omega)$ *that solves* (Elliptic HJB) *pointwise a.e. in* Ω*.*

I. S. & E. Süli, SIAM J. Numer. Anal. 2014:

Discontinuous Galerkin finite element approximation of Hamilton–Jacobi–Bellman equations with Cordes coefficients.

2. PDE theory: proof of well-posedness

Define

$$\gamma^\alpha := \frac{\operatorname{Tr} a^\alpha + c^\alpha/\lambda}{|a^\alpha|^2 + |b^\alpha|^2/2\lambda + (c^\alpha/\lambda)^2}$$
$$F_\gamma[u] := \sup_{\alpha\in\Lambda} [\gamma^\alpha (L^\alpha u - f^\alpha)]$$

Because $\gamma^\alpha > 0$, we can renormalize the operator:

$$F_\gamma[u] = \sup_{\alpha\in\Lambda} [\gamma^\alpha (L^\alpha u - f^\alpha)] = 0 \iff \sup_{\alpha\in\Lambda} [L^\alpha u - f^\alpha] = 0. \tag{3}$$

The problem (Elliptic HJB) for $u \in H^2(\Omega) \cap H^1_0(\Omega)$ is equivalent to

$$\mathcal{A}(u; v) := \int_\Omega F_\gamma[u]\, L_\lambda v \,\mathrm{d}x = 0 \quad \forall\, v \in H^2(\Omega) \cap H^1_0(\Omega), \tag{4}$$

where $L_\lambda v := \Delta v - \lambda v$.

2. PDE theory: proof of well-posedness

Let $H := H^2(\Omega) \cap H_0^1(\Omega)$, $\|v\|_H^2 := \|D^2 u\|^2 + 2\lambda \|\nabla u\|^2 + \lambda^2 \|u\|^2$

1. $\mathcal{A}\colon H \times H \to \mathbb{R}$ is linear in its second argument (only).
2. $\mathcal{A}$ is Lipschitz continuous: there is $C > 0$ such that

$$|\mathcal{A}(u; v) - \mathcal{A}(w, v)| \leq C \|u - w\|_H \|v\|_H \quad \forall u,\, v,\, w \in H,$$

3. We now show that $\mathcal{A}$ is strongly monotone[1] : there exists a positive constant $c = c(\varepsilon) > 0$ such that

$$\frac{1}{c} \|u - v\|_H^2 \leq \mathcal{A}(u; u - v) - \mathcal{A}(v; u - v) \quad \forall\, u, v \in H. \tag{5}$$

On verifying these conditions, we conclude that there exists a unique $u \in H$ that solves $\mathcal{A}(u; v) = 0$ for all $v \in H$ and thus solves (Elliptic HJB).

[1] Remark: must not confuse monotone schemes (i.e. discrete max principle) with strongly monotone operators in functional analytic sense.

Notation: $H := H^2(\Omega) \cap H_0^1(\Omega), \quad \|v\|_H^2 := \|D^2 u\|^2 + 2\lambda\|\nabla u\|^2 + \lambda^2\|u\|^2$

Key ingredients

1. **The Cordes condition**, which implies by direct calculation that

$$\|F_\gamma[u] - F_\gamma[v] - L_\lambda(u-v)\|_{L^2} \leq \sqrt{1-\varepsilon}\|u-v\|_H \tag{6}$$

2. **Miranda–Talenti**: for convex Ω,

$$\|w\|_H \leq \|L_\lambda w\|_{L^2} \qquad \forall\, w \in H^2(\Omega) \cap H_0^1(\Omega) \tag{7}$$

   where $L_\lambda v := \Delta v - \lambda v$ (recall $\lambda > 0$)

   Maugeri, Palagachev & Softova, 2000, and Grisvard 1985

$$\|F_\gamma[u] - F_\gamma[v] - L_\lambda(u-v)\|_{L^2} \leq \sqrt{1-\varepsilon}\|u-v\|_H, \quad \|v\|_H \leq \|L_\lambda v\|_{L^2}$$

