Discontinuous Galerkin finite element methods for Hamilton–Jacobi–Bellman equations with Cordes coefficients

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joint work with
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Talk outline

1. **Introduction**: Hamilton–Jacobi–Bellman (HJB) equations.
2. **Analysis**: Analysis of HJB equations with Cordes coefficients.
3. **Numerical methods**: High-order discontinuous Galerkin methods for HJB equations with Cordes coefficients.
Overview

*Talk outline*

2. *Analysis:* Analysis of HJB equations with Cordes coefficients.
1. Stochastic optimal control

Stochastic differential equation

\[ dX_t = b(X_t, \alpha_t) \, dt + \sigma(X_t, \alpha_t) \, dB_t, \quad X_0 = x, \]

Find a control \( \alpha(\cdot) : t \mapsto \alpha_t \in \Lambda \) that minimises

\[ J(x, \alpha(\cdot)) = \mathbb{E} \left[ \int_0^{\tau_{\text{exit}}} f(X_t, \alpha_t) \, e^{-\int_0^t c(X_s, \alpha_s) \, ds} \, dt \right] \]

- \( b(x, \alpha) \in \mathbb{R}^d \) drift, \( \sigma(x, \alpha) \in \mathbb{R}^{d \times m} \) volatility
- scalar \( f \) and \( c \) : running cost and discount
- \( \alpha_t \) control variable
- \( \Lambda \) controls set (assumed to be a compact metric space).
- stopping time \( \tau_{\text{exit}} \): first exit from bounded domain \( \Omega \subset \mathbb{R}^d \)

Example applications: energy, engineering, finance . . .
1. Dynamic programming principle

The dynamic programming principle (DPP) is a solution process for a stochastic control problem.

Overview: stages of DPP

1. Define the \textit{value function} of the optimal control problem.

2. DPP: the value function is the solution of an HJB equation.

3. The optimal controls can be computed once the value function is available.

Richard Bellman (1920–1984)

Feedback control map \( \alpha_{\text{feedback}} : \Omega \to \Lambda \implies \alpha(t) := \alpha_{\text{feedback}}(X_t). \)
1. Dynamic programming principle

Details in Fleming & Soner 2006

• Define the value function $V$ defined by

$$V(x) := \inf \{ J(x, \alpha(\cdot)) \mid \alpha(\cdot): t \in [0, \infty) \mapsto \alpha_t \in \Lambda, \alpha(\cdot) \in \mathcal{A} \}.$$ 

$\mathcal{A} = \text{set of admissible controls}:$ progressively measurable w.r.t. filtration.

• The function $u := -V$ solves the HJB equation

$$\sup_{\alpha \in \Lambda} [L^\alpha u - f^\alpha] = 0 \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega,$$

(HJB)

where $L^\alpha u := a^\alpha(x) : D^2 u + b^\alpha(x) \cdot \nabla u - c^\alpha(x) u,$ with

$$a^\alpha(x) := \frac{1}{2} \sigma(x, \alpha) \sigma^T(x, \alpha) \in \mathbb{R}^{d \times d},$$

Notation: $a^\alpha : D^2 u = \sum_{i,j=1}^d a^\alpha_{ij}(x) u_{x_i x_j}$
1. Dynamic programming principle

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$$a^\alpha(x) := \frac{1}{2} \sigma(x, \alpha) \sigma^\top(x, \alpha) \in \mathbb{R}^{d \times d},$$

Notation: $a^\alpha : D^2 u = \sum_{i,j=1}^{d} a^\alpha_{ij}(x) u_{x_i x_j},$
1. Hamilton–Jacobi–Bellman Equation

**Elliptic Dirichlet problem:**

\[ \sup_{\alpha \in \Lambda} [L^\alpha u - f^\alpha] = 0 \quad \text{in } \Omega, \]

u = 0 \quad \text{on } \partial \Omega, 

(Elliptic HJB)

where \( L^\alpha u := a^\alpha(x) : D^2 u + b^\alpha(x) \cdot \nabla u - c^\alpha(x) u. \)

**Notation:**

\[ a^\alpha(x) : D^2 u = \sum_{i,j=1}^{d} a_{ij}^\alpha(x) u_{x_i x_j}, \quad b^\alpha(x) \cdot \nabla u = \sum_{i=1}^{d} b_i^\alpha(x) u_{x_i}. \]

**Assumptions in this talk:**

- \( \Omega \subset \mathbb{R}^d \) is bounded and convex, \( \Lambda \) a compact metric space.
- \( a, b, c \) and \( f \) are continuous functions in \( x \in \overline{\Omega}, \alpha \in \Lambda. \)
- \( a^\alpha \) are symmetric positive definite, uniformly on \( \overline{\Omega} \times \Lambda, \) and \( c^\alpha \geq 0. \)
- Cordes coefficients: the coefficient functions \( a, b, c \) satisfy the Cordes condition (coming soon!)
1. Examples

How do HJB equations relate to other PDEs?

$$\sup_{\alpha \in \Lambda} [L^\alpha u - f^\alpha] = 0$$

The HJB equation generalises many other equations:

- Linear nondivergence form elliptic equations

  $$a : D^2 u + b \cdot \nabla u - cu = f,$$
  (assume that $\Lambda$ is a singleton set).

