Mesures de défaut de compacité, contrôle et problèmes inverses pour les ondes

Jérôme Le Rousseau
Université Paris-Nord

Laboratoire Jacques-Louis-Lions
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90', L. Tartar, P. Gérard: H-mesures or microlocal defect mesures.
Goal: localise in space and frequency any obstruction to the strong convergence of a sequence $(u_n) \subset L^2_{loc}(\Omega)$ with $u_n \rightharpoonup 0$.

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Example

$f \in \mathscr{C}_c^\infty(\mathbb{R}^d)$, $\quad u_n(x) = n^{d/2} f(n(x - x_0))$ supported in Ω.

$$\forall \varphi \in L^2_{comp}(\Omega), \quad n^{d/2} \int_\Omega f(n(x - x_0))\varphi(x)dx \longrightarrow 0,$$

Hence $u_n \rightharpoonup 0$. As $\|u_n\|_{L^2} = \|f\|_{L^2}$, we don't have $u_n \not\to 0$.

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Here concentration phenomenum in space at x_0.

$$\forall \varphi \in \mathscr{C}_c^\infty(\Omega), \quad (\varphi u_n, u_n)_{L^2} = n^d \int_\Omega \varphi(x) f^2(n(x - x_0))dx \longrightarrow \|f\|^2_{L^2} \langle \delta, \varphi \rangle,$$

One can also localize in frequency.

Example

$g \in L^2_{loc}(\Omega)$, $\xi_0 \in \mathbb{R}^d \setminus \{0\}$ $\quad u_n(x) = g(x)e^{inx\cdot\xi_0}$. *We have*

$$u_n \rightharpoonup 0$$

However, if we compute

$$(\varphi u_n, u_n)_{L^2} = \int_\Omega |g|^2 \varphi$$

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However, if we compute

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We do not perceive the concentration in frequency with this test function. Need to change the type of test function:

$$\varphi(x) \longrightarrow \varphi(x, \xi)$$

Differential operators in $\mathbb{R}^d$:
With $D = -i\partial$ and $p(x,\xi)$ polynomial in ξ we write

$$p(x,D)u(x) = (2\pi)^{-d} \int e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\, d\xi, \quad u \in \mathscr{S}(\mathbb{R}^d) \text{ or } \mathscr{S}'(\mathbb{R}^d).$$

Pseudo-differential operators in $\mathbb{R}^d$:
If $a(x,\xi)$ is smooth and does not grow faster than a polynomial function in ξ at infinity we can then define

$$a(x,D)u(x) = (2\pi)^{-d} \int e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)\, d\xi, \quad u \in \mathscr{S}(\mathbb{R}^d) \text{ or } \mathscr{S}'(\mathbb{R}^d).$$

if $a(x,\xi)$ of order m, one write $a(x,D) \in \Psi^m(\mathbb{R}^d)$.

Observations:
$a(x,D) : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ cont.
$a(x,D) : \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ cont.

If $a(x,D)$ is of order $m \in \mathbb{R}$, then $a(x,D) : H^s(\mathbb{R}^d) \to H^{s-m}(\mathbb{R}^d)$ cont.

Here, we consider polyhomogeneous symbols

$$a(x,\xi) \sim \sum_{j\in\mathbb{N}} a_{m-j}(x,\xi), \qquad a_{m-j}(x,\xi) \text{ homog. degree } m-j \text{ in } \xi.$$

Principal symbol: $a_m(x,\xi)$.