Strong monotonicity:
Recall $\mathcal{A}(u;v) = \int_\Omega F_\gamma[u] L_\lambda v \mathrm{d}x$.

$$\mathcal{A}(u;u-v) - \mathcal{A}(v;u-v) = \int_\Omega (F_\gamma[u] - F_\gamma[v])\; L_\lambda(u-v)\,\mathrm{d}x.$$

Addition–subtraction of $\|L_\lambda(u-v)\|_{L^2}^2$ gives

$$\mathcal{A}(u;u-v) - \mathcal{A}(v;u-v) = \|L_\lambda(u-v)\|_{L^2}^2 + \underbrace{\int_\Omega (F_\gamma[u] - F_\gamma[v] - L_\lambda(u-v))\; L_\lambda(u-v)\,\mathrm{d}x}_{\geq -\sqrt{1-\varepsilon}\|u-v\|_H \|L_\lambda(u-v)\|_{L^2} \geq -\sqrt{1-\varepsilon}\|L_\lambda(u-v)\|_{L^2}^2}$$

Therefore

$$\mathcal{A}(u;u-v) - \mathcal{A}(v;u-v) \geq (1-\sqrt{1-\varepsilon})\|L_\lambda(u-v)\|_{L^2}^2 \geq (1-\sqrt{1-\varepsilon})\|u-v\|_H^2$$

□

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2. PDE theory: proof of well-posedness

$$\|F_\gamma[u]-F_\gamma[v]-L_\lambda(u-v)\|_{L^2}\le\sqrt{1-\varepsilon}\|u-v\|_H,\quad \|v\|_H\le\|L_\lambda v\|_{L^2}$$

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Recall $\mathcal{A}(u;v)=\int_\Omega F_\gamma[u]L_\lambda v\,\mathrm{d}x$.

$$\mathcal{A}(u;u-v)-\mathcal{A}(v;u-v)=\int_\Omega\left(F_\gamma[u]-F_\gamma[v]\right)L_\lambda(u-v)\,\mathrm{d}x.$$

Addition–subtraction of $\|L_\lambda(u-v)\|_{L^2}^2$ gives

$$\begin{aligned}\mathcal{A}(u;u-v)-\mathcal{A}(v;u-v)&=\|L_\lambda(u-v)\|_{L^2}^2\\&\quad+\underbrace{\int_\Omega\left(F_\gamma[u]-F_\gamma[v]-L_\lambda(u-v)\right)L_\lambda(u-v)\,\mathrm{d}x}_{\ge-\sqrt{1-\varepsilon}\|u-v\|_H\|L_\lambda(u-v)\|_{L^2}\ge-\sqrt{1-\varepsilon}\|L_\lambda(u-v)\|_{L^2}^2}\end{aligned}$$

Therefore

$$\mathcal{A}(u;u-v)-\mathcal{A}(v;u-v)\ge(1-\sqrt{1-\varepsilon})\|L_\lambda(u-v)\|_{L^2}^2\ge(1-\sqrt{1-\varepsilon})\|u-v\|_H^2$$

□

Approach to numerical analysis:

Since the proof of well-posedness hinges on the strong monotonicity of

$$\mathcal{A}(u; v) = \int_\Omega F_\gamma[u] L_\lambda v \mathrm{d}x,$$

we will attempt to discretise the operator $\mathcal{A}$ and conserve its strong monotonicity.

- The Cordes condition carries over straightforwardly to discrete setting
- The Miranda–Talenti inequality does not carry over if the approximation space is not inside $H^2(\Omega) \cap H^1_0(\Omega)$.

Overview

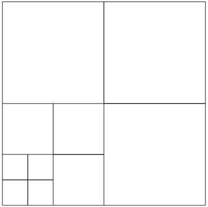
Talk outline

1. *Introduction:* Hamilton–Jacobi–Bellman (HJB) equations.
2. *Analysis:* Analysis of HJB equations with Cordes coefficients.
3. *Numerical methods:* High-order discontinuous Galerkin methods for HJB equations with Cordes coefficients.
    - Design of a consistent, stable and convergent method
    - Error bounds
    - Extension to parabolic problems
    - Numerical experiments

3. Numerics: design of the method

Let $\{\mathcal{T}_h\}_h$ a shape-regular sequence of meshes on Ω.

