Équations compressibles de Navier-Stokes et solutions faibles

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Based on joint works with:
P.–E. JABIN (Maryland USA)
Small paper: D.B., P.-E. Jabin, INdAM series, Springer, 33–54 (2017).
Big paper: D.B., P.E. Jabin, arXiv:1507.04629v2

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State of art: incompressible flows - global weak solutions

Incompressible Navier-Stokes equations:
1933-34 –> J. Leray [1906–1998]

$$[INS] \quad \begin{aligned} \operatorname{div} \mathbf{u} &= 0, \\ \partial_t \mathbf{u} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) - \nu \Delta \mathbf{u} + \nabla \Pi_1 &= \mathbf{0}, \end{aligned}$$

Non-homogeneous incompressible Navier-Stokes equations :

$$[NHINS] \quad \begin{aligned} \operatorname{div} \mathbf{u} &= 0, \\ \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0, \\ \partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2 \operatorname{div}(\mu(\varrho) D(\mathbf{u})) + \nabla \Pi_2 &= 0 \end{aligned}$$

where $D(\mathbf{u}) = (\nabla \mathbf{u} + {}^t\nabla \mathbf{u})/2$.
1974 –> A. Kazhikhov [1947–2005]:

$$0 < C \le \rho_0 \le C^{-1} < +\infty, \qquad \mu(s) = \mu = \mathrm{cte}$$

1990 –> J. Simon [1947–..]: ρ_0 may vanish with $\mu(s) = \mu = \mathrm{cte}$
1993 –> E. Fernández-Cara [1957–..], F. Guillén:
ρ_0 may vanish with $\mu(s) \ge C > 0$
1998 –> P.–L. Lions [1956–..]: see the full details.

See for instance: E. Fernández-Cara,
Discrete & Continuous Dynamical Systems - Series S (2012), 1021-1090

State of art: compressible flows - global weak solutions

What was known until 2015 on global weak solutions with constant viscosities?

Monotonicity assumption on the pressure law

Barotropic case:

$$[CNS] \quad \begin{aligned} &\partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \\ &\partial_t (\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta u - (\lambda + \mu) \nabla \mathrm{div} u + \nabla P(\varrho) = \mathbf{0}, \end{aligned}$$

with a given law $s \mapsto P(s)$, $\mu > 0$ and $\lambda + 2\mu/d > 0$. We assume $\Omega = \Pi^d$ (periodic boundary conditions) and $\rho|_{t=0} = \rho_0$, $\rho u|_{t=0} = m_0$.

The case $P(s) = a\, s^\gamma$ with $a > 0$:

- P.–L. Lions (1993–1998): $\gamma \geq 3d/(d+2)$
- E. Feireisl (2001) with co-authors: $\gamma > d/2$
- Note the recent work: P. Plotnikov-W. Weigant (2015): $d = 2$ and $\gamma = 1$

Some important non-monotone cases

- E. Feireisl (2002)
- B. Ducomet, E. Feireisl, H. Petzeltova, I. Straskarba (2004)

Hypothesis on P with $P'(\rho) \geq C^{-1} \varrho^{\gamma-1} - C$ for all $\varrho \in [0, +\infty)$.

What should the viscous term be?

- Anisotropic viscosities : See work D.B., P.–E. Jabin.
- Density dependent viscosities : See work D. B., B. Desjardins.

What should the pressure law be?

- Thermodynamically the stability of the equilibrium is directly connected to the monotonicity of p
- Monotone laws are also required for hyperbolicity
- However, many physical models have non-monotone pressure
- Its is not clear why a thermodynamical assumption should control the stability of solutions over bounded times
- Non monotone pressure laws: See work D.B., P.–E. Jabin.

Let us explain the steps in the previous proofs with monotone pressure:
Non-trivial extension to the multi-dimensional in space case
of previous ideas introduced in the one-dimensional in space case by:
1986 –> D. Serre [1954–..], 1987 –> D. Hoff [??–...].