- Hamilton–Jacobi: e.g. eikonal equation

  $$\sup_{\alpha \in S^d} [\alpha \cdot \nabla u - 1] = |\nabla u| - 1 = 0.$$  

- Monge–Ampère equation [Krylov 1987, Jensen & Feng 2017]

  \[
  \begin{cases} 
  \det D^2 u - f = 0 \\
  u \text{ convex}
  \end{cases}
  \iff
  \inf_{a \in \mathbb{R}^{d \times d}_{\text{sym,}+}, \text{Tr} a = 1} \left[ a : D^2 u - d (f \det a)^{1/d} \right] = 0.
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1. Examples

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- Monge–Ampère equation [Krylov 1987, Jensen & Feng 2017]
  \[ \begin{cases} \det D^2 u - f = 0 \\ u \text{ convex} \end{cases} \iff \inf_{a \in \mathbb{R}^{d \times d}_{\text{sym}},+} \left[ a : D^2 u - d (f \det a)^{1/d} \right] \geq 0. \]
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*How do HJB equations relate to other PDEs?*

\[
\sup_{\alpha \in \Lambda} [L^\alpha u - f^\alpha] = 0
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  \]
1. Approaches

What are the available approaches to PDE theory and numerical discretization?

- **Weak solutions in** $H^1(\Omega)$: not applicable!
  - → most existing finite element techniques cannot be used!
- **Viscosity solutions**: generally applicable, even to degenerate elliptic problems. Solution space $u \in C(\Omega)$.
  - This leads to *monotone numerical schemes* (c.f. next few slides) . . .
- **Strong solutions in** $H^2(\Omega)$: under the Cordes condition . . .
- Classical solutions in $C^2(\Omega)$: Evans–Krylov Theorem and its developments guarantee interior regularity estimates for the viscosity solution under uniform ellipticity and data regularity assumptions.

Some references:
- Weak, strong and classical solutions
- Viscosity solutions for fully nonlinear PDE
- Regularity theory of viscosity solutions
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- Weak, strong and classical solutions  
  *Gilbarg & Trudinger, 1998.*

- Viscosity solutions for fully nonlinear PDE  
  *Crandall, Lions & Ishii, 1992.*

- Regularity theory of viscosity solutions  
  *Caffarelli & Cabré, 1995.*
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1. Literature

Pre-existing numerical methods:

- **Monotone methods:** low-order methods with discrete maximum principles:
  - mostly finite difference methods: Motzkin & Wasow, Kuo & Trudinger, Kocan, Camilli & Falcone, Barles & Jakobsen, Jakobsen & Debrabant, Fleming & Soner, Bonnans & Zidani, ... 

- **Other pre-existing non-monotone methods** without discrete maximum principles, without convergence theory: Feng, Neilan, Glowinski, Brenner, Lakkis, Pryer, ... [Feng et al. SIAM Rev. 2013].

Is it possible to have stable, consistent and convergent methods for fully nonlinear PDEs without discrete maximum principles?
1. Literature

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  ◦ a few on monotone FEM: Jensen & S., SINUM 2013 and Nochetto & Zhang FOCM 2016.

• Other pre-existing non-monotone methods without discrete maximum principles, without convergence theory: Feng, Neilan, Glowinski, Brenner, Lakkis, Pryer, … [Feng et al. SIAM Rev. 2013].

*Is it possible to have stable, consistent and convergent methods for fully nonlinear PDEs without discrete maximum principles?*
Overview

Talk outline


2. *Analysis*: Analysis of HJB equations with Cordes coefficients.
   - Motivation of Cordes coefficients
   - Existence, Uniqueness, Well-posedness

2. PDE Theory: motivation

Cordes introduced his condition in the context of nondivergence form equations with discontinuous coefficients.

There is a link between HJB equations and nondivergence form equations with discontinuous coefficients:

**Classical solution algorithm: policy iteration, due to Howard and Bellman:**

1. Choose an initial guess $u^0$.

2. For $k \in \mathbb{N}$, given a current guess $u^k$, choose $\alpha_k : \Omega \to \Lambda$, a Lebesgue-measurable selection

\[ \alpha_k(x) \in \arg\max_{\alpha \in \Lambda} (L^\alpha u^k - f^\alpha)(x), \quad \forall x \in \Omega. \]

3. Then, find $u^{k+1}$ as a solution of the PDE

\[ L^\alpha_k u^{k+1} = f^\alpha_k \quad \text{in } \Omega, \quad \text{with } u^{k+1} = 0 \text{ on } \partial\Omega, \]

where $f^\alpha_k : x \mapsto f^\alpha_k(x)(x)$, etc.

Howard 1960, Puterman & Brumelle 1979, Bokanowski et al. 2009, ...
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\[ L^{\alpha_k} u^{k+1} = f^{\alpha_k} \quad \text{in } \Omega, \quad \text{with } u^{k+1} = 0 \text{ on } \partial\Omega, \]

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2. PDE Theory: motivation

Linearized equation:

\[ L^{\alpha_k} u^{k+1} = f^{\alpha_k} \quad \text{in } \Omega, \quad \text{with } u^{k+1} = 0 \text{ on } \partial \Omega, \quad (1) \]

Question: is Eq. (1) a well-posed PDE?

In general, the answer is no: the linearization process (esp. the argmax) leads to discontinuous diffusion coefficients: \( a^{\alpha_k} \in L^\infty(\Omega)^{d \times d} \) and \( a^{\alpha_k} \notin C(\Omega)^{d \times d} \).