- Elements composing the mesh can be parallelepipeds, simplices, or more generally any combination of standard elements.
- The mesh is *not assumed to be quasi-uniform* (very useful for *hp*-refinement).
- Hanging nodes allowed.



3. Numerics: design of the method

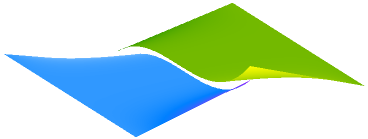
Construction of the discontinuous finite element space

Discontinuous finite element space:

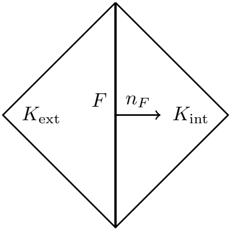
$$V_{h,\mathbf{p}} := \{v_h \in L^2(\Omega) \colon v_h|_K \in \mathcal{P}_{p_K}(K) \ \forall K \in \mathcal{T}_h\}.$$

Polynomial degrees $\mathbf{p} = (p_K)_{K \in \mathcal{T}_h}$

Approximation in H^2 requires $p_K \geq 2$ for all elements K.



Notation of discontinuous Galerkin methods:



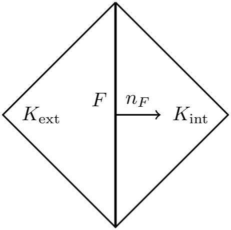
Distinguish interior and boundary faces

$$\mathcal{F}_h^i \text{ interior faces of } \mathcal{T}_h, \qquad \mathcal{F}_h^b \text{ boundary faces of } \mathcal{T}_h,$$
$$\mathcal{F}_h^{i,b} \coloneqq \mathcal{F}_h^i \cup \mathcal{F}_h^b.$$

Jump operators over faces:

$$[\![\phi]\!] \coloneqq \tau_F\left(\phi|_{K_{\text{ext}}}\right) - \tau_F\left(\phi|_{K_{\text{int}}}\right), \quad \{\phi\} \coloneqq \tfrac{1}{2}\tau_F\left(\phi|_{K_{\text{ext}}}\right) + \tfrac{1}{2}\tau_F\left(\phi|_{K_{\text{int}}}\right), \quad \text{if } F \in \mathcal{F}_h^i,$$
$$[\![\phi]\!] \coloneqq \tau_F\left(\phi|_{K_{\text{ext}}}\right), \qquad \{\phi\} \coloneqq \tau_F\left(\phi|_{K_{\text{ext}}}\right), \qquad \text{if } F \in \mathcal{F}_h^b,$$

Notation of discontinuous Galerkin methods:



Let $\{t_i\}_{i=1}^{d-1} \subset \mathbb{R}^d$ be an orthonormal coordinate system on F. Define the *tangential gradient and divergence*

$$\nabla_{\mathrm{T}}\, u := \sum_{i=1}^{d-1} t_i \frac{\partial u}{\partial t_i}, \quad \mathrm{div}_{\mathrm{T}}\, \mathbf{v} := \sum_{i=1}^{d-1} \frac{\partial \mathbf{v}_i}{\partial t_i}. \tag{8}$$

3. Numerics: design of the method

The goal is to discretise

$$\mathcal{A}(u; v) = \int_\Omega F_\gamma[u]\, L_\lambda v \,\mathrm{d}x,$$

whilst conserving the strong monotonicity bound.

Recall main ingredients:

1. The Cordes condition $\rightarrow$ remains unchanged in discrete setting.
2. Miranda–Talenti inequality: not conserved when replacing $H^2(\Omega) \cap H_0^1(\Omega)$ by $V_{h,\mathbf{p}}$.

