The center–focus problem, the integrability of the centers, and the divergence

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Orbits of a differential system

Topologically the orbits of a differential system are:

- points (equilibrium points or singular points),
- circles (periodic solutions), or
- straight lines.

In this talk we are interested in studying the phase portrait in a neighborhood of an equilibrium point of an analytic differential system in the plane.

It is known that the phase portrait in a neighborhood of an equilibrium point of an analytic differential system in the plane is

- either a center,
- or focus,
- or finite union of elliptic, hyperbolic and parabolic sectors.

In this talk we deal with the analytic (mainly polynomial) differential systems of the form

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{1}$$

where the dot denotes derivative with respect to an independent real variable t. We assume that this system always is defined in a neighborhood of the origin and that the origin is a singular point.

If the origin is either a focus or a center, we say that it is a monodromic singular point.

The center–focus problem consists in distinguishing when a monodromic singular point is either a center or a focus.

From now on in this talk we assume that the origin of system (1) is monodromic.

Assume that we have the system

$$
\begin{aligned}
\dot{x} &= P(x,y) = P_n(x,y) + O_{n+1}(x,y), \\
\dot{y} &= Q(x,y) = Q_m(x,y) + O_{m+1}(x,y),
\end{aligned}
\tag{2}
$$

where $n \geq 1$ and $m \geq 1$ are integers and $P_n(x,y)$ and $Q_m(x,y)$ are non–zero homogeneous polynomials of degrees n and m respectively, formed by the lowest order terms of $P(x,y)$ and $Q(x,y)$, respectively.

Define the real polynomial

$$
\Delta(x,y) = \begin{cases} yP_n(x,y) - xQ_m(x,y) & \text{if } n = m, \\ yP_n(x,y) & \text{if } n < m, \\ -xQ_m(x,y) & \text{if } n > m. \end{cases}
$$

A sufficient condition in order that system (2) has a monodromic singular point at the origin is that $\Delta(x, y) = 0$ only if $(x, y) = (0, 0)$.

In this case the origin has no characteristic directions.

A necessary condition in order that system (2) has a monodromic singular point at the origin is that $\Delta(x, y) \geq 0$ or $\Delta(x, y) \leq 0$ for all $(x, y) \in \mathbb{R}^2$.

The center–focus problem started with Poincaré and Dulac 135 years ago, and in the present days many questions remains open.

H. Poincaré, Mémoire sur les courbes définies par une équation differentielle, J. Maths. Pures Appl., 7, 1881, 375–422.

H. Dulac, Détermination et integration d'une certaine classe d'équations différentielle ayant par point singulier un centre, Bull. Sci. Math. Sér. (2) **32** (1908), 230–252.

Assume that the origin of system (1) is a monodromic singular point, but not a strong focus.

It is well-known that, after a linear change of variables and a constant scaling of the time variable (if necessary), such a system can be written in one of the following three forms:

$$\begin{aligned} \dot{x} &= -y + F_1(x,y), & \dot{y} &= x + F_2(x,y), \\ \dot{x} &= y + F_1(x,y), & \dot{y} &= F_2(x,y), \\ \dot{x} &= F_1(x,y), & \dot{y} &= F_2(x,y), \end{aligned}$$

where $F_1(x,y)$ and $F_2(x,y)$ are real analytic functions without constant and linear terms defined in a neighborhood of the origin.

These three kind of monodromic singular points are called linear type, nilpotent or degenerate, respectively.

The linear type monodromic singular points which after changes of variables can be written as

$$\dot{x} = -y + F_1(x,y), \quad \dot{y} = x + F_2(x,y),$$

are characterized by having a pair of imaginary eigenvalues.

The nilpotent monodromic singular points which after changes of variables can be written as

$$\dot{x} = y + F_1(x,y), \quad \dot{y} = F_2(x,y),$$

are characterized by the Andreev theorem (or the nilpotent singular theorem).

The degenerate monodromic singular points which after changes of variables can be written as

$$\dot{x} = F_1(x,y), \quad \dot{y} = F_2(x,y),$$

can be characterized using blow–ups.