• Calderon–Zygmund: If \( a \in C(\Omega)^{d \times d} \) and \( \partial \Omega \in C^{1,1} \), then existence and uniqueness in \( W^{2,p} \) [Gilbarg & Trudinger, 1998]

• If \( a \in L^\infty(\Omega)^{d \times d} \), \( a \notin C(\Omega)^{d \times d} \), then there are counter-examples showing non-uniqueness in general (Maugeri et al, 2000):

\[ \Delta u + \rho \sum_{i,j=1}^{d} \frac{x_i x_j}{|x|^2} u_{x_i} x_j = 0 \text{ in } B \text{ unit ball,} \quad \rho = -1 + \frac{d-1}{1-\theta}, \quad 0 < \theta < 1, \]

If \( d \geq 3 \) and \( d > 2(2 - \theta) > 2 \), two solutions in \( H^2(B) \cap H^1_0(B) \)

\[ u_1(x) = 0 \quad \text{and} \quad u_2(x) = |x|^\theta - 1 \]
2. PDE Theory: motivation

Linearized equation:

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2. PDE theory: Cordes condition

**Cordes condition:**  *Case 1: without advection and reaction*

Assume that there exists \( \varepsilon \in (0, 1] \) s. t.

\[
\frac{|a(x)|^2}{(\text{Tr } a(x))^2} \leq \frac{1}{d - 1 + \varepsilon} \quad \text{a.e. } x \in \Omega,
\]

(Cordes\(_0\))

**Theorem (Cordes, 1956)**

*If \( \Omega \) is convex, and if \( a \in L^\infty(\Omega)^{d \times d} \) unif. ellipt. satisfies (Cordes\(_0\)), then for any \( f \in L^2(\Omega) \) there exists a unique \( u \in H^2(\Omega) \cap H^1_0(\Omega) \) solving

\[
a : D^2 u = f \quad \text{in } \Omega, \quad \text{with } u = 0 \text{ on } \partial \Omega,
\]

**Example**

If dimension \( d = 2 \), (Cordes\(_0\)) \iff uniform ellipticity.
2. PDE theory: Cordes condition

**Cordes condition:**  *Case 1: without advection and reaction*

Assume that there exists $\varepsilon \in (0, 1]$ s. t.

$$\frac{|a^\alpha(x)|^2}{(\text{Tr } a^\alpha(x))^2} \leq \frac{1}{d - 1 + \varepsilon} \quad \text{a.e. } x \in \Omega, \alpha \in \Lambda$$  \hspace{1cm} (2)

**Theorem (Cordes, 1956)**

If $\Omega$ is convex, and if $a^\alpha \in L^\infty(\Omega)^{d \times d}$ unif. ellipt. satisfies (Cordes$_0$), then for any $f \in L^2(\Omega)$ there exists a unique $u \in H^2(\Omega) \cap H^1_0(\Omega)$ solving

$$a^\alpha : D^2 u = f \quad \text{in } \Omega, \quad \text{with } u = 0 \text{ on } \partial \Omega,$$

**Example**

If dimension $d = 2$, (Cordes$_0$) $\iff$ *uniform ellipticity.*
2. PDE theory: Cordes condition

**Cordes condition:** Case 2: extension to $b^\alpha \neq 0$ and $c^\alpha \neq 0$

Assume that there exist $\lambda > 0$ and $\varepsilon \in (0, 1]$ s. t.

\[
\frac{|a^\alpha|^2 + |b^\alpha|^2/2\lambda + (c^\alpha/\lambda)^2}{(\text{Tr } a^\alpha + c^\alpha/\lambda)^2} \leq \frac{1}{d + \varepsilon} \quad \text{in } \Omega, \ \forall \alpha \in \Lambda. \quad (\text{Cordes}_1)
\]
2. PDE theory: well-posedness

Theorem (Strong solutions of HJB equations with Cordes coefficients)

Let \( \Omega \) be a bounded convex open subset of \( \mathbb{R}^d \), and let \( \Lambda \) be a compact metric space.

Let the data be continuous on \( \overline{\Omega} \times \Lambda \), and satisfy (Cordes\(_1\)) with uniformly elliptic \( a^\alpha \) and \( c^\alpha \geq 0 \) for all \( \alpha \in \Lambda \).

Then, there exists a unique \( u \in H^2(\Omega) \cap H^1_0(\Omega) \) that solves (Elliptic HJB) pointwise a.e. in \( \Omega \).


2. PDE theory: proof of well-posedness

Define

\[ \gamma^\alpha := \frac{\text{Tr } a^\alpha + c^\alpha/\lambda}{|a^\alpha|^2 + |b^\alpha|^2/2\lambda + (c^\alpha/\lambda)^2} \]

\[ F_{\gamma}[u] := \sup_{\alpha \in \Lambda} \left[ \gamma^\alpha (L^\alpha u - f^\alpha) \right] \]

Because \( \gamma^\alpha > 0 \), we can renormalize the operator:

\[ F_{\gamma}[u] = \sup_{\alpha \in \Lambda} \left[ \gamma^\alpha (L^\alpha u - f^\alpha) \right] = 0 \iff \sup_{\alpha \in \Lambda} [L^\alpha u - f^\alpha] = 0. \quad (3) \]

The problem (Elliptic HJB) for \( u \in H^2(\Omega) \cap H^1_0(\Omega) \) is equivalent to

\[ A(u; v) := \int_{\Omega} F_{\gamma}[u] L_\lambda v \, dx = 0 \quad \forall \ v \in H^2(\Omega) \cap H^1_0(\Omega), \quad (4) \]

where \( L_\lambda v := \Delta v - \lambda v \).
Let $H := H^2(\Omega) \cap H_0^1(\Omega)$, $\|v\|_H^2 := \|D^2 u\|^2 + 2\lambda \|\nabla u\|^2 + \lambda^2 \|u\|^2$

1. $A : H \times H \to \mathbb{R}$ is linear in its second argument (only).