Our approach:

- Miranda–Talenti inequality was derived from an integration by parts identity (Maugeri et al 2000, Grisvard 1984)
- We will include a discrete *weak form of this identity* in the scheme (next slide)

Numerical scheme: find $u_h \in V_{h,\mathbf{p}}$ such that

$$A_h(u_h; v_h) = 0 \quad \forall\, v_h \in V_{h,\mathbf{p}}. \tag{scheme}$$

$$A_h(u_h; v_h) := \sum_{K\in\mathcal{T}_h} \langle F_\gamma[u_h], L_\lambda v_h \rangle_K + J_h(u_h, v_h) + \frac{1}{2}\left(B_h(u_h, v_h) - \sum_{K\in\mathcal{T}_h} \langle L_\lambda u_h, L_\lambda v_h \rangle_K \right).$$

3. Numerics: design of the method

Numerical scheme: find $u_h \in V_{h,\mathbf{p}}$ such that

$$A_h(u_h; v_h) = 0 \quad \forall\, v_h \in V_{h,\mathbf{p}}. \qquad \text{(scheme)}$$

$$A_h(u_h; v_h) := \sum_{K \in \mathcal{T}_h} \langle F_\gamma[u_h], L_\lambda v_h \rangle_K + J_h(u_h, v_h) + \frac{1}{2}\left(B_h(u_h, v_h) - \sum_{K \in \mathcal{T}_h} \langle L_\lambda u_h, L_\lambda v_h \rangle_K \right).$$

$$\langle F_\gamma[u_h], L_\lambda v_h \rangle_K := \int_K \sup_{\alpha \in \Lambda} [\gamma^\alpha (L^\alpha u_h - f^\alpha)]\, (\Delta v_h - \lambda v_h)\, \mathrm{d}x.$$

3. Numerics: design of the method

Numerical scheme: find $u_h \in V_{h,\mathbf{p}}$ such that

$$A_h(u_h; v_h) = 0 \quad \forall\, v_h \in V_{h,\mathbf{p}}. \qquad \text{(scheme)}$$

$$A_h(u_h; v_h) := \sum_{K\in\mathcal{T}_h} \langle F_\gamma[u_h], L_\lambda v_h\rangle_K + J_h(u_h, v_h) + \frac{1}{2}\left(B_h(u_h, v_h) - \sum_{K\in\mathcal{T}_h} \langle L_\lambda u_h, L_\lambda v_h\rangle_K \right).$$

Jump penalisation with $\mu_F \simeq p_K^2/h_K$ and $\eta_F \simeq p_K^4/h_K^3$ for $F \subset \partial K$:

$$J_h(u_h, v_h) := \sum_{F\in\mathcal{F}_h^{i,b}} \left[\mu_F \langle [\![\nabla_{\mathrm{T}}\, u_h]\!], [\![\nabla_{\mathrm{T}}\, v_h]\!]\rangle_F + \eta_F \langle [\![u_h]\!], [\![v_h]\!]\rangle_F\right] + \sum_{F\in\mathcal{F}_h^{i}} \mu_F \langle [\![\nabla u_h \cdot n_F]\!], [\![\nabla v_h \cdot n_F]\!]\rangle_F.$$

Numerical scheme: find $u_h \in V_{h,\mathbf{p}}$ such that

$$A_h(u_h; v_h) = 0 \quad \forall\, v_h \in V_{h,\mathbf{p}}. \tag{scheme}$$

$$A_h(u_h; v_h) := \sum_{K \in \mathcal{T}_h} \langle F_\gamma[u_h], L_\lambda v_h \rangle_K + J_h(u_h, v_h) + \frac{1}{2}\left(B_h(u_h, v_h) - \sum_{K \in \mathcal{T}_h} \langle L_\lambda u_h, L_\lambda v_h \rangle_K \right).$$

$$\langle L_\lambda u_h, L_\lambda v_h \rangle_K := \int_K (\Delta u_h - \lambda u_h)\,(\Delta v_h - \lambda v_h)\,\mathrm{d}x.$$

