Fonction propre principale optimale pour des opérateurs elliptiques avec un terme de transport grand

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Plan of the talk

- Isoperimetric problem
  - Faber-Krahn inequality
  - Adding a drift
- Optimization in a fixed domain
  - Optimal drift
  - Principal eigenvalue for nonlinear operators
- Asymptotics of the optimal principal eigenfunction
  - The conjectures
  - The (partial) answers

Isoperimetric problem for the principal eigenvalue

Let λ_Ω denote the principal eigenvalue of $-\Delta$ in a bounded domain Ω, i.e.,

$$\begin{cases} -\Delta\varphi = \lambda_\Omega \varphi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \\ \varphi > 0 & \text{in } \Omega. \end{cases}$$

Question (Rayleigh $\sim$1890)

Which Ω minimizes λ_Ω under the constraint $|\Omega| = 1$?

Answer (Faber, Krahn 1920s): $\Omega =$ the ball.

Tool: Schwarz symmetrization.

Adding a drift $v \in L^\infty(\Omega)$. Let $\lambda_{\Omega,v}$ denote the principal eigenvalue:

$$\begin{cases} -\Delta\varphi - v \cdot \nabla\varphi = \lambda_{\Omega,v}\varphi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \\ \varphi > 0 & \text{in } \Omega. \end{cases}$$

Question

Which Ω and v minimize $\lambda_{\Omega,v}$ under the constraints $|\Omega| = 1$, $\|v\|_\infty \le \tau$?

Answer (Hamel-Nadirashvili-Russ, Ann. of Math. 2011):

$$\Omega = \text{ the ball}, \qquad v(x) = -\tau\frac{x}{|x|}.$$

Tool: new type of symmetrization.

Optimization in a fixed domain

Let Ω be a given bounded smooth domain.
For $v \in L^\infty(\Omega)$, let λ_v be the principal eigenvalue:

$$\begin{cases} -\Delta\varphi - v \cdot \nabla\varphi = \lambda_v \varphi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \\ \varphi > 0 & \text{in } \Omega. \end{cases}$$

$$\forall \tau \geq 0, \qquad \lambda(\tau) := \inf\{\lambda_v \ : \|v\|_{L^\infty(\Omega)} \leq \tau\}.$$

Theorem (Hamel-Nadirashvili-Russ)

The infimum is achieved by a unique $\underline{v}$. Furthermore

$$\underline{v} = \tau \frac{\nabla\varphi_\tau}{|\nabla\varphi_\tau|} \qquad \textit{whenever } \nabla\varphi_\tau \neq 0,$$

where φ_τ is the associated principal eigenfunction.

Principal eigenvalue for nonlinear operators

$\varphi_\tau \in C^2(\Omega)$ satisfies the nonlinear eigenvalue problem

$$\begin{cases} -\Delta\varphi_\tau - \tau|\nabla\varphi_\tau| = \lambda(\tau)\varphi_\tau & \text{in } \Omega \\ \varphi_\tau = 0 & \text{on } \partial\Omega \\ \varphi_\tau > 0 & \text{in } \Omega. \end{cases}$$

Can we say that $\lambda(\tau)$ is THE principal eigenvalue?

Difficulty: nonlinear $\Longrightarrow$ Krein-Rutman theory does not directly apply.

But: 1-homogeneous $\Longrightarrow$ Berestycki-Nirenberg-Varadhan approach does!

$$\lambda_{gen} = \sup\{\lambda \ : \ (-\Delta - \tau|\nabla| - \lambda I) \text{ admits a positive supersolution}\}.$$

- Pucci (Felmer-Quaas)
- Bellman (Quaas-Sirakov)
- *p* and ∞ Laplacian (Kawohl-Lindqvist, Birindelli-Demengel, Juutinen)
- Truncated Laplacian (Birindelli-Galise-Ishii)
- Arbitrary degenerate ellipticity (Berestycki-Capuzzo Dolcetta-Porretta-R.)

Proposition

$$\lambda(\tau) = \max\{\lambda \ : \exists\varphi > 0,\ -\Delta\varphi - \tau|\nabla\varphi| \geq \lambda\varphi \text{ in } \Omega\}$$

$$= \min\{\lambda \ : \exists\varphi > 0,\ -\Delta\varphi - \tau|\nabla\varphi| \leq \lambda\varphi \text{ in } \Omega,\ \varphi = 0 \text{ on } \partial\Omega\}.$$

Furthermore, the above extrema are attained only by (a multiple of) φ_τ.

Asymptotics as $\tau \to +\infty$

Theorem (Hamel-Nadirashvili-Russ)

$$e^{-R\tau} \le \lambda(\tau) \le e^{-r\tau}, \qquad B_r(x_1) \subset \Omega \subset B_R(x_2).$$

$d(\cdot) := \mathrm{dist}(\cdot, \partial\Omega)$

$\mathcal{C}$:= *cut locus* = {points where d is not differentiable}

Conjecture 1 (—)

Let $(x_\tau)_\tau$ be maximal points for $(\varphi_\tau)_\tau$. Then

$$d(x_\tau) \to \max_{\overline{\Omega}} d \quad \text{as } \tau \to +\infty.$$

Conjecture 2 (—)

$$\forall x \notin \mathcal{C}, \quad \left| \frac{\nabla\varphi_\tau(x)}{|\nabla\varphi_\tau(x)|} - \nabla d(x) \right| \to 0 \quad \text{as } \tau \to +\infty.$$

Answers

Theorem (Hamel-R.-Russ)

Let $(x_\tau)_\tau$ be maximal points for $(\varphi_\tau)_\tau$. Then

$$d(x_\tau) \to \max_{\overline{\Omega}} d \quad \text{as } \tau \to +\infty.$$

Theorem (—)

For any $M > 0$,

$$\sup_{d<M/\tau} \left| \frac{\nabla \varphi_\tau}{|\nabla \varphi_\tau|} - \nabla d \right| \to 0 \quad \text{as } \tau \to +\infty.$$

Theorem (—)

$$\lim_{\tau \to +\infty} \frac{\varphi_\tau(x)}{\|\varphi_\tau\|_\infty (1 - e^{-\tau d(x)})} = 1, \qquad \text{uniformly w.r.t. } x \in \Omega.$$

Idea of the proof of Theorem 1

Construction of barriers (positive supersolutions) increasing w.r.t. d:

$\forall r < \max_{\overline{\Omega}} d,\ \tau \gg 1,\ \exists \Psi \in C^2((0,r)) \cap C^0([0,r])$ positive increasing,

$$-\Psi'' - (\tau+\varepsilon)\Psi' \geq \lambda(\tau)\Psi \qquad \text{in } (0,r)$$

with $\varepsilon > 0$ independent of τ (using $\lambda(\tau) \leq e^{-r\tau}$ because $B_r(x_1) \subset \Omega$).

$$\psi := \Psi \circ d$$

$$-\Delta\psi - \tau|\nabla\psi| \geq \lambda(\tau)\psi + \underbrace{(\varepsilon\tau - \Delta d)}_{>0?}|\nabla\psi| \qquad \text{for } d < r.$$

Lemma

$$\sup_{\Omega} \Delta d < \infty.$$