2. $A$ is Lipschitz continuous: there is $C > 0$ such that
   \[ |A(u; v) - A(w; v)| \leq C \|u - w\|_H \|v\|_H \quad \forall u, v, w \in H, \]

3. We now show that $A$ is strongly monotone\(^1\): there exists a positive constant $c = c(\varepsilon) > 0$ such that
   \[ \frac{1}{c} \|u - v\|_H^2 \leq A(u; u - v) - A(v; u - v) \quad \forall u, v \in H. \]

On verifying these conditions, we conclude that there exists a unique $u \in H$ that solves $A(u; v) = 0$ for all $v \in H$ and thus solves (Elliptic HJB).

\(^1\)Remark: must not confuse monotone schemes (i.e. discrete max principle) with strongly monotone operators in functional analytic sense.
Notation: $H := H^2(Ω) \cap H^1_0(Ω)$, $\|v\|_H^2 := \|D^2 u\|^2 + 2\lambda \|\nabla u\|^2 + \lambda^2 \|u\|^2$

Key ingredients

1. **The Cordes condition**, which implies by direct calculation that

$$\|F_\gamma [u] - F_\gamma [v] - L_\lambda (u - v)\|_L^2 \leq \sqrt{1 - \varepsilon} \|u - v\|_H$$  \hspace{1cm} (6)

2. **Miranda–Talenti**: for convex $Ω$,

$$\|w\|_H \leq \|L_\lambda w\|_L^2 \quad \forall \ w \in H^2(Ω) \cap H^1_0(Ω)$$  \hspace{1cm} (7)

where $L_\lambda v := \Delta v - \lambda v$ (recall $\lambda > 0$)

Maugeri, Palagachev & Softova, 2000, and Grisvard 1985
2. PDE theory: proof of well-posedness

\[ \| F_\gamma[u] - F_\gamma[v] - L_\lambda(u - v) \|_{L^2} \leq \sqrt{1 - \varepsilon} \| u - v \|_H, \quad \| v \|_H \leq \| L_\lambda v \|_{L^2} \]

**Strong monotonicity:**
Recall \( \mathcal{A}(u; v) = \int_{\Omega} F_\gamma[u] L_\lambda v \, dx \).

\[
\mathcal{A}(u; u - v) - \mathcal{A}(v; u - v) = \int_{\Omega} (F_\gamma[u] - F_\gamma[v]) L_\lambda(u - v) \, dx.
\]

Addition–subtraction of \( \| L_\lambda(u - v) \|_{L^2}^2 \) gives

\[
\mathcal{A}(u; u - v) - \mathcal{A}(v; u - v) = \| L_\lambda(u - v) \|_{L^2}^2
+ \int_{\Omega} (F_\gamma[u] - F_\gamma[v] - L_\lambda(u - v)) L_\lambda(u - v) \, dx
\geq -\sqrt{1 - \varepsilon} \| u - v \|_H \| L_\lambda(u - v) \|_{L^2} \geq -\sqrt{1 - \varepsilon} \| L_\lambda(u - v) \|_{L^2}^2
\]

Therefore

\[
\mathcal{A}(u; u - v) - \mathcal{A}(v; u - v) \geq (1 - \sqrt{1 - \varepsilon}) \| L_\lambda(u - v) \|_{L^2}^2 \geq (1 - \sqrt{1 - \varepsilon}) \| u - v \|_H^2
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2. PDE theory: proof of well-posedness

\[ \| F_\gamma[u] - F_\gamma[v] - L_\lambda(u - v) \|_{L^2} \leq \sqrt{1 - \varepsilon} \| u - v \|_H, \quad \| v \|_H \leq \| L_\lambda v \|_{L^2} \]

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\geq -\sqrt{1 - \varepsilon} \| u - v \|_H \| L_\lambda(u - v) \|_{L^2} \geq -\sqrt{1 - \varepsilon} \| L_\lambda(u - v) \|_{L^2}^2
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\[ A(u; u - v) - A(v; u - v) = \int_\Omega (F_\gamma[u] - F_\gamma[v]) L_\lambda (u - v) dx. \]

Addition–subtraction of \( \|L_\lambda(u - v)\|_{L^2}^2 \) gives

\[ A(u; u - v) - A(v; u - v) = \|L_\lambda(u - v)\|_{L^2}^2 \]

\[ \quad + \int_\Omega (F_\gamma[u] - F_\gamma[v] - L_\lambda(u - v)) L_\lambda(u - v) dx \]

\[ \geq -\sqrt{1 - \varepsilon}\|u - v\|_H\|L_\lambda(u - v)\|_{L^2} \geq -\sqrt{1 - \varepsilon}\|L_\lambda(u - v)\|_{L^2}^2 \]

Therefore

\[ A(u; u - v) - A(v; u - v) \geq (1 - \sqrt{1 - \varepsilon})\|L_\lambda(u - v)\|_{L^2}^2 \geq (1 - \sqrt{1 - \varepsilon})\|u - v\|_H^2 \]
2. PDE theory

*Approach to numerical analysis:*

Since the proof of well-posedness hinges on the strong monotonicity of

$$A(u; v) = \int_{\Omega} F_\gamma[u] L_\lambda v \, dx,$$

we will attempt to discretise the operator $A$ and conserve its strong monotonicity.