Pseudo-differential operators operators in Ω:
$A : \mathscr{C}_c^\infty(\Omega) \to D'(\Omega)$ is said to be in $\Psi^m(\Omega)$ if for all $\chi, \tilde{\chi} \in \mathscr{C}_c^\infty(\Omega)$, $\chi A \tilde{\chi} \in \Psi^m(\mathbb{R}^d)$.
Then $A : \mathscr{C}_c^\infty(\Omega) \to \mathscr{C}^\infty(\Omega)$ and more important

$$A : H^s_{comp}(\Omega) \to H^{s-m}_{loc}(\Omega)$$

A is said to be properly supported if $A : \mathscr{C}_c^\infty(\Omega) \to \mathscr{C}_c^\infty(\Omega)$. Then, $A : \mathscr{C}^\infty(\Omega) \to \mathscr{C}^\infty(\Omega)$ and more important:

$$A : H^s_{comp}(\Omega) \to H^{s-m}_{comp}(\Omega)$$

and

$$A : H^s_{loc}(\Omega) \to H^{s-m}_{loc}(\Omega)$$

Pseudo-differential operators operators in Ω:
We say that $A \in \Psi^m_{comp}(\Omega)$ if $A \in \Psi^m(\Omega)$ and there exists $\chi, \tilde{\chi} \in \mathscr{C}^\infty_c(\Omega)$ such that $A = \chi A \tilde{\chi}$.
Then, up to a smoothing operator

$$A = \chi a(x, D) \tilde{\chi}$$

for some $a(x, \xi)$ of order m.

Then $A : H^s_{loc}(\Omega) \to H^{s-m}_{comp}(\Omega)$.

Principal symbol: $\sigma(A)(x, \xi) = \chi(x)\tilde{\chi}(x) a_m(x, \xi)$, homogeneous of degree m in ξ

Theorem (Gérard, Tartar)

Let $(u_n) \subset L^2_{loc}(\Omega)$ with $u_n \rightharpoonup 0$. There exists a subsequence (u_{n_k}) and μ a positive Radon measure on $\Omega \times \mathbb{S}^{d-1}$ such that for all $A \in \Psi^0_{comp}(\Omega)$ we have

$$(Au_{n_k}, u_{n_k})_{L^2(\Omega)} \underset{k\to\infty}{\longrightarrow} \langle \mu, \sigma(A) \rangle_{\Omega\times\mathbb{S}^{d-1}} = \int\limits_{\Omega\times\mathbb{S}^{d-1}} \sigma(A)(x,\xi)\, d\mu(x,\xi).$$

Example

$u_n = n^{d/2} f(n(x-x_0))$	$\mu = \delta(x-x_0)h(\xi)d\sigma(\xi)$, *with* $h(\xi) = (2\pi)^{-d} \int_0^\infty \lvert\hat{f}(r,\xi)\rvert^2 r^{d-1} dr.$
$u_n = g(x)e^{inx\cdot\xi_0)}$	$\mu = \lvert g(x)\rvert^2 \delta(\xi - \xi_0/\lvert\xi_0\rvert)$
$u_n = n^{d/2} f(n(x-x_0))e^{in^2 x\cdot\xi_0)}$	$\mu = \lVert f\rVert^2_{L^2} \delta(x-x_0)\delta(\xi-\xi_0/\lvert\xi_0\rvert)$

Two generalizations

1. $(u_n) \subset H^s_{loc}(\Omega), \quad u_n \rightharpoonup 0,$

   There exists μ positive Radon measure on $\Omega \times \mathbb{S}^{d-1}$ such that for all $A \in \Psi^{2s}_{comp}(\Omega)$ we have

$$(Au_{n_k}, u_{n_k})_{L^2(\Omega)} \underset{k\to\infty}{\longrightarrow} \langle \mu, \sigma(A) \rangle_{\Omega\times\mathbb{S}^{d-1}} = \int\limits_{\Omega\times\mathbb{S}^{d-1}} \sigma(A)(x,\xi)\, d\mu(x,\xi).$$

2.

Two generalizations

1.

2. Let $(u_n) \subset (H^s_{loc}(\Omega))^N$, $\quad u_n \rightharpoonup 0$,
There exists $\mu = (\mu_{ij})$, $N \times N$ matrix of complex Radon measures such that for all $A \in (\Psi^{2s}_{comp}(\Omega))^N$ we have

$$(Au_{n_k}, u_{n_k})_{L^2(\Omega)} \underset{k\to\infty}{\longrightarrow} \int_{\Omega\times\mathbb{S}^{d-1}} \operatorname{tr}\big(\sigma(A)(x,\xi)\, d\mu(x,\xi)\big).$$

The measure μ is Hermitian semi-definite positive.