Case $P(\varrho) = a\varrho^\gamma$ (Estimates):

Energy estimates:

$$\sup_{t\in[0,T]} \Big(\frac{1}{2}\int_{\Pi^d} \varrho|u|^2 + \frac{1}{\gamma-1}\int_{\Pi^d} \varrho^\gamma\Big) + \mu\int_0^T\int_{\Pi^d} |\nabla u|^2 + (\lambda+\mu)\int_0^T\int_{\Pi^d} |\mathrm{div} u|^2$$

$$\le \frac{1}{2}\int_{\Pi^d} \frac{|m_0|^2}{\varrho_0} + \frac{1}{\gamma-1}\int_{\Pi^d} \varrho_0^\gamma$$

Extra integrability on the density (Bogovskii operator):

$$\int_{\Pi^d} \varrho^p = \int_{\Pi^d} \varrho^{\gamma+\theta} \le C < +\infty$$

with

$$\theta \le 2\gamma/d - 1.$$

This estimate is obtained testing the momentum equation by a test function satisfying $\mathrm{div}\varphi = \rho^\theta - \overline{\rho^\theta}$ where $\bar{\cdot}$ is the mean value. Using the energy estimate to prove that the other quantities are controlled ($\Longrightarrow$ constraint on θ).

Remark. We have ϱ square integrable namely $p \ge 2$ if $\gamma \ge 3d/(d+2)$ (P.–L. Lions constraint)

To prove global existence of weak solutions:

1) Stability: Assume there exists a sequence satisfying the energy estimates uniformly and the equations in a weak sense. Is it possible to extract a subsequence converging in some sense to a weak solution of the system and satisfying the energy inequality?

2) Construction of approximate solutions: regularization, fixed point, Galerkin method etc...

We focus on stability in this talk!

Compactness to pass to the limit in ϱu and $\varrho u \otimes u$ mostly relies on

- compactness (negative sobolev space) on $\varrho_k u_k$: Aubin-Lions-Simon Lemma
- convergence in norm to have compactness on $\sqrt{\varrho_k} u_k$ in $L^2((0,T) \times \Pi^d)$

Quite similar to non-homogeneous incompressible Navier-Stokes equations.

The main difficulty in the proof: passage to the limit in ϱ_k^γ in weak formulation

How to get compactness on ϱ in Lebesgue spaces?

The main step where the monotonicity is required (case $\gamma \geq 3d/(d+2)$)
We use the fact that $\rho \ln \rho$ satisfies (renormalization technic due to Di-Perna and Lions) the equation

$$\partial_t(\varrho \ln \varrho) + \mathrm{div}(\varrho \ln \varrho u) + \varrho \mathrm{div} u = 0.$$

noticing that

$$s \mapsto s \ln s$$

is a strictly convex function and

$$s \mapsto p(s)$$

is an increasing function.

Goal: show that

$$\overline{\varrho \ln \varrho} = \varrho \ln \varrho$$

$\Longrightarrow$ commutation between stricly convex function and weak limit
$\Longrightarrow$ compactness in L^1.

Renormalization of limit:

$$\partial_t(\varrho \ln \varrho) + \mathrm{div}(\varrho \ln \varrho u) + \varrho \mathrm{div} u = 0.$$

Limit of Renormalization (denoting $\bar{\cdot}$ the weak limit):

$$\partial_t(\overline{\varrho \ln \varrho}) + \mathrm{div}(\overline{\varrho \ln \varrho u}) + \overline{\varrho \mathrm{div} u} = 0.$$

This uses the property (effective flux property): weak compactness

$$\overline{\rho\, \mathrm{div} u} - \frac{\overline{P(\rho)\rho}}{\lambda + 2\mu} = \rho\, \mathrm{div} u - \frac{\overline{P(\rho)}\rho}{\lambda + 2\mu}$$

which gives

$$\overline{\rho \mathrm{div} u} - \rho \mathrm{div} u = \frac{\overline{P(\rho)\rho} - \overline{P(\rho)}\rho}{\lambda + 2\mu} \qquad \Longrightarrow \text{ appropriate sign due to monotonicity}$$

$\Longrightarrow$ If no defect measure initially then compactness

For more general γ, use a clever troncature procedure: see E. Feireisl.

