Fractional diffusion as macroscopic limit of kinetic models

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A modified moment method for diffusive limits [Mellet, 2010]

$$\varepsilon^2 \partial_t f + \varepsilon v \cdot \nabla_x f = \rho_f M - f$$

with $x, v \in \mathbb{R}^d$, $M(v)$ even, and $\rho_f(x,t) = \int f(x,v,t)dv$.

Modified test function: Let $\chi(x,v,t)$ satisfy

$$\chi - \varepsilon v \cdot \nabla_x \chi = \phi(x,t)$$

This implies

$$\chi = \phi + \varepsilon v \cdot \nabla_x \phi + \varepsilon^2 v^{tr} \nabla_x^2 \phi \, v + O(\varepsilon^3)$$

and

$$\varepsilon^2 \int \chi \partial_t f \, d(x,v,t) = \int \rho_f M(\chi - \phi) d(x,v,t) \approx \varepsilon^2 \int \rho_f D \Delta_x \phi \, d(x,t)$$

with $D = 1/d \int |v|^2 M \, dv$.

Mellet's moment method, ctd.

Now divide by ε^2 and pass to the limit:

$$\partial_t \rho = D \Delta_x \rho$$

All you need to make this rigorous is a uniform bound for f (e.g. from an entropy estimate)

... and bounded second order moments of M.

What if this is not the case?

Then the macroscopic limit might give fractional diffusion:
[Jara, Komorowski, Olla, 2009], [Mellet, 2010],
[Mellet, Mischler, Mouhot, 2011]

Derivation of fractional diffusion 1: transform method

$$\varepsilon^\alpha \partial_t f + \varepsilon v \cdot \nabla_x f = \rho_f M - f \,, \qquad f(t=0) = f_0 \,,$$

with $M(v) \sim |v|^{-d-\alpha}$ as $|v| \to \infty$, $0 < \alpha < 2$.

Fourier-Laplace transform $\hat{f}(\xi, v, \tau) = \mathcal{L}_t \mathcal{F}_x f$, integration with respect to v:

$$\hat{\rho}_f \int M \frac{\tau + \varepsilon^\alpha \tau^2 + \varepsilon^{2-\alpha}(v\cdot\xi)^2}{(1+\varepsilon^\alpha\tau)^2 + \varepsilon^2 (v\cdot\xi)^2} dv = \int \frac{\mathcal{F}_x f_0}{1 + \varepsilon^\alpha \tau + \varepsilon i v \cdot \xi} dv$$

Limit $\varepsilon \to 0$:

$$\hat{\rho}(\tau + A|\xi|^\alpha) = \mathcal{F}_x \rho_0$$

[Mellet, Mischler, Mouhot, 2011]

Derivation of fractional diffusion 2: moment method

$$\int \chi \partial_t f \, d(x,v,t) = \int \rho_f \int \varepsilon^{-\alpha} M(\chi - \phi) dv \, d(x,t)$$

with

$$\chi(x,v,t) = \int_0^\infty e^{-s} \phi(x + \varepsilon s v, t) ds$$

This leads to the integral representation of the fractional Laplacian:

$$\begin{aligned} \lim_{\varepsilon \to 0} \varepsilon^{-\alpha} M(\chi - \phi) dv &= c_{d,\alpha} P.V. \int \frac{\phi(x+w,t) - \phi(x,t)}{|w|^{d+\alpha}} dw \\ &=: \mathcal{L}_\alpha \phi \end{aligned}$$

[Mellet, 2010]

Derivation of fractional diffusion 3: Claude's method

Invert transport+loss term:

$$f(x,v,t) \approx M(v)\int_0^\infty e^{-s}\rho_f(x-\varepsilon vs,t)ds$$

Use this for computing the distributional formulation of the flux

$$\varepsilon^{1-\alpha}\int f\, v\cdot\nabla\phi\, dv\, dx\, dt = \int \rho_f \mathcal{L}_{\alpha,\varepsilon}\phi\, dx\, dt$$

with

$$\mathcal{L}_{\alpha,\varepsilon}\phi \to \mathcal{L}_\alpha\phi = c_{\alpha,d}\nabla_x\cdot\int\frac{\nabla_y\phi(y)}{|x-y|^{d+\alpha-2}}dy$$

[Bardos, private communication, last week]

More general models

- Decaying collision frequencies instead of fat-tailed equilibrium distributions:
  [Mellet, Mischler, Mouhot, 2011]
- Convection terms: formal asymptotics in [Mellet, 2010], but lack of uniform estimates

Derivation of fractional diffusion with drift

Weakly biased collision operator:

$$Q_\varepsilon(f) = \int [T_\varepsilon(v' \to v, x, t) f' - T_\varepsilon(v \to v', x, t) f] dv'$$

where

$$T_\varepsilon(v' \to v, x, t) = M(v)(1 + \varepsilon^{\alpha-1}\Phi(v, v', c(x,t))$$

with a given vector field c.