Proposition

$\mu = M\nu$ with $\nu = \operatorname{tr}(\mu)$ and M borelian Hermitian semi-definite positive matrix defined ν a.e.

(meaning $|\mu_{ij}| \ll \nu$)

First properties

Proposition

Let $P \in \Psi^m(\Omega)^{L\times N}$ properly supported and (u_n) a pure sequence in $(H^s_{loc}(\Omega))^N$ associated with μ. Then (Pu_n) is pure in $(H^{s-m}_{loc}(\Omega))^L$ and

$$\mu_{[Pu_n]} = \sigma(P)\ \mu_{[u_n]}\ \sigma(P^*).$$

In particular, if $Pu_n \to 0$ in $(H^{s-m}_{loc}(\Omega))^L$ if and only if $\sigma(P)\ \mu_{[u_n]} = 0$.

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In particular, if $Pu_n \to 0$ in $(H^{s-m}_{loc}(\Omega))^L$ <u>if and only if</u> $\sigma(P)\ \mu_{[u_n]} = 0$.

A first consequence is the compensated compactness theorem.

Theorem

Let $P \in \Psi^m(\Omega)^{L\times N}$ and $Q \in (\Psi^{2s}_{ps}(\Omega))^{N\times N}$ such that

$$\sigma(P)h = 0 \quad \Rightarrow \quad \langle \sigma(Q)h, h\rangle = 0.$$

Let $(u_n) \rightharpoonup u$ in $(H^s_{loc}(\Omega))^N$ such that $Pu_n \to 0$ in $(H^{s-m}_{loc}(\Omega))^L$. Then,

$$(Qu_n(x), u_n(x))_{\mathbb{C}^N} \quad \longrightarrow \quad (Qu(x), u(x))_{\mathbb{C}^N} \qquad (\text{in } \mathscr{D}'(\Omega))$$

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In particular, if $Pu_n \to 0$ in $(H^{s-m}_{loc}(\Omega))^L$ if and only if $\sigma(P)\ \mu_{[u_n]} = 0$.

The so-called div-curl lemma is a particular case.

Theorem

$\Omega \subset \mathbb{R}^3$. Let $(E_n), (B_n) \subset (L^2_{loc}(\Omega)^3)$ be such that $E_n \rightharpoonup E$ and $B_n \rightharpoonup B$ with moreover

$$\mathrm{div} E_n \to 0, \text{ in } H^{-1}_{loc}(\Omega)$$
$$\mathrm{curl} B_n \to 0 \text{ in } (H^{-1}_{loc}(\Omega))^3$$

Then $E_n(x)\cdot B_n(x) \to E(x)\cdot B(x)$ in $\mathscr{D}'(\Omega)$.

We saw $u_n \rightharpoonup 0$ in L^2_{loc} and $Pu_n \to 0$ in H^{-m}_{loc} implies $\text{supp}(\mu) \subset \text{Char}(P)$.

Theorem

Let $P \in \Psi^m_{ps}(\Omega)$ such that $\sigma(P) \in \mathbb{R}$ and $\sigma(P - P^*) \in i\mathbb{R}$. Then if $Pu_n \to 0$ in $H^{1-m}(\Omega)$ we have $H_p\mu = 0$.

The measure μ is invariant along the bicharacteristic flow.

The controlled wave equation on Ω reads:

$$\partial_t^2 y - \Delta y = \chi_\omega f, \qquad y_{|t=0} = y_0,\ \partial_t y_{|t=0} = y_1, \qquad y_{|(0,T)\times\partial\Omega} = 0,$$

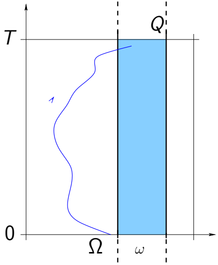
that is, with $Y = (y, \partial_t y)$,

$$\partial_t Y + AY = \mathcal{B} f, \qquad \mathcal{B} = \begin{pmatrix} 0 \\ \chi_\omega \end{pmatrix}$$

with $A = \begin{pmatrix} 0 & -\operatorname{Id} \\ -\Delta & 0 \end{pmatrix}$.