- The Cordes condition carries over straightforwardly to discrete setting
- The Miranda–Talenti inequality does not carry over if the approximation space is not inside $H^2(\Omega) \cap H^1_0(\Omega)$. 
Overview

Talk outline


2. *Analysis*: Analysis of HJB equations with Cordes coefficients.

   - Design of a consistent, stable and convergent method
   - Error bounds
   - Extension to parabolic problems
   - Numerical experiments
3. Numerics: design of the method

Let \( \{T_h\}_h \) a shape-regular sequence of meshes on \( \Omega \).

- Elements composing the mesh can be parallelepipeds, simplices, or more generally any combination of standard elements.
- The mesh is *not assumed to be quasi-uniform* (very useful for \( hp \)-refinement).
- Hanging nodes allowed.
3. Numerics: design of the method

Construction of the discontinuous finite element space

Discontinuous finite element space:

\[ V_{h,p} := \{ v_h \in L^2(\Omega) : v_h|_K \in \mathcal{P}_{p_K}(K) \ \forall K \in \mathcal{T}_h \}. \]

Polynomial degrees \( p = (p_K)_{K \in \mathcal{T}_h} \)

Approximation in \( H^2 \) requires \( p_K \geq 2 \) for all elements \( K \).
3. Numerics: design of the method

*Notation of discontinuous Galerkin methods:*

\[ \begin{align*}
  &F_{\text{int}} \\
  &F_{\text{ext}} \\
  &F_{\text{int}} \quad n_F \\
  &F_{\text{ext}}
\end{align*} \]

Distinguish interior and boundary faces

\[ \mathcal{F}_h^i \text{ interior faces of } \mathcal{T}_h, \quad \mathcal{F}_h^b \text{ boundary faces of } \mathcal{T}_h, \]

\[ \mathcal{F}_h^{i,b} := \mathcal{F}_h^i \cup \mathcal{F}_h^b. \]

Jump operators over faces:

\[ \begin{align*}
  [\phi] &:= \tau_F (\phi|_{K_{\text{ext}}}) - \tau_F (\phi|_{K_{\text{int}}}), \quad \{\phi\} := \frac{1}{2} \tau_F (\phi|_{K_{\text{ext}}}) + \frac{1}{2} \tau_F (\phi|_{K_{\text{int}}}), & \text{if } F \in \mathcal{F}_h^i, \\
  [\phi] &:= \tau_F (\phi|_{K_{\text{ext}}}), \quad \{\phi\} := \tau_F (\phi|_{K_{\text{ext}}}), & \text{if } F \in \mathcal{F}_h^b,
\end{align*} \]
3. Numerics: design of the method

*Notation of discontinuous Galerkin methods:*

Let \( \{ t_i \}_{i=1}^{d-1} \subset \mathbb{R}^d \) be an orthonormal coordinate system on \( F \). Define the **tangential gradient and divergence**

\[
\nabla_T u := \sum_{i=1}^{d-1} t_i \frac{\partial u}{\partial t_i}, \quad \text{div}_T \mathbf{v} := \sum_{i=1}^{d-1} \frac{\partial \mathbf{v}_i}{\partial t_i}.
\]
3. Numerics: design of the method

The goal is to discretise

\[ \mathcal{A}(u; v) = \int_\Omega F_\gamma[u] L_\lambda v \, dx, \]

whilst conserving the strong monotonicity bound.

Recall main ingredients:

1. The Cordes condition remains unchanged in discrete setting.

2. Miranda–Talenti inequality: not conserved when replacing \( H^2(\Omega) \cap H^1_0(\Omega) \) by \( V_{h,p} \).

Our approach:

• Miranda–Talenti inequality was derived from an integration by parts identity (Maugeri et al 2000, Grisvard 1984)

• We will include a discrete weak form of this identity in the scheme (next slide)
3. Numerics: design of the method

Numerical scheme: find $u_h \in V_{h,p}$ such that

$$A_h(u_h; v_h) = 0 \quad \forall \, v_h \in V_{h,p}.$$  
(scheme)

$$A_h(u_h; v_h) := \sum_{K \in T_h} \langle F_\gamma[u_h], L_\lambda v_h \rangle_K + J_h(u_h, v_h)$$

$$+ \frac{1}{2} \left( B_h(u_h, v_h) - \sum_{K \in T_h} \langle L_\lambda u_h, L_\lambda v_h \rangle_K \right).$$
3. Numerics: design of the method

**Numerical scheme:** find \( u_h \in V_{h,p} \) such that

\[
A_h(u_h; v_h) = 0 \quad \forall \, v_h \in V_{h,p}. \tag{scheme}
\]

\[
A_h(u_h; v_h) := \sum_{K \in T_h} \langle F_\gamma[u_h], L_\lambda v_h \rangle_K + J_h(u_h, v_h)
\]

\[
+ \frac{1}{2} \left( B_h(u_h, v_h) - \sum_{K \in T_h} \langle L_\lambda u_h, L_\lambda v_h \rangle_K \right).
\]

\[
\langle F_\gamma[u_h], L_\lambda v_h \rangle_K := \int_K \sup_{\alpha \in \Lambda} \left[ \gamma^\alpha (L^\alpha u_h - f^\alpha) \right] (\Delta v_h - \lambda v_h) \, dx.
\]
3. Numerics: design of the method

