Ground states of a two phase model with cross and self attractive interactions

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Description of the problem I

Two phases, represented by E_1, $E_2 \subset \mathbb{R}^N$, with $E_1 \cap E_2 = \emptyset$ and (fixed) masses m_1 and m_2 respectively, interact trying to minimize an energy of the form

$$\mathcal{F}_K^{c_{ij}}(E_1, E_2) := c_{11}\, J_K(E_1, E_1) + c_{22}\, J_K(E_2, E_2) + c_{12}\, J_K(E_1, E_2),$$

where $c_{ij} \leq 0$, $K \in L^1_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R})$ is a non-increasing radially symmetric potential, and

$$J_K(E_i, E_j) = J_K(\chi_{E_i}, \chi_{E_j}) := \int_{\mathbb{R}^N \times \mathbb{R}^N} \chi_{E_i}(x)\, \chi_{E_j}(y)\, K(x - y)\, \mathrm{d}x\, \mathrm{d}y.$$

Problem: For any fixed $m_1, m_2 > 0$,

$$\min_{\substack{|E_i| = m_i \\ E_1 \cap E_2 = \emptyset}} \mathcal{F}_K^{c_{ij}}(E_1, E_2). \tag{P}$$

Goals:

- existence of minimizers;
- qualitative properties of minimizers;
- characterization of the shape of minimizers (K Coulomb).

Trivial case. $c_{12} = 0 \rightsquigarrow \min_{|E_i| = m_i} c_{ii}\, J_K(E_i, E_i)$, for $i = 1, 2$.
If $c_{ii} < 0$, the solution is $E_i = B^{m_i}$ (Riesz Lemma).

Riesz Lemma

Let $f, g \in L^1(\mathbb{R}^N; [0,1])$ with $\|f\|_{L^1}, \|g\|_{L^1} > 0$. Then,

$$J_K(f,g) \overset{(1)}{\leq} J_K(f^*, g^*) \overset{(2)}{\leq} J_K(\chi_{B^f}, \chi_{B^g}) \equiv J_K(B^f, B^g),$$

where

(1) holds with "$=$" if and only if $f(\cdot) = f^*(\cdot - x_0)$ and $g(\cdot) = g^*(\cdot - x_0)$ for some $x_0 \in \mathbb{R}^N$,

and

(2) holds with "$=$" if and only if $f^* = \chi_{B^f}$ and $g^* = \chi_{B^g}$.

Description of the problem II

Wlog we assume that $c_{12} = -2$, and set

$$\mathcal{F}_K^{c_1,c_2}(E_1, E_2) := c_1\, J_K(E_1, E_1) + c_2\, J_K(E_2, E_2) - 2\, J_K(E_1, E_2),$$

the problem is

$$\min_{\substack{|E_i|=m_i \\ E_1 \cap E_2 = \emptyset}} \mathcal{F}_K^{c_1,c_2}(E_1, E_2), \tag{P}$$

Example: Let $c_1 = c_2 = 0$, $m_1 = m_2 =: m$ and let $K \geq 0$.

Then, $J_K(f,f)$ is a strictly convex functional (depending on f), and hence

$$J_K(f_1, f_2) = 2\, J_K\left(\tfrac{f_1+f_2}{2}, \tfrac{f_1+f_2}{2}\right) - \tfrac{J_K(f_1,f_1)}{2} - \tfrac{J_K(f_2,f_2)}{2} \leq J_K\left(\tfrac{f_1+f_2}{2}, \tfrac{f_1+f_2}{2}\right). \quad \text{(StrConv)}$$

It follows that

$$\mathcal{E}_K^{0,0}(f_1, f_2) = -2\, J_K(f_1, f_2) \geq -2\, J_K\left(\tfrac{f_1+f_2}{2}, \tfrac{f_1+f_2}{2}\right).$$

The rhs is minimized for $f_1 + f_2 = \chi_{B^{2m}}$ (Riesz Lemma), but

$$\text{(StrConv)} \rightsquigarrow f_1 = f_2 = \frac{1}{2}\chi_{B^{2m}} \quad \text{if } (f_1, f_2) \text{ minimizes } \mathcal{E}_K^{0,0}. \qquad \square$$