3. Numerics: design of the method

Numerical scheme: find $u_h \in V_{h,\mathbf{p}}$ such that

$$A_h(u_h; v_h) = 0 \quad \forall\, v_h \in V_{h,\mathbf{p}}. \qquad \text{(scheme)}$$

$$A_h(u_h; v_h) := \sum_{K\in\mathcal{T}_h} \langle F_\gamma[u_h], L_\lambda v_h\rangle_K + J_h(u_h, v_h) + \frac{1}{2}\left(B_h(u_h, v_h) - \sum_{K\in\mathcal{T}_h} \langle L_\lambda u_h, L_\lambda v_h\rangle_K \right).$$

$$B_h(u_h, v_h) := \sum_{K\in\mathcal{T}_h} \left[\langle D^2 u_h, D^2 v_h\rangle_K + 2\lambda\langle \nabla u_h, \nabla v_h\rangle_K + \lambda^2 \langle u_h, v_h\rangle_K \right]$$
$$+ \sum_{F\in\mathcal{F}_h^i} \left[\langle \operatorname{div}_{\mathrm{T}} \nabla_{\mathrm{T}}\{u_h\}, [\![\nabla v_h \cdot n_F]\!]\rangle_F + \langle \operatorname{div}_{\mathrm{T}} \nabla_{\mathrm{T}}\{v_h\}, [\![\nabla u_h \cdot n_F]\!]\rangle_F \right]$$
$$- \sum_{F\in\mathcal{F}_h^{i,b}} \left[\langle \nabla_{\mathrm{T}}\{\nabla u_h \cdot n_F\}, [\![\nabla_{\mathrm{T}} v_h]\!]\rangle_F + \langle \nabla_{\mathrm{T}}\{\nabla v_h \cdot n_F\}, [\![\nabla_{\mathrm{T}} u_h]\!]\rangle_F \right]$$

$-\lambda \sum_{F\in\mathcal{F}_h^{i,b}} [\langle \{\nabla u_h \cdot n_F\}, [\![v_h]\!]\rangle_F + \langle \{\nabla v_h \cdot n_F\}, [\![u_h]\!]\rangle_F] - \lambda \sum_{F\in\mathcal{F}_h^{i}} [\langle \{u_h\}, [\![\nabla v_h \cdot n_F]\!]\rangle_F + \langle \{v_h\}, [\![\nabla u_h \cdot n_F]\!]\rangle_F]$

3. Numerics: design of the method

Numerical scheme: find $u_h \in V_{h,\mathbf{p}}$ such that

$$A_h(u_h; v_h) = 0 \quad \forall\, v_h \in V_{h,\mathbf{p}}. \qquad \text{(scheme)}$$

$$A_h(u_h; v_h) := \sum_{K\in\mathcal{T}_h} \langle F_\gamma[u_h], L_\lambda v_h\rangle_K + J_h(u_h, v_h) + \frac{1}{2}\left(B_h(u_h, v_h) - \sum_{K\in\mathcal{T}_h} \langle L_\lambda u_h, L_\lambda v_h\rangle_K \right).$$

Key consistency result: If $u \in H^2(\Omega) \cap H^1_0(\Omega)$ has well-defined second derivative traces on faces F of the mesh, then

$$B_h(u, v_h) = \sum_K \langle L_\lambda u, L_\lambda v_h\rangle_K, \quad J_h(u, v_h) = 0 \quad \forall\, v_h \in V_{h,\mathbf{p}}.$$

Technical point: a sufficient condition is that $u \in H^s(K)$ with $s > 5/2$ for every $K \in \mathcal{T}_h$.