The study of the linear type centers and of the centers without characteristic directions can be done with the so called Poincaré–Liapunov constants, because for such centers the Poincaré map in a neighborhood of them is analytic.

Suppose that the origin is a monodromic singular point. Then we have the Poincaré map $\mathcal{P} : [0, x^*) \to [0, \infty)$, being $\mathcal{P}(x)$ the point in $[0, \infty)$ corresponding to the first cut with $[0, \infty)$ of the orbit through the point $(x, 0)$ in positive time.

It is clear that the origin of system (1) is a center if and only if this Poincaré map is the identity.

For linear type centers or the centers without characteristic directions always the Poincaré map is analytic at $x = 0$ and writes as

$$\mathcal{P}(x) = x + \sum_{i=3}^{\infty} \alpha_i x^i,$$

where α_i are algebraic polynomials in the coefficients of (P, Q).

Knowing the Poincaré map, the origin is stable if the first non-zero α_i is negative, and unstable if $\alpha_i > 0$.

If all $\alpha_i = 0$ the origin is a center.

The α_{2k} are function of the previous α_i with $i < 2k$. Therefore the interesting expressions are the α_{2k+1}'s.

We define the $2k+1$ Poincaré–Liapunov constant as the expression α_{2k+1} modulus the vanishing of all the previous α_i with $i < 2k+1$.

But the computation of the Poincaré–Liapunov constants usually is a very difficult computational problem.

The technique of the Poincaré–Liapunov constants for solving the focus–center problem for linear type centers and of the centers without characteristic directions can be extended to nilpotent centers.

J. GINÉ AND J. LLIBRE, A method for characterizing nilpotent centers, J. Math. Anal. Appl. **413** (2014), 537–545.

I.A. GARCÍA, H. GIACOMINI, J. GINÉ AND J. LLIBRE, Analytic nilpotent centers as limits of linear type centers revisited, J. Math. Anal. and Appl. **441** (2016), 893–899.

In short, now we have algorithms based in the computation of Poincaré–Liapunov constants for solving the focus–center problem for linear type centers, nilpotent centers and centers without characteristic directions. Moreover, for such centers the Poincaré–Liapunov constants are algebraic polynomials in function of the coefficients of the polynomial differential systems.

The conditions on the coefficients of a polynomial differential system which determine when such a system has a center, which is neither of linear type, nor nilpotent and with characteristic directions, can be non-algebraic as it was proved in

Yu. S. Il'yashenko, Algebraic unsolvability and almost algebraic solvability of the problem for the center–focus, Funkcion. Anal. Priloz. **6** (1972), No 3, 30–37.

J. Llibre and H. Zoladek, The Poincaré center problem, J. Dynamical and Control Systems **14** (2008), 505–535.

LOCAL INTEGRABILITY AT A CENTER.

LINEAR TYPE CENTER THEOREM. A polynomial differential system has a center at the origin if and only if there exists a local analytic first integral of the form $H = x^2 + y^2 + F(x, y)$ defined in a neighborhood of the origin, where F starts with terms of order higher than 2.

H. Poincaré, Mémoire sur les courbes définies par une équation differentielle, J. Maths. Pures Appl., 7, 1881, 375–422.

The Poincaré's Theorem was extended to analytic differential systems by Liapunov.

M.A. Liapunov, Problème général de la stabilité du mouvement, Ann. of Math. Stud. **17**, Princeton University Press, 1947.

NILPOTENT CENTER THEOREM. Assume that a polynomial differential system has a nilpotent center at the origin.

(a) If the system has a formal first integral, then it has a formal first integral of the form $H = y^2 + F(x, y)$, where F starts with terms of order higher than two.

(b) If the system has a local analytic first integral defined at the origin, then it has a local analytic first integral of the form $H = y^2 + F(x, y)$, where F starts with terms of order higher than two.

(c) A polynomial system $\dot{x} = y(1 + f(x, y^2))$, $\dot{y} = g(x, y^2)$, with $O(f(x, y^2)) = O(x, y^2)$ and $O(g(x, y^2)) = O(x, y^2)$, has a local analytic first integral of the form $H = y^2 + F(x, y)$, where F starts with terms of order higher than two.