**Numerical scheme:** find $u_h \in V_{h,p}$ such that

$$A_h(u_h; v_h) = 0 \quad \forall \, v_h \in V_{h,p}. \quad \text{(scheme)}$$

$$A_h(u_h; v_h) := \sum_{K \in T_h} \langle F_\gamma[u_h], L_\lambda v_h \rangle_K + J_h(u_h, v_h)$$

$$+ \frac{1}{2} \left( B_h(u_h, v_h) - \sum_{K \in T_h} \langle L_\lambda u_h, L_\lambda v_h \rangle_K \right).$$

Jump penalisation with $\mu_F \simeq p_k^2/h_K$ and $\eta_F \simeq p_k^4/h_K^3$ for $F \subset \partial K$:

$$J_h(u_h, v_h) := \sum_{F \in \mathcal{F}_h} \left[ \mu_F \langle \left[ \nabla_T u_h \right], \left[ \nabla_T v_h \right] \rangle_F + \eta_F \langle \left[ u_h \right], \left[ v_h \right] \rangle_F \right]$$

$$+ \sum_{F \in \mathcal{F}_h} \mu_F \langle \left[ \nabla u_h \cdot n_F \right], \left[ \nabla v_h \cdot n_F \right] \rangle_F.$$
3. Numerics: design of the method

**Numerical scheme:** find $u_h \in V_{h,p}$ such that

$$A_h(u_h; v_h) = 0 \quad \forall v_h \in V_{h,p}. \quad \text{(scheme)}$$

$$A_h(u_h; v_h) := \sum_{K \in T_h} \langle F_\gamma[u_h], L_\lambda v_h \rangle_K + J_h(u_h, v_h)$$

$$+ \frac{1}{2} \left( B_h(u_h, v_h) - \sum_{K \in T_h} \langle L_\lambda u_h, L_\lambda v_h \rangle_K \right).$$

$$\langle L_\lambda u_h, L_\lambda v_h \rangle_K := \int_K (\Delta u_h - \lambda u_h)(\Delta v_h - \lambda v_h) \, dx.$$
3. Numerics: design of the method

**Numerical scheme:** find $u_h \in \mathcal{V}_{h,p}$ such that

$$A_h(u_h; v_h) = 0 \quad \forall v_h \in \mathcal{V}_{h,p}.$$  \hspace{1cm} \text{(scheme)}

$$A_h(u_h; v_h) := \sum_{K \in \mathcal{T}_h} \langle \mathcal{F}_\gamma[u_h], L\lambda v_h \rangle_K + J_h(u_h, v_h)$$

$$+ \frac{1}{2} \left( B_h(u_h, v_h) - \sum_{K \in \mathcal{T}_h} \langle L\lambda u_h, L\lambda v_h \rangle_K \right).$$

$$B_h(u_h, v_h) := \sum_{K \in \mathcal{T}_h} \left[ \langle D^2 u_h, D^2 v_h \rangle_K + 2\lambda \langle \nabla u_h, \nabla v_h \rangle_K + \lambda^2 \langle u_h, v_h \rangle_K \right]$$

$$+ \sum_{F \in \mathcal{F}_h^i} \left[ \langle \text{div}_T \nabla_T \{u_h\}, \llbracket \nabla v_h \cdot n_F \rrbracket_F \rangle_F + \langle \text{div}_T \nabla_T \{v_h\}, \llbracket \nabla u_h \cdot n_F \rrbracket_F \rangle_F \right]$$

$$- \sum_{F \in \mathcal{F}_h^{i,b}} \left[ \langle \nabla_T \{\nabla u_h \cdot n_F\}, \llbracket \nabla_T v_h \rrbracket_F \rangle_F + \langle \nabla_T \{\nabla v_h \cdot n_F\}, \llbracket \nabla_T u_h \rrbracket_F \rangle_F \right]$$

$$- \lambda \sum_{F \in \mathcal{F}_h^i \setminus b} \langle \{\nabla u_h \cdot n_F\}, \llbracket v_h \rrbracket_F \rangle_F + \langle \{\nabla v_h \cdot n_F\}, \llbracket u_h \rrbracket_F \rangle_F - \lambda \sum_{F \in \mathcal{F}_h^i} \langle \{u_h\}, \llbracket \nabla v_h \cdot n_F \rrbracket_F \rangle_F + \langle \{v_h\}, \llbracket \nabla u_h \cdot n_F \rrbracket_F \rangle_F.$$
3. Numerics: design of the method

**Numerical scheme:** find \( u_h \in V_{h,p} \) such that

\[
A_h(u_h; v_h) = 0 \quad \forall \ v_h \in V_{h,p}.
\]

(scheme)

\[
A_h(u_h; v_h) := \sum_{K \in \mathcal{T}_h} \langle F_\gamma[u_h], L_\lambda v_h \rangle_K + J_h(u_h, v_h)
\]

\[
+ \frac{1}{2} \left( B_h(u_h, v_h) - \sum_{K \in \mathcal{T}_h} \langle L_\lambda u_h, L_\lambda v_h \rangle_K \right).
\]

**Key consistency result:** If \( u \in H^2(\Omega) \cap H^1_0(\Omega) \) has well-defined second derivative traces on faces \( F \) of the mesh, then

\[
B_h(u, v_h) = \sum_K \langle L_\lambda u, L_\lambda v_h \rangle_K, \quad J_h(u, v_h) = 0 \quad \forall \ v_h \in V_{h,p}.
\]

**Technical point:** a sufficient condition is that \( u \in H^s(K) \) with \( s > 5/2 \) for every \( K \in \mathcal{T}_h \).
3. Numerics: consistency, stability and error bounds