(P) has (in general) no minimizer in the class of characteristic functions

⇝ Relaxation

$$\min_{(f_1, f_2) \in \mathcal{A}_{m_1, m_2}} \mathcal{E}_K^{c_1, c_2}(f_1, f_2), \tag{Prel}$$

where

$$\mathcal{E}_K^{c_1, c_2}(f_1, f_2) := c_1 \, J_K(f_1, f_1) + c_2 \, J_K(f_2, f_2) - 2 \, J_K(f_1, f_2),$$

and

$$\mathcal{A}_{m_1, m_2} := \{(f_1, f_2) \, : \, f_i \in L^1(\mathbb{R}^N; \mathbb{R}^+), \int_{\mathbb{R}^N} f_i \, \mathrm{d}x = m_i, \; f_1 + f_2 \le 1\}.$$

Existence of minimizers for (Prel)

Theorem

(Prel) is well-posed for any $c_1, c_2 \leq 0$ and for any $m_1, m_2 > 0$.
More precisely, if $\{(f_{1,n}, f_{2,n})\}_n$ is a minimizing sequence, then

$$\exists\, \{\tau_n\}_n \subset \mathbb{R}^N,\ \exists\,(f_1, f_2) \in \mathcal{A}_{m_1,m_2} \ \text{s.t.}\ f_{i,n}(\cdot - \tau_n) \stackrel{tight}{\longrightarrow} f_i \ \text{(up to s.s.)} \tag{1}$$

and (f_1, f_2) is a minimizer of $\mathcal{E}_K^{c_1,c_2}$ in $\mathcal{A}_{m_1,m_2}$.

Proof:

- $\lim_{|x|\to+\infty} K(x) = -\infty$. Fix $\varepsilon > 0$. Let $A_{1,n}, A_{2,n} \subset \mathbb{R}^N$ be s.t. $\int_{A_{i,n}} f_{i,n}\,\mathrm{d}x \geq \varepsilon$, then $\mathrm{dist}\,(A_{1,n}, A_{2,n}) \leq C$. Analogously, if $B_{1,n}, B_{2,n}$ are s.t. $\int_{B_{i,n}} f_{i,n}\,\mathrm{d}x \geq \varepsilon$, then $\mathrm{dist}\,(A_{i,n}, B_{i,n}) \leq C$.

  Diagunal argument $\rightsquigarrow$ (1).

  L.s.c. of $\mathcal{E}_K^{c_1,c_2}(\cdot,\cdot)$ w.r.t. tight convergence $\rightsquigarrow$ (f_1, f_2) is a minimizer.
- $\lim_{|x|\to+\infty} K(x) \in \mathbb{R}$ (similar idea). □

Proposition

The minimizers of (Prel) *have compact support.*

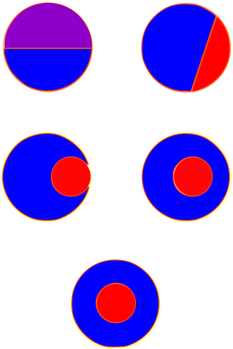
Strongly attractive case: $c_1 + c_2 \leq -2$.

Theorem

(i) *For* $c_1 = c_2 = -1$, (f_1, f_2) *is a minimizer if and only if* $f_1 + f_2 = \chi_{B^{m_1+m_2}(x_0)}$.

(ii) *For* $c_1 = -1$ *and* $c_2 < -1$, (f_1, f_2) *is a minimizer if and only if* $f_1 + f_2 = \chi_{B^{m_1+m_2}(x_0)}$ *and* $f_2 = \chi_{B^{m_2}(y_0)}$ *with* $B^{m_2}(y_0) \subset B^{m_1+m_2}(x_0)$.

(iii) *For* $-1 < c_1 < 0$ *and* $c_2 < -1$, (f_1, f_2) *is a minimizer if and only if* $f_1 + f_2 = \chi_{B^{m_1+m_2}(x_0)}$ *and* $f_2 = \chi_{B^{m_2}(x_0)}$.