If defect measures present initially (multi-fluid systems):
See D. Serre, A.A. Amosov and A.A. Zlotnik , E. Feireisl, D.B. and M. Hillairet.

How to get the effective flux property?

To understand let us consider a simplified momentum equation

$$-\mu\Delta u_n - (\lambda+\mu)\nabla \mathrm{div}\, u_n + \nabla P(\rho_n) = S_n$$

that means suppress the total derivative in the compressible Navier-Stokes momentum equation. Taking the divergence of the momentum equation we get

$$\Delta((\lambda+2\mu)\mathrm{div}\, u_n + P(\rho_n)) = \text{ div } S_n$$

and therefore

$$(\lambda+2\mu)\mathrm{div}\, u_n + P(\rho_n)) = \Delta^{-1} \text{ div } S_n \tag{1}$$

Passing to the limit and multiplying by ρ, we get

$$(\lambda+2\mu)\rho\mathrm{div}\, u + \overline{P(\rho)}\rho = (\Delta^{-1} \text{ div } S)\rho.$$

Multiplying Equation (1) by ρ_n and then passing to the limit, we get

$$(\lambda+2\mu)\overline{\rho\mathrm{div}\, u} + \overline{P(\rho)\rho} = (\Delta^{-1} \text{ div } S)\rho.$$

Substracting the two last equations provides the conclusion.

In the anisotropic case

$$-\mu_x \Delta_x u - \mu_z \partial_z^2 u - \lambda \nabla \mathrm{div}\, u \text{ with } \mu_x \neq \mu_z \text{ const}$$

Then

$$\overline{\varrho \mathrm{div}\, u} - \varrho \mathrm{div}\, u = \frac{\overline{\varrho}\,\overline{A_\mu \varrho^\gamma} - \overline{\varrho A_\mu \varrho^\gamma}}{\mu_x + \lambda}$$

where $A_\mu = a_\mu(\Delta - (\mu_x - \mu_z)\partial_z^2)^{-1}\partial_z^2$ with $a_\mu = (\mu_x - \mu_z)$.

No *a priori* sign on the right-hand side: Non-local effects.

$\Longrightarrow$ difficulty: Possible mixing phenomena (small/large value of density)

See discussions in D.B., B. Desjardins, D. Gérard-Varet (2004).

Remark: Density dependent viscosities:
An other story D.B., B. Desjardins (2004): BD entropy is the starting point.
–> The viscosities vanishes when ρ vanishes.
–> Not covered actually for non-homogeneous incompressible NS eqs.

The first compressible Navier-Stokes system under consideration (Non-monotone pressure)

D.B., P.-E. Jabin: arXiv:1507.04629 Submitted (2015).

Consider the following barotropic system in periodic box:

$$[CNS] \quad \begin{aligned} \partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \\ \partial_t (\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta u - (\lambda + \mu) \nabla \mathrm{div} u + \nabla P(\varrho) = \mathbf{0}, \end{aligned}$$

with the pressure P locally Lipschitz on $[0, +\infty)$, with $P(0) = 0$ and

$$C^{-1} \varrho^\gamma - C \leq P(\varrho) \leq C \varrho^\gamma + C$$

and for all $s \geq 0$, we only assume

$$|P'(s)| \leq s^{\widetilde{\gamma} - 1}$$

for some $\widetilde{\gamma} > 1$.