**Numerical scheme:** find \( u_h \in V_{h,p} \) such that

\[
A_h(u_h; v_h) = 0 \quad \forall v_h \in V_{h,p}.
\]

(scheme)

Full theoretical justification given in [S. & S"uli, SINUM 2014]:

- **Consistency Theorem:** sufficiently regular solution of (Elliptic HJB) solves:

\[
A_h(u; v_h) = 0 \quad \forall v_h \in V_{h,p}.
\]

- **Discrete Stability Theorem:** Existence & uniqueness of numerical solution since the nonlinear form \( A_h \) is strongly monotone: provided \( \mu_F \gtrsim p^2/h \) and \( \eta_F \gtrsim p^2/h \)

\[
\| u_h - v_h \|_h^2 \lesssim A_h(u_h; u_h - v_h) - A_h(v_h; u_h - v_h) \quad \forall u_h, v_h \in V_{h,p},
\]

where

\[
\| v_h \|_h^2 := \sum_{K \in T_h} \left[ |v_h|_{H^2(K)}^2 + 2\lambda |v_h|_{H^1(K)}^2 + \lambda^2 \| v_h \|_{L^2(K)}^2 \right] + J_h(v_h, v_h)
\]

- **Consistency+Stability \( \implies \) error bounds and convergence.**
3. Numerics: error bounds

\[ \|v_h\|^2_h := \sum_{K \in T_h} \left[ |v_h|^2_{H^2(K)} + 2\lambda |v_h|^2_{H^1(K)} + \lambda^2 \|v_h\|^2_{L^2(K)} \right] + J_h(v_h, v_h). \]

**Theorem (High-order convergence rates)**

(Under previous assumptions & standard assumptions for DG meshes...)

Assume that \( u \in H^s(\Omega; T_h) \), with \( s_K > 5/2 \) for all \( K \in T_h \).

\[ \|u - u_h\|^2_h \lesssim \sum_{K \in T_h} \left[ \frac{h^{t_K - 2}_K}{p^{s_K - 5/2}_K} \|u\|_{H^{s_K}(K)} \right]^2, \]

where \( t_K = \min(p_K + 1, s_K) \) for each \( K \in T_h \).

Simplified form:

\[ \|u - u_h\|_h \lesssim \frac{h^{\min(s,p+1)-2}}{p^{s-5/2}} \|u\|_{H^s(\Omega)}. \]

- Optimal in \( h \), half-order subopt. in \( p \)
- High-order convergence rates.
- Higher efficiency on well-chosen meshes and \( hp \)-refinement.
3. Numerics: error bounds

If $u$ has only minimal regularity, then we have the following quasi-optimal approximation property with respect to the $H^2$-conforming subspace:

**Theorem (Minimal regularity error bound)**

*Under previous assumptions...*  
Let $u \in H^2(\Omega) \cap H^1_0(\Omega)$ be the solution of (Elliptic HJB). Then

$$\|u - u_h\|_h \leq \inf_{z_h \in V_{h,p} \cap H^2(\Omega) \cap H^1_0(\Omega)} \|u - z_h\|_h.$$  

Note however that DG method requires only quadratic polynomials, whereas $H^2$-conforming methods may require higher (e.g. Argyris elements require quintic polynomials).
3. Numerics: extensions to parabolic problems

S. & Süli, Num. Math. 2016: extension to parabolic HJB equations

- Generalisation of the Cordes condition and the PDE theory: existence and uniqueness of the strong solution
  \[ u \in L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \cap H^1(0, T; L^2(\Omega)). \]

- Numerical scheme: \( hp-\tau q \)-version space-time DGFEM using tensor product of \( V_{h,p} \) with piecewise polynomials in time.

- Stability, consistency and convergence rates that are:
  - \( h \)-optimal,
  - \( p \)-suboptimal by \( p^{3/2} \),
  - \( \tau \)-optimal,
  - \( q \)-optimal.

- Exponential convergence rates under \( hp-\tau q \) refinement verified experimentally.
3. Numerics: experiment 1: $h$-refinement

Experiment 1: Test of high order convergence rates under $h$-refinement, fixed $p$.

Example (Control of correlated diffusions)

$$a^\alpha := \frac{1}{2} R^\top \begin{pmatrix} 1 + \sin^2 \theta & \sin \theta \cos \theta \\ \sin \theta \cos \theta & \cos^2 \theta \end{pmatrix} R$$

$$\alpha := (\theta, R) \in [0, \frac{\pi}{3}] \times \text{SO}(2) =: \Lambda.$$  

Remark: $a^\alpha$ becomes increasingly anisotropic as $\theta \to \pi/3$; rotation matrices $R \in \text{SO}(2)$ prevent monotone schemes from aligning the grid with the anisotropy.
Example (Control of correlated diffusions)

Uniform $h$-refinement on smooth solution $u(x, y) = \exp(xy) \sin(\pi x) \sin(\pi y)$:
3. Numerics: experiment 2: \textit{hp}-refinement

Experiment 2: \textit{test of exponential convergence rates under \textit{hp}-refinement}

Example (Strong anisotropy + boundary layer)
Let \( \Omega = (0, 1)^2 \), \( b^\alpha \equiv (0, 1) \), \( c^\alpha \equiv 10 \) and define

\[
a^\alpha := \alpha^\top \begin{pmatrix} 20 & 1 \\ 1 & 0.1 \end{pmatrix} \alpha, \quad \alpha \in \Lambda := \text{SO}(2), \quad \lambda = \frac{1}{2}.
\]