Goal:
Under the Geometrical Control Condition, that is,

every (generalized) geodesic travelled at speed one enters the control region ω,



Goal:
Under the Geometrical Control Condition, prove that the solution to

$$-\partial_t U + A^* U = 0, \qquad (S^*)$$

satisfies

$$\|U(T)\|_H \leq C_{obs} \|\mathcal{B}^* U\|_{L^2(0,T;U)},$$

Goal:

Under the Geometrical Control Condition, prove that the solution to

$$\partial_t^2 u - \Delta u = 0, \qquad u_{|t=0} = u_0,\ \partial_t u_{|t=0} = u_1, \qquad u_{|(0,T)\times\partial\Omega} = 0,$$

satisfies

$$\|u_0\|_{L^2}^2 + \|u_1\|_{H^{-1}}^2 \leq C \int_0^T \|\chi_\omega u(t,.)\|_{L^2}^2 \, dt.$$

Goal:

$$\|u_0\|_{L^2}^2 + \|u_1\|_{H^{-1}}^2 \leq C \int_0^T \|\chi_\omega u(t,.)\|_{L^2}^2 \, dt.$$

1) A compactness argument: proof (by contradiction) of a relaxed observability inequality

$$\|u_0\|_{L^2}^2 + \|u_1\|_{H^{-1}}^2 \leq C \int_0^T \|\chi_\omega u(t,.)\|_{L^2}^2 \, dt + C\Big(\|u_0\|_{H^{-1}}^2 + \|u_1\|_{H^{-2}}^2\Big)$$

Assume $\|u_0\|_{L^2}^2 + \|u_1\|_{H^{-1}}^2 = 1$ and $RHS \to 0$.
Then u weakly converges to 0. Associated with u is a microlocal defect measure μ that is invariant along bicharacteristics and that vanishes above ω

If all geodesics travelled at speed one enters ω within the time interval $(0, T)$
(Geometrical Control Condition), then $\mu \equiv 0$.
$\to$ contradiction.

Goal:

$$\|u_0\|_{L^2}^2 + \|u_1\|_{H^{-1}}^2 \leq C \int_0^T \|\chi_\omega u(t,.)\|_{L^2}^2 \, dt.$$

2) A uniqueness argument.

There are no invisible solution:

$$\chi_\omega u \equiv 0 \quad \Rightarrow u_0 = 0 \text{ and } u_1 = 0.$$

Key argument:

1) space of invisible solutions is finite dimensional by relaxed inequality

2) invariant by the semigroup generator A

3) unique continuation argument for eigenvalues of the Laplace operator.

Goal:

$$\|u_0\|_{L^2}^2 + \|u_1\|_{H^{-1}}^2 \leq C \int_0^T \|\chi_\omega u(t,.)\|_{L^2}^2 \, dt.$$

3) Proof (by contradiction) of the observability inequality Assume $\|u_0\|_{L^2}^2 + \|u_1\|_{H^{-1}}^2 = 1$ and $RHS \to 0$.
then
1) u weakly converges to an invisible solution, that is, zero. In particular, the initial conditions u_0 and u_1 wealy converge to 0.
2) Contradiction with the relaxed inequality.

Case of a moving control domain.

Above the control domain is in fact $Q = (0, T) \times \omega$.

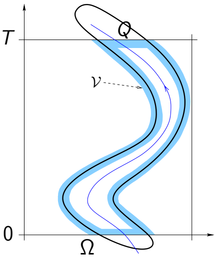
Now we consider control domains is $Q \subset (0, T) \times \Omega$.

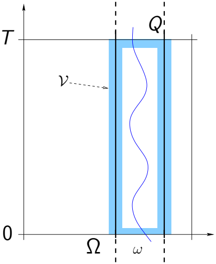
This requires a modification of the original proof for step 2): "the set of invisible solutions" is trivial.