3. Numerics: consistency, stability and error bounds

Numerical scheme: find $u_h \in V_{h,\mathbf{p}}$ such that

$$A_h(u_h; v_h) = 0 \quad \forall\, v_h \in V_{h,\mathbf{p}}. \tag{scheme}$$

Full theoretical justification given in [S. & Süli, SINUM 2014]:

- Consistency Theorem: sufficiently regular solution of (Elliptic HJB) solves:
$$A_h(u; v_h) = 0 \quad \forall\, v_h \in V_{h,\mathbf{p}}.$$
- Discrete Stability Theorem: Existence & uniqueness of numerical solution since the nonlinear form A_h is strongly monotone: provided $\mu_F \gtrsim p^2/h$ and $\eta_F \gtrsim p^2/h$
$$\|u_h - v_h\|_h^2 \lesssim A_h(u_h; u_h - v_h) - A_h(v_h; u_h - v_h) \quad \forall\, u_h,\, v_h \in V_{h,\mathbf{p}},$$
where
$$\|v_h\|_h^2 := \sum_{K\in\mathcal{T}_h} \left[|v_h|^2_{H^2(K)} + 2\lambda |v_h|^2_{H^1(K)} + \lambda^2 \|v_h\|^2_{L^2(K)} \right] + J_h(v_h, v_h)$$
- Consistency+Stability $\Longrightarrow$ error bounds and convergence.

$$\|v_h\|_h^2 := \sum_{K\in\mathcal{T}_h}\left[|v_h|^2_{H^2(K)} + 2\lambda|v_h|^2_{H^1(K)} + \lambda^2\|v_h\|^2_{L^2(K)}\right] + J_h(v_h, v_h).$$

Theorem (High-order convergence rates)

(Under previous assumptions & standard assumptions for DG meshes...)

Assume that $u \in H^{\mathbf{s}}(\Omega;\mathcal{T}_h)$, *with* $s_K > 5/2$ *for all* $K \in \mathcal{T}_h$.

$$\|u - u_h\|_h^2 \lesssim \sum_{K\in\mathcal{T}_h}\left[\frac{h_K^{t_K-2}}{p_K^{s_K-5/2}}\|u\|_{H^{s_K}(K)}\right]^2,$$

where $t_K = \min(p_K + 1, s_K)$ *for each* $K \in \mathcal{T}_h$.

Simplified form:

$$\|u - u_h\|_h \lesssim \frac{h^{\min(s,p+1)-2}}{p^{s-5/2}}\|u\|_{H^s(\Omega)}.$$

- Optimal in h, half-order subopt. in p
- High-order convergence rates.
- Higher efficiency on well-chosen meshes and *hp*-refinement.

If u has only minimal regularity, then we have the following quasi-optimal approximation property with respect to the H^2-conforming subspace:

Theorem (Minimal regularity error bound)

Under previous assumptions. . .
Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ *be the solution of* (Elliptic HJB)*. Then*

$$\|u - u_h\|_h \leq \inf_{z_h \in V_{h,\mathbf{p}} \cap H^2(\Omega) \cap H_0^1(\Omega)} \|u - z_h\|_h.$$

Note however that DG method requires only quadratic polynomials, whereas H^2-conforming methods may require higher (e.g. Argyris elements require quintic polynomials).

3. Numerics: extensions to parabolic problems

S. & Süli, Num. Math. 2016: extension to parabolic HJB equations

- Generalisation of the Cordes condition and the PDE theory: existence and uniqueness of the strong solution $u \in L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega)) \cap H^1(0,T;L^2(\Omega))$.
- Numerical scheme: hp-τq-version space-time DGFEM using tensor product of $V_{h,\mathbf{p}}$ with piecewise polynomials in time.
- Stability, consistency and convergence rates that are:
  - h-optimal,
  - p-suboptimal by $p^{3/2}$,
  - τ-optimal,
  - q-optimal.
- Exponential convergence rates under hp-τq refinement verified experimentally.

3. Numerics: experiment 1: h-refinement

Experiment 1 : Test of high order convergence rates under h-refinement, fixed p.