This theorem also works for analytic differential systems.

It is known that there are nilpotent centers which do not have a local analytic first integral defined in their neighborhood.

J. Chavarriga, H. Giacomini, J. Giné and J. Llibre, Local analytic integrability for nilpotent centers, Ergodic Theory and Dynamical Systems **23** (2003), 417–428.

Mazzi–Sabatini THEOREM. Any analytic differential system having a center has a local C^∞ first integral defined in its neighborhood.

L. Mazzi and M. Sabatini, A characterization of centres via first integrals, J. Differential Equations **76** (1988), 222–237.

THEOREM. Any analytic differential system having a center at the point p has a local analytic first integral defined in a punctured neighborhood of p, but such an analytic first integral does not need to be defined at the center point.

Weigu Li, J. Llibre, M. Nicolau and X. Zhang, On the differentiability of first integrals of two dimensional flows, Proc. Amer. Math. Soc. **130** (2002), 2079–2088.

This completes our knowledge on the local integrability of centers.

The divergence of system (1), denoted by $\mathrm{div}(x, y)$, is the function

$$\mathrm{div}(x, y) = \frac{\partial P}{\partial x}(x, y) + \frac{\partial Q}{\partial y}(x, y).$$

System (1) is a Hamiltonian system if $\mathrm{div}(x, y) \equiv 0$. In such a case if there exists a neighborhood $\mathcal{U}$ of the origin and an analytic function $H : \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$, called the Hamiltonian, such that

$$\dot{x} = P(x, y) = -\frac{\partial H}{\partial y}, \quad \dot{y} = Q(x, y) = \frac{\partial H}{\partial x},$$

then the monodromic singular point at the origin of this Hamiltonian system always is a center.

Our aim is to show other results relating the divergence of system (1) with the solution of the center problem.

The four next results that I will present in the following are proved in the paper:

M. GRAU AND J. LLIBRE, Divergence and Poincaré–Liapunov constants for analytic differential systems, J. Differential Equations **258** (2015), 4348–4367.

Given an analytic function $f : \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$, where $\mathcal{U}$ is a neighborhood of the origin, we consider its Taylor expansion at the origin:

$$f(x, y) = f_d(x, y) + \mathcal{O}_{d+1}(x, y),$$

where $d \geq 0$ is an integer and $f_d(x, y)$ is a non–zero homogeneous polynomial of degree d.

When $f_d(x, y) > 0$ (resp. $f_d(x, y) < 0$) for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ we say that f is positive definite (resp. negative definite).

It is clear that a necessary condition for $f(x, y)$ to be of sign definite is that d is even.

PROPOSITION 1. Assume that the origin of an analytic differential system (1) is a monodromic singular point, and that the divergence $\mathrm{div}(x, y)$ of system (1) is of sign definite.

Then the origin of system (1) is a focus; either unstable if the divergence is positive definite or stable if it is negative definite.

We remark that in the case that the origin of system (1) is a strong focus (i.e. with eigenvalues $\alpha \pm \beta i$ and $\alpha \neq 0$), then the divergence $\mathrm{div}_d(x, y) = \mathrm{div}(0, 0) = 2\alpha \neq 0$ and the focus is unstable if $\mathrm{div}(0, 0) > 0$, and stable if $\mathrm{div}(0, 0) < 0$.

PROPOSITION 1 is a generalization of this result for the strong focus to any monodromic singular point.

THEOREM 2. Consider an analytic differential system

$$\dot{x} = -y + F_1(x,y), \qquad \dot{y} = x + F_2(x,y),$$

where $F_1(x,y)$ and $F_2(x,y)$ have no constant and linear terms and are defined in a neighborhood of the origin. Denote by $\mathrm{div}_d(x,y)$ the lowest order terms of the divergence $\mathrm{div}(x,y)$ of the system. Define

$$\alpha_{d+1} = \frac{1}{d+2}\int_0^{2\pi} \mathrm{div}_d(\cos t, \sin t)\, dt.$$

If $\alpha_{d+1} \neq 0$, then it is the non–zero first Poincaré–Liapunov constant, and consequently the origin is a focus.