GCC becomes:

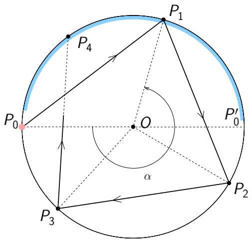
every (generalized) bicharacteristic enters the region Q

[LR, Lebeau, Trélat, Terpolilli]

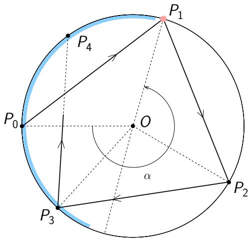




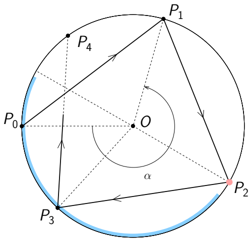
Case of a moving control domain.



$t = 0$

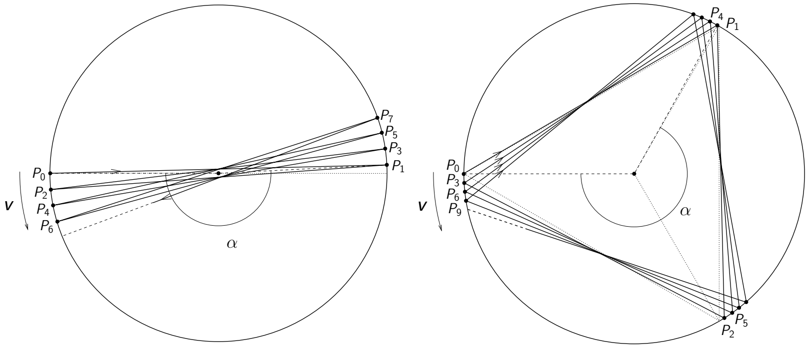


$t = 2\sin(\alpha/2)$



$t = 4\sin(\alpha/2)$

Case of a moving control domain.



Outline

Let $(\mathcal{M}, g)$ be a Riemannian manifold.

Laplace-Beltrami operator :

$$\Delta_g = -|g|^{-1/2} \sum_{1 \leq i,j \leq d} D_i\big(|g|^{1/2} g^{ij} D_j\big), \qquad D = -i\partial.$$

Let $(\mathcal{M}, g)$ be a Riemannian manifold.

Laplace-Beltrami operator with a time-varying magnetic potential $A(t, x)$:

$$\Delta_{g,A} = -|g|^{-1/2} \sum_{1 \leq i,j \leq d} (D_i - A_i)\big(|g|^{1/2} g^{ij} (D_j - A_j)\big), \qquad D = -i\partial.$$

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Associated wave equation with boundary data:

$$(\partial_t^2 - \Delta_{g,A}) y = 0 \text{ in } (0,T) \times \mathcal{M}, \qquad y_{|t=0} = \partial_t y_{|t=0} = 0 \text{ in } \mathcal{M},$$
$$y_{|\Sigma} = f \text{ in } \Sigma = (0,T) \times \partial\mathcal{M}$$

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If $f \in H^{1/2}(\Sigma)$, then $\nu \cdot (D - A) y_{|\Sigma} \in H^{-1/2}(\Sigma)$.

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$$y_{|\Sigma} = f \text{ in } \Sigma = (0,T)\times\partial\mathcal{M}$$

Define the Dirichlet-to-Neumann map

$$\Lambda_A : H^{1/2}(\Sigma) \to H^{-1/2}(\Sigma),$$
$$f \mapsto \nu\cdot(D-A)y_{|\Sigma}.$$

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Associated wave equation with boundary data:

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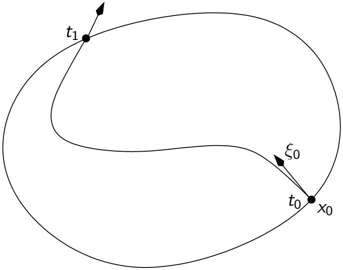
Inverse problem question:

Can we recover A from the knowledge of Λ_A?

Here, we shall be more modest:

We attempt to recover the light-ray transform of A.