Mathematical result

Theorem. Let (ϱ_0, u_0) such that

$$E(\varrho_0, u_0) = \int_{\Pi^d} \frac{|m^0|^2}{2\varrho_0} + \varrho_0 e(\varrho_0) < +\infty$$

with $e(s) = \int_0^s P(\tau)/\tau^2 d\tau$. Let P satisfying the previous hypothesis with

$$\gamma > (\max(2, \tilde{\gamma}) + 1)\, d/(d+2)$$

then there exists a global weak solution to the compressible Barotropic Navier-Stokes equations (CNS).

Remark:

- If $\tilde{\gamma} = \gamma$ then $\gamma > 3d/(d+2)$.
- Truncated procedure as introduced by E. Feireisl could give $\gamma > d/2$.
- Work with pressure $P(\rho, t, x)$ with appropriate (t, x) dependency
  Importance for heat-conducting NS els
  Importance of such pressure: biology, solar events........

The second compressible Navier-Stokes system under consideration (Anisotropic viscosity)

D.B., P.-E. Jabin: arXiv:1507.04629 Submitted (2015).

Consider the following barotropic system in periodic box:

$$[ACNS] \quad \begin{array}{c} \partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \\ \partial_t (\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \mathrm{div}(A(t)\nabla u) - (\lambda + \mu)\nabla \mathrm{div} u + \nabla P(\varrho) = \mathbf{0}, \end{array}$$

with the pressure P locally Lipschitz on $[0, +\infty)$, with $P(0) = 0$ and

$$C^{-1}\rho^{\gamma-1} - C \le P'(\rho) \le C\rho^{\gamma-1} + C$$

and a $d \times d$ matrix $A = \mu \mathrm{Id} + \delta A(t)$ with time dependent smooth coefficient.

Remarks:

- Case usually encountered in geophysics: $-\nu_x \Delta_x u - \nu_z \partial_z^2 u$ (see Handbook R. Temam and M. Ziane).
- We can consider: $-\mathrm{div}(A(t)D(u)) + \lambda \nabla \mathrm{div} u$.
- Incompressible flows - weak sol.: anisotropy no problem if not degenerate.
- Compressible feature: Possible "density mixing"due to non-local operator.

Mathematical result

Theorem. Let (ϱ_0, u_0) such that

$$E(\varrho_0, u_0) = \int_{\Pi^d} \frac{|m^0|^2}{2\varrho_0} + \varrho_0 e(\varrho_0) < +\infty$$

with $e(s) = \int_0^s P(\tau)/\tau^2 d\tau$. Let P satisfying the monotonicity assumption and assume that

$$\gamma > \frac{d}{2}\Big[\big(1 + \frac{1}{d}\big) + \sqrt{1 + \frac{1}{d^2}}\Big].$$

There exists a universal constant $C_\star > 0$ such that if

$$\|\delta A\|_\infty \leq C_\star (2\mu + \lambda).$$

then there exists a global weak solution to the compressible Barotropic Navier-Stokes equations (CNS).

Remark. Seems a straightforward perturbation result......
BUT it is trickier than the non-monotone pressure case due to non-local terms!!

Compactness on the density: An idea

Propagate some explicit regularity on ρ by computing

$$\int \frac{|\rho(t,x)-\rho(t,y)|}{(|x-y|+h)^k}\,dx\,dy$$

for some $k \geq d$.

However this corresponds to a Sobolev like regularity on ρ which cannot work.

The problem:

Weak solutions:
No Sobolev regularity propagation on ϱ for compressible Navier-Stokes Eqs.

The frame:

- Weak regularity on the velocity field
- Vacuum state for the density.

The new idea:

- Introduce some appropriate weights w_k in the quantity to be controlled
  Precise the rate of convergence in terms of h.
- Derive appropriate properties on the weights
  Go back to the definition without weights without too much lost in h.

The new idea

Propagate some explicit regularity on ρ by computing

$$\int \frac{|\rho(t,x)-\rho(t,y)|}{(|x-y|+h)^k}(w(t,x)+w(t,y))dx\,dy$$

for some $k \geq d$ where the weight w solve the same transport equation

$$\partial_t w(t,x) + u(t,x)\cdot\nabla_x w(t,x) = -\lambda D^x w(t,x)$$

for a well chosen penalization D^x and appropriate parameter λ (idem for $w(t,y)$). Then explain that $w(t,x)$ (and $w(t,y)$) cannot be too small, too often to bound

$$\int \frac{|\rho(t,x)-\rho(t,y)|}{(|x-y|+h)^k}dx\,dy$$

in terms of h.