Theorem: The problem

$$Q_\varepsilon(F_\varepsilon) = 0\,, \qquad \int F_\varepsilon dv = 1$$

has a unique solution satisfying $F_\varepsilon = M(1 + O(\varepsilon^{\alpha-1}))$.

Derivation of fractional diffusion with drift

Entropy inequality:

$$\frac{\varepsilon^\alpha}{2}\frac{d}{dt}\|f\|^2 \le \varepsilon^\alpha \lambda \|f\|^2 - \nu\|f - \rho_f F_\varepsilon\|^2$$

with $\|\cdot\|$ the norm in $L^2(dx\,dv/F_\varepsilon)$, coercivity constant ν of Q_ε, and bound λ for x- and t-derivatives of c.

Consequence: Uniform bound for f and convergence to $\rho(x,t)M(v)$.

Moment method or, for constant c, transform method: limit equation

$$\partial_t \rho + \nabla_x \cdot (\rho u(c)) = \mathcal{L}_\alpha \rho$$

[Aceves Sanchez, CS, 2016]

Extension 1: drift from acceleration

$$\varepsilon^\alpha \partial_t f + \varepsilon v \cdot \nabla_x f + \varepsilon^{\alpha-1} E(x,t) \cdot \nabla_v f = Q(f)$$

with

$$Q(f) = \int [\sigma(v,v')Mf' - \sigma(v',v)M'f]dv'$$

Results for $\alpha \geq 1$ (for $\alpha = 1$ drift and diffusion both 1st order).

Entropy inequality for

$$Q_\varepsilon(f) = Q(f) - \varepsilon^{\alpha-1} E \cdot \nabla_v f$$

[Aceves Sanchez, Mellet, 2016]

Extension 2: fractional Fokker-Planck + acceleration

$$\varepsilon^\alpha \partial_t f + \varepsilon v \cdot \nabla_x f + \varepsilon^{\alpha-1} E(x,t) \cdot \nabla_v f = Q_\alpha(f)$$

with

$$Q_\alpha(f) = \nabla_v(vf) + \mathcal{L}_{\alpha,v} f$$

Results and approach like above.

[Aceves Sanchez, Cesbron, 2016]

Fractional diffusion on bounded domains

$$\varepsilon^\alpha \partial_t f + \varepsilon v \cdot \nabla_x f = \rho_f M - f\,, \qquad f(t=0) = f_0\,,$$

for $v \in \mathbb{R}^d$, $x \in \Omega \subset \mathbb{R}^d$. Ω bounded, smooth.

Homogeneous inflow boundary conditions:

$$f(x,v,t) = 0\,, \qquad x \in \partial\Omega\,,\ v \cdot \nu(x) < 0\,.$$

Modified test function:

$$\chi - \varepsilon v \cdot \nabla_x \chi = \phi(x,t)\,,$$

$$\chi(x,v,t) = 0\,, \qquad x \in \partial\Omega\,,\ v \cdot \nu(x) > 0\,.$$

Fractional diffusion on bounded domains

Limiting equation:

$$\partial_t \rho = \mathcal{L}_\alpha(\rho) - h_\alpha(x)\rho$$

with

$$\mathcal{L}_\alpha(\rho) = c_{d,\alpha} P.V. \int_{S_\Omega(x)} \frac{\rho(y) - \rho(x)}{|x-y|^{d+\alpha}} dy$$

where $S_\Omega(x)$ is the largest star shaped domain with center in x contained in Ω, and

$$h_\alpha(x) = O(\mathrm{dist}(x, \partial\Omega)^{-\alpha})$$

[Aceves Sanchez, CS, 2016]

Specular reflection on spherical domains: [Cesbron, 2016]

To do list

- Nonlinear problems, e.g. self consistent acceleration field
- Other boundary conditions, e.g. inhomogeneous inflow data