Let (x_0, ξ_0) be a the boundary and such that the bicharacteristic shot from it at time $t_0 \in (0, T)$ enters $\mathcal{M}$ and exists it at $t^+ \in (0, T)$.



Let $(t, x(t), \tau, \xi(t))$ be that bicharacteristic. Then

$$\mathcal{L}^1 A(x_0, t_0; \xi_0) = \int_{t_0}^{t^+} \xi(t) \cdot A(t, x(t))\, dt.$$

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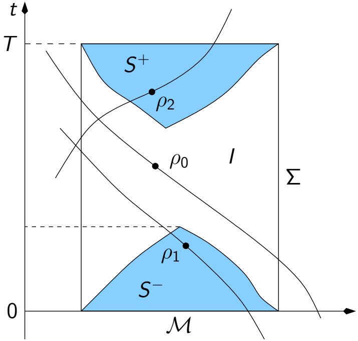

A natural question:

Can we recover $\mathcal{L}^1 A$ in a stable manner?

Lemma

The ajoint operator $\Lambda_A^* : H^{1/2}(\Sigma) \to H^{-1/2}(\Sigma)$ *is given by* $\Lambda_A^* g = \nu \cdot (D - A) u_{|\Sigma}$ *with*

$$(\partial_t^2 - \Delta_A) u = 0, \quad u_{|t=T} = 0, \; \partial_t u_{|t=T} = 0, \quad u_{|\Sigma} = g \in H^{1/2}(\Sigma).$$

Let A_1 and A_2 be two magnetic potentials.

Lemma

We have

$$\langle (\Lambda_{A_1} - \Lambda_{A_2}) f, g \rangle_{H^{-1/2}(\Sigma), H^{1/2}(\Sigma)}$$
$$= \int_0^T \Big(\langle u_1, (A_2 - A_1) \cdot D\, u_2 \rangle_{H^{1/2}(\mathcal{M}), H^{-1/2}(\mathcal{M})}$$
$$+ \langle (A_2 - A_1) \cdot D\, u_1, u_2 \rangle_{H^{-1/2}(\mathcal{M}), H^{1/2}(\mathcal{M})} \Big)\, dt + R,$$

with $R = (r\, u_1, u_2)_{L^2((0,T)\times\mathcal{M})}$, *where* $r = |A_1|^2 - |A_2|^2$.

Here,

$$P_1 u_1 = 0, \quad u_{1|t=0} = 0,\ \partial_t u_{1|t=0} = 0, \quad u_{1|\Sigma} = f \in H^{1/2}(\Sigma),$$

with $P_1 = \partial_t^2 - \Delta_{g,A_1}$ and

$$P_2 u_2 = 0, \quad u_{2|t=T} = 0,\ \partial_t u_{2|t=T} = 0, \quad u_{2|\Sigma} = g \in H^{1/2}(\Sigma),$$

with $P_2 = \partial_t^2 - \Delta_{g,A_2}$.

Idea:

- Concentrate $f^n \in H^{1/2}(\Sigma)$ to a Dirac mass at $(x_0, t_0, \tau_0, \xi_0) \in \mathrm{Char}\, P$, with $\|f^n\|_{H^{1/2}(\Sigma)} \sim 1$
- Then u_1^n solution to

$$P_1 u_1^n = 0, \quad u^n_{1\,|t=0} = 0,\ \partial_t u^n_{1\,|t=0} = 0, \quad u^n_{1\,|\Sigma} = f^n \in H^{1/2}(\Sigma),$$

  with $P_1 = \partial_t^2 - \Delta_{g,A_1}$, is associated with a $H^{1/2}$-measure μ only supported on the selected bicharacteristic γ.
- As $H_p \mu = 0$, then we have a Dirac transported along γ. Thus

$$\int B(x,t,\tau,\xi) d\mu = \int_{t_0}^{t^+} B(x(t),t,\tau,\xi(t))\, dt.$$

- Thus

$$\int_0^T \Big(\langle u_1^n, (A_2 - A_1)\cdot D\, u_1^n \rangle_{H^{1/2}(\mathcal{M}), H^{-1/2}(\mathcal{M})}$$