Example (Control of correlated diffusions)

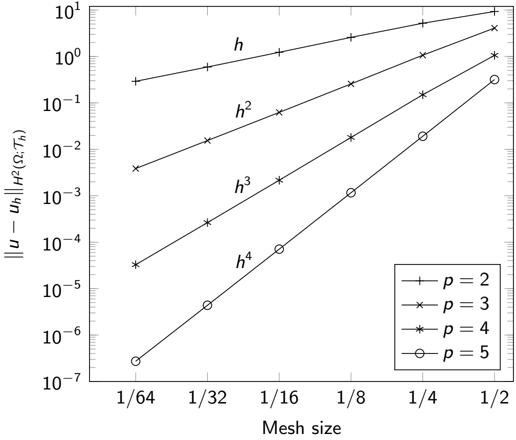
$$a^{\alpha} := \frac{1}{2} R^{\top} \begin{pmatrix} 1+\sin^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \cos^2\theta \end{pmatrix} R$$

$$\alpha := (\theta, R) \in [0, \tfrac{\pi}{3}] \times \mathrm{SO}(2) =: \Lambda.$$

Remark: a^{α} becomes increasingly anisotropic as $\theta \to \pi/3$; rotation matrices $R \in \mathrm{SO}(2)$ prevent monotone schemes from aligning the grid with the anisotropy.

Example (Control of correlated diffusions)

Uniform h-refinement on smooth solution $u(x, y) = \exp(xy)\sin(\pi x)\sin(\pi y)$:



Experiment 2: test of exponential convergence rates under hp-refinement

Example (Strong anisotropy + boundary layer)

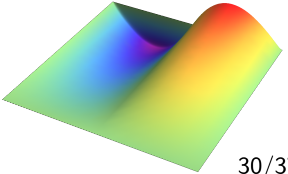
Let $\Omega = (0,1)^2$, $b^\alpha \equiv (0,1)$, $c^\alpha \equiv 10$ and define

$$a^\alpha := \alpha^\top \begin{pmatrix} 20 & 1 \\ 1 & 0.1 \end{pmatrix} \alpha, \quad \alpha \in \Lambda := \mathrm{SO}(2), \quad \lambda = \frac{1}{2}.$$

(Cordes$_1$) holds with $\varepsilon \approx 0.0024$ and $\lambda = 1/2$. Choose solution:

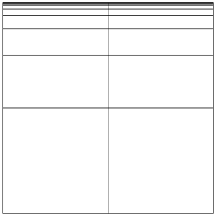
$$u(x,y) = (2x-1)\left(\mathrm{e}^{1-|2x-1|} - 1\right)\left(y + \frac{1-\mathrm{e}^{y/\delta}}{\mathrm{e}^{1/\delta}-1}\right), \quad \delta := 0.005 = \mathrm{O}(\varepsilon)$$

- Near-degenerate and anisotropic diffusion.
- Sharp boundary layer.
- Non-smooth solution.

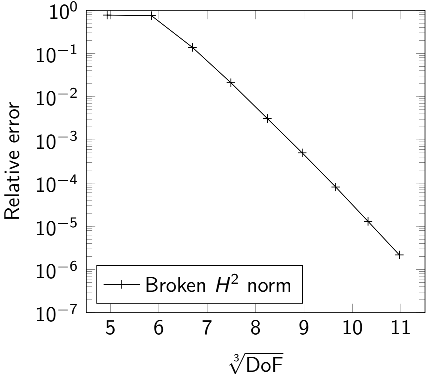


Example (Strong anisotropy + boundary layer)

We use boundary layer adapted meshes with p-refinement: $2 \leq p_K \leq 10$, from 100 to 1320 DoFs.



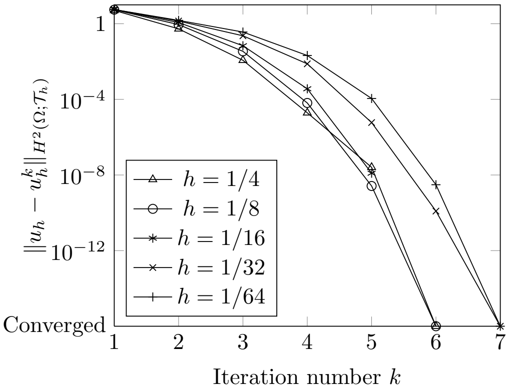
Boundary layer adapted mesh.