A compactness Lemma

Let ϱ_k bounded in $L^p((0,T)\times\Pi^d)$ (with $1\le p<+\infty$) and

$$\partial_t\varrho_k\in L^q(0,T;W^{-1,q}(\Pi^d))$$

with $q>1$. Let K_h positive, bounded functions, compactly suppported in Π^d s.t.

$$\forall\eta>0 \text{ small },\qquad \sup_h\int_{\Pi^d}1_{|x|\ge\eta}K_h(x)\,dx<+\infty$$

and

$$\|K_h\|_{L^1(\Pi^d)}\to+\infty \text{ when } h\to+0$$

If

$$\limsup_k\sup_{t\in[0,T]}\Big[\frac{1}{\|K_h\|_{L^1}}\int_{\Pi^d}K_h(x-y)|\varrho_k(t,x)-\varrho_k(t,y)|^p\,dxdy\Big]\to 0,\qquad \text{as } h\to 0$$

Then ϱ_k compact in $L^p((0,T)\times\Pi^d)$.

A compactness Lemma

Some references:

- J. Bourgain, H. Brézis, P. Mironescu: Functional spaces (2001)
- A.C. Ponce: Functional spaces (2004)
- F. Ben Belgacem, P.–E. Jabin: Nonlinear continuity equations (2013)

Remark: Let us denote

$$\overline{K}_h(x) = \frac{K_h(x)}{\|K_h\|_{L^1}}.$$

For $0 < h_0 < 1$, then

$$\mathcal{K}_{h_0}(x) = \int_{h_0}^{1} \overline{K}_h(x) \frac{dh}{h}$$

where

$$K_h(x) = \frac{1}{(h + |x|)^a} \text{ for } |x| \le 1/2$$

is eligible for the compactness lemma. Remark that

$$\|\mathcal{K}_{h_0}\|_{L^1} \approx |\log h_0|.$$

How it works on a more simple case?

See D.B., P.-E. Jabin: Guy Métivier's Birthday - Springer-INdAM-Series (2017).
Let us consider the following system

$$[CS] \quad \begin{aligned} \partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \\ -\mu \Delta \mathbf{u} + \alpha \mathbf{u} + \nabla P(\varrho) = S \end{aligned}$$

where $\mu, \alpha > 0$ with a given pressure law $s \mapsto P(s)$:
System encountered in biology, petroleum ingeneering for instance.

We assume the pressure P locally Lipschitz on $[0, +\infty)$, with $P(0) = 0$ and

$$C^{-1}\varrho^\gamma - C \le P(\varrho) \le C\varrho^\gamma + C$$

and for all $s \geq 0$, we only assume

$$|P'(s)| \le s^{\gamma-1}$$

with $\gamma > 1$.

Result: Global existence of weak solutions !!

Energy bounds:

$$\rho_k^\gamma \in L^\infty(0, T; L^1(\mathbb{T}^d)), \qquad u_k \in L^2(0, T; H^1(\mathbb{T}^d))$$

Extra integrability bounds

$$\rho_k \in L^p((0, T) \times \mathbb{T}^d) \text{ with } p > 2 \text{ because} \gamma > 1.$$

