Optimized Schwarz Methods for Time-Harmonic Wave Problems

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http://onelab.info/wiki/GetDDM
http://onelab.info/wiki/DDM_for_Waves

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Outline

1. Introduction to domain decomposition method
2. The Helmholtz case
3. The Maxwell case
4. ONELAB and GetDDM
5. Conclusion
1 Introduction to domain decomposition method

2 The Helmholtz case

3 The Maxwell case

4 ONELAB and GetDDM

5 Conclusion
Reference problem

Scattering of an acoustic wave on an obstacle

\[ \begin{align*}
\n\mathbb{R}^3 \setminus \overline{\Omega^-} \\
(\Delta + k^2)u &= 0 \\
\lim_{||x|| \to +\infty} \left( \nabla u \cdot \frac{x}{||x||} - iku \right) &= 0
\end{align*} \]

With...

- \( k \): wavenumber; \( u^{inc} = e^{ikx \cdot \alpha} \): incident plane wave
- Sommerfeld radiation condition at infinity

Practical applications

- Communication between submarines
- Electromagnetic waves in urban environment
Reference problem

**FE: truncation of the domain**

\[
\begin{align*}
(\Delta + k^2)u &= 0 \quad (\Omega) \\
u &= -u^{inc} \quad (\Gamma) \\
\lim_{\|\mathbf{x}\| \to +\infty} (\nabla u \cdot \frac{\mathbf{x}}{\|\mathbf{x}\|} - iku) &= 0 \\
\partial_{\mathbf{n}} u - iku &= 0 \quad (\Gamma^\infty)
\end{align*}
\]

With . . .

- \(\mathbf{n}\): unit outwardly directed vector to \(\Omega\)
- Simple **Absorbing Boundary Condition (ABC)** on \(\Gamma^\infty\) (not the topic here)
Domain decomposition method

Numerical solution: major problems

- **Solution is a wave**: mesh refinement (typical element size: $\pi/(5k)$)
- **High frequency** ($\lambda := \frac{2\pi}{k} \ll L$): direct solving impossible
- **Indefinite operator**: iterative solving hard if not impossible
Numerical solution: major problems

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- **Indefinite operator**: iterative solving hard if not impossible

**Hybrid method: Domain Decomposition Method (DDM)**

- Mesh decomposition
- Iterative algorithm
- Result
Domain decomposition method: principle and origin

\[ \begin{align*}
\Omega \\
\left\{ \begin{array}{ll}
-\Delta u & = f \quad (\Omega) \\
u & = 0 \quad (\Gamma)
\end{array} \right.
\end{align*} \]
Domain decomposition method: principle and origin

\[ \begin{align*}
\Omega_1 & \quad \Omega_2 \\
\Sigma_{1,2} & \quad \Sigma_{2,1}
\end{align*} \]

\[ \begin{cases}
-\Delta u = f & (\Omega) \\
u = 0 & (\Gamma)
\end{cases} \]

**Schwarz alternating method (H. Schwarz (1870))**

\[ \begin{align*}
-\Delta u_1^{n+1} &= f & (\Omega_1) \\
u_1^{n+1} &= 0 & (\Gamma_1) \\
u_1^{n+1} &= u_2^n & (\Sigma_{1,2})
\end{align*} \]

\[ \begin{align*}
-\Delta u_2^{n+1} &= f & (\Omega_2) \\
u_2^{n+1} &= 0 & (\Gamma_2) \\
u_2^{n+1} &= u_1^{n+1} & (\Sigma_{2,1})
\end{align*} \]

And glue the solutions in the overlap.
Domain decomposition method: principle and origin

- The Helmholtz case
- The Maxwell case
- ONELAB and GetDDM
- Conclusion

Domain decomposition method: principle and origin

\[ \begin{align*}
\Omega_1 & \quad \Omega_2 \\
\Sigma_{1,0} & \quad \Sigma_{2,1}
\end{align*} \]

Additive Schwarz method

\[ \begin{align*}
\begin{cases}
-\Delta u & = f \\
u & = 0 \quad (\Gamma)
\end{cases}
\end{align*} \]

\[
\begin{align*}
\begin{cases}
-\Delta u_1^{n+1} & = f \\
u_1^{n+1} & = 0 \\
u_1^{n+1} & = u_2^n 
\end{cases} \quad \begin{cases}
-\Delta u_2^{n+1} & = f \\
u_2^{n+1} & = 0 \\
u_2^{n+1} & = u_1^n 
\end{cases}
\end{align*} \]

And glue the solutions in the overlap.
Schwarz method and Helmholtz equation

Limitations

- (Very) slow convergence
- Overlap is mandatory
- Even with overlap, the algorithm does not converge for Helmholtz equation
Schwarz method and Helmholtz equation

Limitations

- (Very) **slow convergence**
- Overlap is **mandatory**
- Even with overlap, the algorithm **does not converge for Helmholtz equation**