THEOREM 3 Consider an analytic differential system

$$\dot{x} = y + F_1(x, y), \qquad \dot{y} = F_2(x, y),$$

where $F_1(x, y)$ and $F_2(x, y)$ have no constant and linear terms, are defined in a neighborhood of the origin, and assume that the origin is a monodromic nilpotent singular point. Denote by $\mathrm{div}_d(x, y)$ the lowest order terms of the divergence $\mathrm{div}(x, y)$ of the system. Define

$$V_{d+1}(\varepsilon) = \int_0^{2\pi/\sqrt{\varepsilon}} \mathrm{div}_d \left(\cos(\sqrt{\varepsilon}\, t), \ -\sqrt{\varepsilon} \sin(\sqrt{\varepsilon}\, t)\right) \, dt,$$

where $\varepsilon > 0$, and define the constant v_{d+1} through the series expansion $V_{d+1}(\varepsilon) = \dfrac{v_{d+1}}{\sqrt{\varepsilon}} + O(1)$.

(a) If the origin is a center, then $v_{d+1} = 0$ for all $\varepsilon > 0$.

(b) If $v_{d+1} > 0$ (resp. $v_{d+1} < 0$), then the origin is an unstable (resp. stable) focus.

Assume that there is a monodromic singular point at the origin of system (1) without characteristic directions. Then the polynomial $\Delta(x, y)$ defined previously satisfies that $\Delta(x, y) = 0$ only if $(x, y) = (0, 0)$.

In this case the degree of the lowest order terms of $P(x, y)$ and $Q(x, y)$ must coincide, that is,

$$P(x, y) = P_n(x, y) + \mathcal{O}_{n+1}(x, y),$$
$$Q(x, y) = Q_n(x, y) + \mathcal{O}_{n+1}(x, y).$$

We define

$$v(\theta) \;=\; \exp\left[\int_0^\theta \frac{\cos\sigma P_n(\cos\sigma, \sin\sigma) + \sin\sigma Q_n(\cos\sigma, \sin\sigma)}{\cos\sigma Q_n(\cos\sigma, \sin\sigma) - \sin\sigma P_n(\cos\sigma, \sin\sigma)}\, d\sigma\right].$$

THEOREM 4. Consider an analytic differential system (1) whose origin is monodromic and has no characteristic directions. Denote by $\mathrm{div}_d(x, y)$ the lowest order terms of degree d of the divergence $\mathrm{div}(x, y)$ of the system. Assume that $v(2\pi) = 1$ and

$$\alpha = \int_0^{2\pi} \frac{\mathrm{div}_d(\cos\theta, \sin\theta)\; v(\theta)^{d-n+1}}{\cos\theta Q_n(\cos\theta, \sin\theta) - \sin\theta P_n(\cos\theta, \sin\theta)}\, d\theta \neq 0.$$

Then the origin is a focus which is stable (resp. unstable) if $\alpha < 0$ (resp. $\alpha > 0$).

We remark that from PROPOSITION 1, if $v(2\pi) > 1$ then the origin is an unstable focus, and if $v(2\pi) < 1$ then the origin is a stable focus. So this theorem is useful to establish the stability of the origin when $v(2\pi) = 1$.

PROPOSITION 1. Assume that the origin of an analytic differential system (1) is a monodromic singular point, and that the divergence $\mathrm{div}(x, y)$ of system (1) is of sign definite.

Then the origin of system (1) is a focus; either unstable if the divergence is positive definite or stable if it is negative definite.

Proof of PROPOSITION 1.

The Bendixson criterium: If the divergence of a system (1) is not identically zero and does not change sign in a simply connected region in $\mathbb{R}^2$, then there is no closed orbit lying entirely in this simply connected region.

If the divergence of system (1) is of sign definite, then there is a neighborhood $\mathcal{U}_O$ of the origin in which $\mathrm{div}(x, y) > 0$ or $\mathrm{div}(x, y) < 0$ for all $(x, y) \in \mathcal{U}_O$.