We write g^x for $g(t,x)$ everywhere (idem g^y). One has

$$\partial_t|\rho_k^x-\rho_k^y|+\mathrm{div}_x\left(u_k^x\,|\rho_k^x-\rho_k^y|\right)+\mathrm{div}_y\left(u_k^y\,|\rho_k^x-\rho_k^y|\right)$$
$$=\frac{1}{2}(\mathrm{div}_x u_k^x+\mathrm{div}_y u_k^y)\,|\rho_k^x-\rho_k^y|-\frac{1}{2}(\mathrm{div}_x u_k^x-\mathrm{div}_y u_k^y)\,(\rho_k^x+\rho_k^y)\,s_k,$$

where $s_k = sign\,(\rho_k^x-\rho_k^y)$ and therefore

$$\begin{aligned}\frac{d}{dt}R(t) = \quad & \int_{\mathbb{T}^{2d}}\nabla K_h(x-y)\cdot(u_k^x-u_k^y)\,|\rho_k^x-\rho_k^y|\,(w^x+\,w^y) \qquad (2)\\ & -\int_{\mathbb{T}^{2d}}K_h(x-y)\,(\mathrm{div}\,u_k^x-\mathrm{div}\,u_k^y)\,(\rho_k^x+\rho_k^y+(\rho_k^x-\rho_k^y)\,)\,s_k\,w^x\\ & +2\int_{\mathbb{T}^{2d}}K_h(x-y)\,|\rho_k^x-\rho_k^y|\,\left(\partial_t w_k^x+u_k^x\cdot\nabla_x w^x+\mathrm{div}_x u_k^x\,w_k^x\right)\\ = \quad & A_1+A_2+A_3\end{aligned}$$

where

$$R(t)=\int_{\mathbb{T}^{2d}}K_h(x-y)|\rho_k^x-\rho_k^y|(w^x+w^y)\,dxdy.$$

Note that

$$A_1 = \int_{\mathbb{T}^{2d}} \nabla K_h(x-y)\cdot(u_k^x - u_k^y)\,|\rho_k^y - \rho_k^y|\,(w_k^x + w_k^y) \quad (3)$$

$$\leq C \int_{\mathbb{T}^{2d}} K_h(x-y)\,(D_{|x-y|}\,u_k^x + D_{|x-y|}\,u_k^y)\,|\rho_k^x - \rho_k^y| w_k^x,$$

where we have used here the inequality

$$|u(x) - u(y)| \leq C\,|x-y|\,(D_{|x-y|}\,u_k^x + D_{|x-y|}\,u_k^y),$$

with the square function

$$D_h u_k^x = \frac{1}{h}\int_{|z|\leq h} \frac{|\nabla u_k^{x+z}|}{|z|^{d-1}}\,dz.$$

$$\int_{h_0}^1 \int_0^t \frac{A_1}{\|K_h\|_L^1} \frac{dh}{h} \qquad \le C \int_{h_0}^1 \int_0^t \int_{\mathbb{T}^d} \overline{K_h}(z) \|D_{|z|} u_k(\cdot) - D_{|z|} u_k(\cdot + z)\|_{L^2} \frac{dh}{h}$$

$$+C \int_0^t \int_{\mathbb{T}^{2d}} \mathcal{K}_{h_0}(x-y) M|\nabla u_k^x| \, |\rho_k^x - \rho_k^y| w_k^x$$

where Mf is the maximal function defined as follows

$$Mf(x) = \sup_{r \le 1} \frac{1}{|B(0,r)|} \int_{B(0,r)} f(x+z)\, dz.$$

Use (translation property of the square function) that

$$\int_{h_0}^1 \int_0^t \int_{\mathbb{T}^d} \overline{K_h}(z) \|D_{|z|} u_k(\cdot) - D_{|z|} u_k(\cdot + z)\|_{L^2} \frac{dh}{h} \le C\, |\log h_0|^{1/2} \int_0^t \|u(\tau, .)\|_{H_x^1} \, d\tau$$