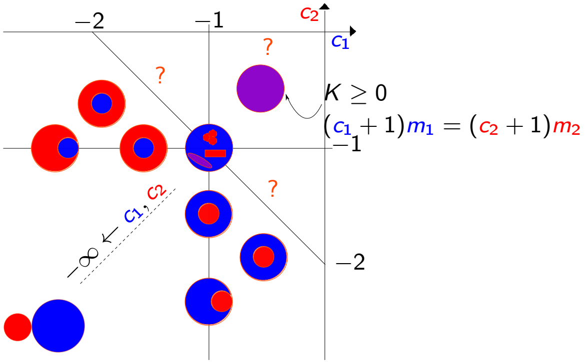


Proof: It is enough to notice that

$$\mathcal{E}_K^{c_1,c_2}(f_1, f_2) = c_1\, J_K(f_1+f_2, f_1+f_2) - 2(c_1+1)\, J_K(f_2, f_1+f_2) + (c_1+c_2+2) J_K(f_2, f_2)$$

and to apply Riesz Lemma. □

General K



Proposition

If $N = 1$ and if $c_1, c_2 < -1$, then

$$(f_1, f_2) = (\chi_{[-m_1,0]}, \chi_{[0,m_2]}) \;\; and \;(f_1, f_2) = (\chi_{[0,m_1]}, \chi_{[-m_2,0]})$$

are the unique (up to translations) minimizers of $\mathcal{E}_K^{c_1,c_2}$ in $\mathcal{A}_{m_1,m_2}$.

Coulombic K: Consequences of the first variation

Let (f_1, f_2) be a minimizer of $\mathcal{E}_K^{c_1,c_2}(f_1, f_2)$ in $\mathcal{A}_{m_1,m_2}$.

Set $V_1 := f_1 * K$ and $V_2 := f_2 * K$. Then, $-\Delta V_i = f_i$ for $i = 1, 2$.

First Var. $\rightsquigarrow$ $\begin{array}{l} (c_1+1)V_1 - (c_2+1)V_2 = \text{const} \quad \text{in } \{0 < f_1, f_2 < 1\} \\ (c_1-1)V_1 + (c_2-1)V_2 = \text{const}' \quad \text{in } \{0 < f_1, f_2 < 1\} \setminus \{f_1 + f_2 = 1\}. \end{array}$

Coulombic K: Doing $\Delta \rightsquigarrow$

$$(c_1+1)f_1 - (c_2+1)f_2 = 0 \quad \text{in } \{0 < f_1, f_2 < 1\} \qquad \text{(FVar1)}$$

$$(c_1-1)f_1 + (c_2-1)f_2 = 0 \quad \text{in } \{0 < f_1, f_2 < 1\} \setminus \{f_1 + f_2 = 1\} \qquad \text{(FVar2)}$$

* (FVar1) $\Rightarrow$ NO MIXING if $(c_1+1)(c_2+1) < 0$, or $c_1 = -1 \neq c_2$ or $c_2 = -1 \neq c_1$.
* (FVar2) $\Rightarrow$ $\{0 < f_1, f_2 < 1\} \subset \{f_1 + f_2 = 1\}$ and

$$f_1 = \frac{c_2+1}{c_1+c_2+2} \text{ and } f_2 = \frac{c_1+1}{c_1+c_2+2} \quad \text{in } \{0 < f_1, f_2 < 1\}.$$

* Moreover, if $-1 < c_1, c_2 < 0$, then either $\operatorname{supp} f_1 \setminus \{f_2 = 0\}$ is empty or $\operatorname{supp} f_2 \setminus \{f_1 = 0\}$ is empty.

Coulombic K

Theorem

Let $-1 < c_1, c_2 \leq 0$. If $(c_2+1)m_2 \leq (c_1+1)m_1$, then the unique minimizer is

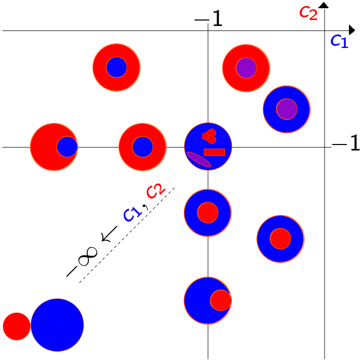
$$(f_1, f_2) = (\chi_{B^{m_1+m_2}} - a\,\chi_{B^{\frac{m_2}{a}}}, a\,\chi_{B^{\frac{m_2}{a}}}),$$

with $a := \frac{c_2+1}{c_1+c_2+2}$.

Theorem

If $-1 \leq c_1 \leq 0$ and $c_2 \leq -1$ and $c_1 + c_2 > -2$, then the unique minimizer is

$$(f_1, f_2) = (\chi_{B^{m_1+m_2}} - \chi_{B^{m_2}}, \chi_{B^{m_2}}).$$



Proposition

Let $c_1, c_2 < -1$. If (f_1, f_2) is a minimizer, then $f_1 = \chi_{E_1}$ and $f_2 = \chi_{E_2}$.