$$\to \int (A_2 - A_1)(t,x)\cdot \xi\, d\mu = \int_{t_0}^{t^+} (A_2 - A_1)(t, x(t))\cdot \xi(t)\, dt$$

$$= \mathcal{L}_1(A_2 - A_1).$$

- Thus

$$\int_0^T \Big(\langle u_1^n, (A_2 - A_1) \cdot D\, u_1^n \rangle_{H^{1/2}(\mathcal{M}), H^{-1/2}(\mathcal{M})}$$

$$\to \int (A_2 - A_1)(t,x) \cdot \xi \, d\mu = \int_{t_0}^{t^+} (A_2 - A_1)(t, x(t)) \cdot \xi(t)\, dt$$

$$= \mathcal{L}_1(A_2 - A_1).$$

- However, in the integral indentity we have

$$\int_0^T \Big(\langle u_1^n, (A_2 - A_1) \cdot D\, u_2^n \rangle_{H^{1/2}(\mathcal{M}), H^{-1/2}(\mathcal{M})}$$

u_2^n???

- Set $g^n = u^n_{1|\Sigma} \in H^{1/2}(\Sigma)$ localised near the exit point x_1.
- g^n is bounded in $H^{1/2}(\Sigma)$ and $\|g^n\|_{H^{1/2}(\Sigma)} \gtrsim \|f^n\|_{H^{1/2}(\Sigma)} \sim 1$.
- Use g^n (localised away from $t = T$) to generate u^n_2 in a backward manner:

$$P_2 u^n_2 = 0, \quad u^n_{2|t=T} = 0,\ \partial_t u^n_{2|t=T} = 0, \quad u^n_{2|\Sigma} = g^n \in H^{1/2}(\Sigma),$$

  with $P_2 = \partial_t^2 - \Delta_{g,A_2}$.
- Then the cross-measure associated with u^n_1 and u^n_2 is precisely μ.

- In the identity

$$\begin{aligned}\langle(\Lambda_{A_1}-\Lambda_{A_2})f,g\rangle_{H^{-1/2}(\Sigma),H^{1/2}(\Sigma)} \\ = \int_0^T \Big(\langle u_1,(A_2-A_1)\cdot D\,u_2\rangle_{H^{1/2}(\mathcal{M}),H^{-1/2}(\mathcal{M})} \\ + \langle(A_2-A_1)\cdot D\,u_1,u_2\rangle_{H^{-1/2}(\mathcal{M}),H^{1/2}(\mathcal{M})}\Big)\,dt + R,\end{aligned}$$

  The RHS converges to $2\mathcal{L}^1(A_1-A_2)(x_0,t_0,\xi_0)$.
- As we have $\|g^n\|_{H^{1/2}(\Sigma)} \eqsim \|f^n\|_{H^{1/2}(\Sigma)} \sim 1$, for the LHS we find

$$|\langle(\Lambda_{A_1}-\Lambda_{A_2})f,g\rangle_{H^{-1/2}(\Sigma),H^{1/2}(\Sigma)}| \lesssim \|\Lambda_{A_1}-\Lambda_{A_2}\|_{H^{1/2}(\Sigma),H^{-1/2}(\Sigma)}$$

We thus obtain

Theorem (Dos Santos Ferreira, Laurent, LR)

For a non-capured bicharacteristic starting at (x_0, t_0, ξ_0) *we have*

$$|\mathcal{L}^1(A_1 - A_2)(x_0, t_0, \xi_0)| \lesssim \|\Lambda_{A_1} - \Lambda_{A_2}\|_{H^{1/2}(\Sigma), H^{-1/2}(\Sigma)}.$$

- Evidently, microlocalized version of the DtN map can be used $\rightarrow$ partial data.
- Before the exit point x_1 the generalized bicharacteristic may have interacted with the boundary: hyperbolic points, glancing points etc... We only ask for the entrance and exit points to be hyperbolic.