Splitting in different parts the integral, we prove that

$$A_2 \leq C \int_{\mathbb{T}^{2d}} K_h(x-y)\,(1+(\rho_k^x)^\gamma))\,|\rho_k^x - \rho_k^y|\, w_k^x.$$

and we have

$$\begin{aligned} A_3 = &\int_{\mathbb{T}^{2d}} K_h(x-y)\,|\rho_k^x - \rho_k^y|\,\left(\partial_t w_k^x + u_k^x \cdot \nabla_x w^x + \mathrm{div}_x u_k^x\, w_k^x\right) \\ &\leq \int_{\mathbb{T}^{2d}} K_h(x-y)\,|\rho_k^x - \rho_k^y|\,\left(-\lambda D_k^x + \mathrm{div}_x u_k^x\right) w_k^x. \end{aligned}$$

Therefore we get

$$\limsup_k \Big[\frac{1}{|\log h_0|}\int_{\mathbb{T}^{2d}} \mathcal{K}_{h_0}(x-y)\,|\rho_k^x-\rho_k^y|\,(w_k^x+w_k^y)dx\,dy\Big] \to 0 \text{ as } h_0\to 0$$

where

$$\partial_t \log w_k + u_k\cdot\nabla \log w_k + \lambda D_k = 0, \qquad w_k|_{t=0} = 1$$

with

$$D_k = M|\nabla u_k| + |\mathrm{div}\, u_k| + (\rho_k)^\gamma.$$

Remark. If transport equation considered with compactness properties on $\operatorname{div} u_k$ then in many respect: Equivalent of the method of G. Crippa and C. De Lellis at the PDE level instead of ODE level: No weight needed.

See paper by F. Ben Belgacem and P.–E. Jabin:
Nice results on non-linear continuity Eq using the compactness Lemma.

We now have to control the weights so as to remove them. Namely we want to prove that

$$\limsup_k \Big[\frac{1}{|\log h_0|}\int_{\mathbb{T}^{2d}} \mathcal{K}_{h_0}(x-y)\,|\rho_k^x-\rho_k^y|\,dx\,dy\Big] \to 0 \text{ as } h_0 \to 0$$

and not only

$$\limsup_k \Big[\frac{1}{|\log h_0|}\int_{\mathbb{T}^{2d}} \mathcal{K}_{h_0}(x-y)\,|\rho_k^x-\rho_k^y|\,(w_k^x+w_k^y)dx\,dy\Big] \to 0 \text{ as } h_0 \to 0.$$

We have $0 \leq w_k \leq 1$ and

$$\frac{d}{dt} \int_{\mathbb{T}^d} \rho_k |\log w_k| \leq \lambda \int_{\mathbb{T}^d} \rho_k D_k < +\infty$$

then (let t be fixed) split the integral into two parts using $\omega_\eta = \{x : w_k \leq \eta\}$:

$$\{x \in \omega_\eta^c \text{ or } y \in \omega_\eta^c\}$$

and

$$\{x \in \omega_\eta \text{ and } y \in \omega_\eta\}$$

with η chosen in terms of h_0.

Show that we can get rid the weight without loosing to much decay in h_0 to continue to ensure the convergence to 0 as h_0 go to zero.

One get the conclusion: Stability of sequence of global weak solutions.

Construction of approximate solutions:

$$\begin{cases} \partial_t \rho_k + \mathrm{div}(\rho_k u_k) = 0, \\ -\mu\Delta u_k - (\lambda+\mu)\nabla \mathrm{div}\, u_k + \nabla P_\epsilon(\rho_k) = S, \end{cases} \tag{4}$$

with the fixed source term S and the fixed initial data

$$\rho_k|_{t=0} = \rho^0. \tag{5}$$

The pressure P_ε is defined as follows:

$$P_\varepsilon(\rho) = p(\rho) \text{ if } \rho \le c_{0,\varepsilon}, \qquad P_\varepsilon(\rho) = p(C_{0,\varepsilon}) + C(\rho - c_{0,\varepsilon})^\beta \text{ if } \rho \ge c_{0,\varepsilon},$$

with large enough β.

Global existence of weak solutions for fixed ε: See E. Feireisl (2002).

Global existence of weak solutions of the Stokes problem:
Let ε go to zero using the stability process to get the result

Merci pour votre attention !!