(Cordes$_1$) holds with \( \varepsilon \approx 0.0024 \) and \( \lambda = 1/2 \). Choose solution:

\[
u(x, y) = (2x - 1) \left( e^{1-|2x-1|} - 1 \right) \left( y + \frac{1 - e^{y/\delta}}{e^{1/\delta} - 1} \right), \quad \delta := 0.005 = O(\varepsilon)
\]

- Near-degenerate and anisotropic diffusion.
- Sharp boundary layer.
- Non-smooth solution.
3. Numerics: experiment 2: \textit{hp}-refinement

Example (Strong anisotropy + boundary layer)

We use boundary layer adapted meshes with \textit{p}-refinement: $2 \leq p_K \leq 10$, from 100 to 1320 DoFs.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{boundary_layer_mesh.png}
\caption{Boundary layer adapted mesh.}
\end{figure}

Exponential rate: $\| u - u_h \|_h \lesssim \exp(-c^{3/\sqrt{\text{DoF}}})$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{exponential_rate.png}
\caption{Broken $H^2$ norm}
\end{figure}
3. Numerics: Experiment 3: linearization and algebraic solvers

Solution of nonlinear equation by a superlinearly convergent semismooth Newton method. [S. & Süli, SINUM 2014, Sect. 8]
3. Numerics: experiment 3: linearization and algebraic solvers

Nonoverlapping domain decomposition preconditioners with GMRES: (all tolerances $10^{-6}$ in discrete $H^2$-type norm)

<table>
<thead>
<tr>
<th>DoF</th>
<th>$h$</th>
<th>Average GMRES iterations (Newton steps)</th>
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<tbody>
<tr>
<td></td>
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<td>4 Subdomains</td>
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<td>$H = 2h$ $H = 4h$ $H = 8h$</td>
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<tr>
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<td>16 Subdomains</td>
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<td></td>
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<td>$H = 2h$ $H = 4h$ $H = 8h$</td>
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<td>144</td>
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<td>14.3 (6) 18.8 (5) 26.8 (5)</td>
</tr>
<tr>
<td>576</td>
<td>1/8</td>
<td>15.2 (5) 18.8 (5) 26.8 (5)</td>
</tr>
<tr>
<td>2304</td>
<td>1/16</td>
<td>15.4 (5) 20.0 (5) 26.8 (5)</td>
</tr>
<tr>
<td>9216</td>
<td>1/32</td>
<td>16.3 (6) 19.7 (6) 29.5 (6)</td>
</tr>
<tr>
<td>36864</td>
<td>1/64</td>
<td>16.0 (6) 18.3 (6) 26.3 (6)</td>
</tr>
<tr>
<td>147456</td>
<td>1/128</td>
<td>16.3 (6) 18.3 (6) 23.0 (6)</td>
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<tr>
<th>$h$</th>
<th>$p = 2$</th>
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<td>1/64</td>
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</table>
Example (Strongly anisotropic parabolic problem)
Let $\Omega = (0, 1)^2$, $I = (0, 1)$, $\Lambda = SO(2)$,

$$a^\alpha := \alpha \begin{pmatrix} 1 & 1/40 \\ 1/40 & 1/800 \end{pmatrix} \alpha^\top, \quad \alpha \in \Lambda.$$

For $\omega = 1$, Cords condition holds with $\varepsilon \approx 1.25 \times 10^{-3}$.
Solution $u = (1 - e^{-t}) \exp(xy) \sin(\pi x) \sin(\pi y)$.
Uniform refinement with $q = p - 1$, $h \approx \tau$.

Remark (Monotone FDM)
Consistency requires (at least) stencil width $\geq 20$, with more than 1529 stencil points.
3. Numerics: experiment 4: Parabolic

\[ \| u - u_h \|_h^n / \| u \|_h^n \]

<table>
<thead>
<tr>
<th>Mesh size ( h \approx \tau )</th>
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<tbody>
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<td>( 1/32 )</td>
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<td>( 1/16 )</td>
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\[ \| u(T) - u_h(T) \|_{H^1(\Omega; T_h)} / \| u(T) \|_{H^1(\Omega; T_h)} \]

<table>
<thead>
<tr>
<th>Mesh size ( h \approx \tau )</th>
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\[ \| v \|_h^2 := \sum_{n=1}^{N} \int_{I_n} \sum_{K \in T_h} \left[ \omega^2 \| \partial_t v \|_{L^2(K)}^2 + \| v \|_{H^2(K)}^2 \right] dt. \]
Summary and outlook

Is it possible to have stable, consistent and convergent methods for fully nonlinear PDEs without discrete maximum principles?

- For equations with Cordes coefficients as presented here
  - Consistency & Stability of non-conforming discretisations
  - Convergence rates for sufficiently regular solutions
  - Non-structured meshes, varying polynomial degrees, etc.
References

Linear nondivergence form PDE: *Discontinuous Galerkin finite element approximation of nondivergence form elliptic equations with Cordes coefficients*,

Elliptic HJB: *Discontinuous Galerkin finite element approximation of Hamilton–Jacobi–Bellman equations with Cordes coefficients*,

Parabolic HJB: *Discontinuous Galerkin finite element methods for time-dependent Hamilton–Jacobi–Bellman equations with Cordes coefficients*,

Solvers: *Nonoverlapping Domain Decomposition Preconditioners for Discontinuous Galerkin Approximations of Hamilton–Jacobi–Bellman Equations*,

Thank you!