Pb: What's about the shapes of E_1 and E_2?

Proof of the Coulombic case for $c_1 = c_2 = 0$

Claim: If $c_1 = c_2 = 0$ and $m_2 \leq m_1$ then (f_1, f_2) is a minimizer iff

$$(f_1, f_2) = (\chi_{B^{m_1+m_2}} - \tfrac{1}{2}\chi_{B^{2m_2}}, \tfrac{1}{2}\chi_{B^{2m_2}}).$$

Proof.

- First Variation $\rightsquigarrow$ $f_1 = g\,\chi_E - \frac{1}{2}\chi_M$ and $f_2 = \frac{1}{2}\chi_M$, with $M \subset E$, $0 < g \leq 1$ and $g = 1$ in M.
- Let V_2^* be the spherical rearrangement of $V_2 (:= f_2 * K)$ and let $\tilde{f}_2$ be s.t.

$$\int_{\{V_2^* > t\}} \tilde{f}_2 \, \mathrm{d}x = \int_{\{V_2 > t\}} f_2 \, \mathrm{d}x \quad \text{for any } t \in \mathbb{R}.$$

- Set $\tilde{V}_2 := \tilde{f}_2 * K$, show that $\tilde{V}_2 \geq V_2^*$ and that "$>$" holds true in $\{V_2^* > \bar{t}\}$, where $\bar{t}$ is the maximal level such that $\{V_2 > t\}$ is a ball.
- Show that $\exists\, \hat{t} < \max \tilde{V}_2$ s.t. $\int_{\{\tilde{V}_2 > \hat{t}\}} (1 - \tilde{f}_2) = m_1$ and set $\tilde{f}_1 := 1 - \tilde{f}_2$ in $\{\tilde{V}_2 > \hat{t}\}$ and zero outisde.
- Show that $\mathcal{E}_K^{0,0}(\tilde{f}_1, \tilde{f}_2) \leq \mathcal{E}_K^{0,0}(f_1, f_2)$ and that $\tilde{V}_2 = V_2^*$, so that the superlevels of V_2 are balls. Conclude taht $f_2 = \frac{1}{2}\chi_{B^{2m_2}}$. □

Proof of the Coulombic case: triangles and square

Triangle(s): Let $c_1 \leq -1$, $-1 \leq c_2 \leq 0$ and $c_1 + c_2 > -2$. Then the unique minimizer is

$$(f_1, f_2) = (\chi_{B^{m_1}}, \chi_{B^{m_1+m_2}} \setminus \chi_{B^{m_1}}).$$

Proof: Set $g_1 := \frac{f_1}{2}$, $g_2 := \frac{f_1}{2} + f_2$, $\tilde{m}_1 := \frac{m_1}{2}$ and $\tilde{m}_2 := \frac{m_1}{2} + m_2 > \tilde{m}_1$. Then

$$\mathcal{E}_K^{c_1,c_2}(f_1, f_2) = 4(c_1+1)J_K(g_1, g_1) + c_2 J_K(g_1+g_2, g_1+g_2) + 2(1+c_2)\mathcal{E}_K^{0,0}(g_1, g_2).$$

Previous slide result + Riesz Lemma + going back to (f_1, f_2) $\rightsquigarrow$ the claim. $\square$

Square: similar argument.

Open problems

- shape of minimizers for $c_1, c_2 < -1$ in the Coulombic case;
- $J_{K_{ij}}(f_i, f_j) = \int_{\mathbb{R}^N \times \mathbb{R}^N} f_i(x)\, f_j(y)\, K_{ij}(x-y)\,\mathrm{d}x\,\mathrm{d}y$ for $i, j = 1, 2$;
- n-phase problem with $n \geq 3$;
- dynamics (Berendsen-Burger-Pietschmann, September 2016);
- $c_1 \geq 0$ and/or $c_2 \geq 0$ (if $c_1, c_2 > 1$, there exist no minimizers for all values of m_1 and m_2).

References

M. Cicalese, L. De Luca, M. Novaga, M. Ponsiglione,
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SIAM J. Math. Anal., **48** (2016), no. 5, pp. 3412–3443.