Exponential rate: $\|u - u_h\|_h \lesssim \exp(-c\sqrt[3]{\mathrm{DoF}})$.

3. Numerics: Experiment 3: linearization and algebraic solvers

Solution of nonlinear equation by a superlinearly convergent semismooth Newton method. [S. & Süli, SINUM 2014, Sect. 8]



3. Numerics: experiment 3: linearization and algebraic solvers

Nonoverlapping domain decomposition preconditioners with GMRES: (all tolerances 10^{-6} in discrete H^2-type norm)

		Average GMRES iterations (Newton steps)					
		4 Subdomains			16 Subdomains		
DoF	h	$H = 2h$	$H = 4h$	$H = 8h$	$H = 2h$	$H = 4h$	$H = 8h$
144	1/4	14.3 (6)					
576	1/8	15.2 (5)	18.8 (5)		17.8 (5)		
2304	1/16	15.4 (5)	20.0 (5)	26.8 (5)	18.0 (5)	25.0 (5)	
9216	1/32	16.3 (6)	19.7 (6)	29.5 (6)	17.3 (6)	24.0 (6)	36.5 (6)
36864	1/64	16.0 (6)	18.3 (6)	26.3 (6)	17.2 (6)	22.0 (6)	32.8 (6)
147456	1/128	16.3 (6)	18.3 (6)	23.0 (6)	17.0 (6)	19.8 (6)	28.0 (6)

h	$p = 2$	$p = 3$	$p = 4$	$p = 5$
1/4	18	21	21	22
1/8	19	20	19	20
1/16	19	19	19	19
1/32	18	19	17	18
1/64	17	19	16	17

Example (Strongly anisotropic parabolic problem)

Let $\Omega = (0,1)^2$, $I = (0,1)$, $\Lambda = \mathrm{SO}(2)$,

$$a^\alpha := \alpha \begin{pmatrix} 1 & 1/40 \\ 1/40 & 1/800 \end{pmatrix} \alpha^\top, \quad \alpha \in \Lambda.$$

For $\omega = 1$, Cords condition holds with $\varepsilon \approx 1.25 \times 10^{-3}$.

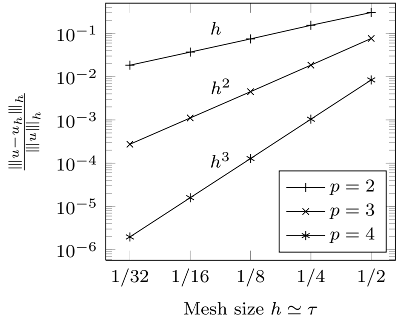
Solution $u = (1 - \mathrm{e}^{-t}) \exp(xy) \sin(\pi x) \sin(\pi y)$.

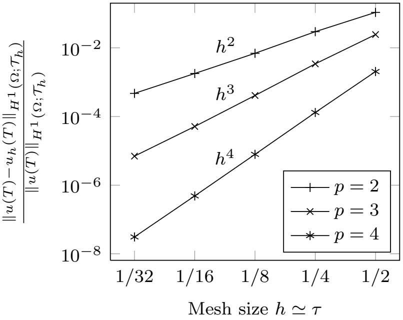
Uniform refinement with $q = p - 1$, $h \simeq \tau$.

Remark (Monotone FDM)

Consistency requires (at least) stencil width ≥ 20, with more than 1529 stencil points.

3. Numerics: experiment 4: Parabolic





$$|||v|||_h^2 := \sum_{n=1}^{N} \int_{I_n} \sum_{K \in \mathcal{T}_h} \left[\omega^2 \|\partial_t v\|_{L^2(K)}^2 + \|v\|_{H^2(K)}^2 \right] \mathrm{d}t.$$

Summary and outlook

Is it possible to have stable, consistent and convergent methods for fully nonlinear PDEs without discrete maximum principles?

- For equations with Cordes coefficients as presented here
  - Consistency & Stability of non-conforming discretisations
  - Convergence rates for sufficiently regular solutions
  - Non-structured meshes, varying polynomial degrees, etc.

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Thank you!