If the origin is a center, then there is a continuum of periodic orbits completely contained in $\mathcal{U}_O$ which contradicts the Bendixson criterium. Hence, the origin is a focus. This proves the first part of the PROPOSITION.

It is well–known that if the divergence is negative (resp. positive) the area decreases (resp. increases) under the flow, and consequently the focus will be stable (resp. unstable). This completes the proof of PROPOSITION 1.

THEOREM 2. Consider an analytic differential system

$$\dot{x} = -y + F_1(x, y), \qquad \dot{y} = x + F_2(x, y),$$

where $F_1(x, y)$ and $F_2(x, y)$ have no constant and linear terms and are defined in a neighborhood of the origin. Denote by $\mathrm{div}_d(x, y)$ the lowest order terms of the divergence $\mathrm{div}(x, y)$ of the system. Define

$$\alpha_{d+1} = \frac{1}{d+2} \int_0^{2\pi} \mathrm{div}_d\,(\cos t, \sin t)\; dt.$$

If $\alpha_{d+1} \neq 0$, then it is the non–zero first Poincaré–Liapunov constant, and consequently the origin is a focus.

Its proof uses the Birkhoff normal form of a center provided in

G. BELITSKIĬ, Smooth equivalence of germs of vector fields with one zero or a pair of purely imaginary eigenvalues, Funct. Anal. Appl. **20** (1986), 253–259.

THEOREM 3 Consider an analytic differential system

$$\dot{x} = y + F_1(x,y), \qquad \dot{y} = F_2(x,y),$$

where $F_1(x,y)$ and $F_2(x,y)$ have no constant and linear terms, are defined in a neighborhood of the origin, and assume that the origin is a monodromic nilpotent singular point. Denote by $\mathrm{div}_d(x,y)$ the lowest order terms of the divergence $\mathrm{div}(x,y)$ of the system. Define

$$V_{d+1}(\varepsilon) = \int_0^{2\pi/\sqrt{\varepsilon}} \mathrm{div}_d\left(\cos(\sqrt{\varepsilon}\,t),\ -\sqrt{\varepsilon}\sin(\sqrt{\varepsilon}\,t)\right)\, dt,$$

where $\varepsilon > 0$, and define the constant v_{d+1} through the series expansion $V_{d+1}(\varepsilon) = \dfrac{v_{d+1}}{\sqrt{\varepsilon}} + O(\varepsilon)$.

(a) If the origin is a center, then $v_{d+1} = 0$ for all $\varepsilon > 0$.

(b) If $v_{d+1} > 0$ (resp. $v_{d+1} < 0$), then the origin is an unstable (resp. stable) focus.

The proof of Theorem 3 uses results from

J. GINÉ AND J. LLIBRE, A method for characterizing nilpotent centers, J. Math. Anal. Appl. **413** (2014), 537–545.

I.A. GARCÍA, H. GIACOMINI, J. GINÉ AND J. LLIBRE, Analytic nilpotent centers as limits of nondegenerated centers revisited, J. Math. Anal. and Appl. **441** (2016), 893–899.

THEOREM 4 Consider an analytic differential system (1) whose origin is monodromic and has no characteristic directions. Denote by $\mathrm{div}_d(x,y)$ the lowest order terms of degree d of the divergence $\mathrm{div}(x,y)$ of the system. Assume that $v(2\pi)=1$ and

$$\alpha = \int_0^{2\pi} \frac{\mathrm{div}_d\,(\cos\theta, \sin\theta)\ \ v(\theta)^{d-n+1}}{\cos\theta Q_n(\cos\theta, \sin\theta) - \sin\theta P_n(\cos\theta, \sin\theta)}\, d\theta \neq 0.$$

Then the origin is a focus which is stable (resp. unstable) if $\alpha < 0$ (resp. $\alpha > 0$).

The proof follows by direct computations.

For more details on the proofs of THEOREMS 2 and 3 see the paper:

M. GRAU AND J. LLIBRE, Divergence and Poincaré–Liapunov constants for analytic differential systems, J. Differential Equations **258** (2015), 4348–4367.

THANK YOU VERY MUCH FOR YOUR ATTENTION