Simple case

\[
\begin{cases}
(\partial_{xx} + \partial_{yy})u_1^{n+1} + k^2u_1^{n+1} = 0 & x \in (-\infty, L), y \in \mathbb{R}, \\
u_1^{n+1}(L, y) = u_2^n(L, y),
\end{cases}
\]

\[
\begin{cases}
(\partial_{xx} + \partial_{yy})u_2^{n+1} + k^2u_2^{n+1} = 0 & x \in (0, +\infty), y \in \mathbb{R}, \\
u_2^{n+1}(0, y) = u_1^n(0, y).
\end{cases}
\]
Schwarz method and Helmholtz equation

Fourier transform in the $y$ direction ($\xi = \text{Fourier variable}$)

\[
\begin{align*}
\partial_{xx} \hat{u}_{1}^{n+1} + (k^2 - \xi^2) \hat{u}_{1}^{n+1} &= 0, \quad x \in (-\infty, L), \xi \in \mathbb{R}, \\
\hat{u}_{1}^{n+1}(L, \xi) &= \hat{u}_{2}^{n}(L, \xi), \\
\partial_{xx} \hat{u}_{2}^{n+1} + (k^2 - \xi^2) \hat{u}_{2}^{n+1} &= 0, \quad x \in (0, +\infty), \xi \in \mathbb{R}, \\
\hat{u}_{2}^{n+1}(0, \xi) &= \hat{u}_{1}^{n}(0, \xi),
\end{align*}
\]

Solution of the ODE

\[
\begin{align*}
\hat{u}_{1}^{n+1}(0, x) &= e^{-2\sqrt{\xi^2 - k^2}L} \hat{u}_{1}^{n-1}(0, x), \\
\hat{u}_{2}^{n+1}(L, x) &= e^{-2\sqrt{\xi^2 - k^2}L} \hat{u}_{2}^{n-1}(L, x),
\end{align*}
\]

Convergence factor

\[
\rho := e^{-2\sqrt{\xi^2 - k^2}L} = \begin{cases} 
  e^{-2i\sqrt{k^2 - \xi^2}L} & \text{if } \xi^2 \leq k^2, \\
  e^{-2\sqrt{\xi^2 - k^2}L} & \text{otherwise}.
\end{cases}
\]
Absolute value of the convergence factor

\[ |\rho| := \begin{cases} 
1 & \text{if } \xi^2 \leq k^2 \\
\exp\left(-2\sqrt{\xi^2 - k^2}L\right) & \text{otherwise.}
\end{cases} \]

(Propagative modes)

(Evanescent modes)
**Schwarz method and Helmholtz equation**

**Absolute value of the convergence factor**

\[ |\rho| \begin{cases} 
1 & \text{if } \xi^2 \leq k^2 \\
 e^{-2\sqrt{\xi^2-k^2} L} & \text{otherwise.}
\end{cases} \]

- (Propagative modes)
- (Evanescent modes)

**Solution**

P-L. Lions algorithm: Fourier-Robin type transmission condition
Non-overlapping domain decomposition method

Decompose the domain (here $N = 2$)

\[ \Omega \quad \Gamma \quad \Omega^\infty \]

Recast the system into 2 coupled systems

\[
\begin{cases}
(\Delta + k^2)u_1 &= 0 \quad (\Omega_1) \\
u_1 &= -u^{inc} \quad (\Gamma_1) \\
(\partial_n - ik)u_1 &= 0 \quad (\Gamma_1^\infty) \\
(\partial_n + S_1)u_1 &= (\partial_n + S_1)u_2 \quad (\Sigma_{1,2})
\end{cases}
\]

\[
\begin{cases}
(\Delta + k^2)u_2 &= 0 \quad (\Omega_2) \\
u_2 &= -u^{inc} \quad (\Gamma_2) \\
(\partial_n - ik)u_2 &= 0 \quad (\Gamma_2^\infty) \\
(\partial_n + S_2)u_2 &= (\partial_n + S_2)u_1 \quad (\Sigma_{2,1})
\end{cases}
\]

$S_j$: Transmission operators
Non-overlapping domain decomposition method

Parallel Schwarz algorithm

Introducing surface unknown $g_{ij} := (\partial_n i + S_i)u_j$, the algorithm reads (iteration $n$ to $n+1$):

1. Solve the $N$ independant problems

\[
\begin{align*}
(\Delta + k^2)u_1^{n+1} & = 0 \quad (\Omega_1) \\
\partial_n u_1^{n+1} & = -u^{inc} \quad (\Gamma_1) \\
(\partial_n - ik)u_1^{n+1} & = 0 \quad (\Gamma_1^\infty) \\
(\partial_n + S_1)u_1^{n+1} & = g_{12} \quad (\Sigma_{1,2})
\end{align*}
\]

\[
\begin{align*}
(\Delta + k^2)u_2^{n+1} & = 0 \quad (\Omega_2) \\
\partial_n u_2^{n+1} & = -u^{inc} \quad (\Gamma_2) \\
(\partial_n - ik)u_2^{n+1} & = 0 \quad (\Gamma_2^\infty) \\
(\partial_n + S_2)u_2^{n+1} & = g_{21} \quad (\Sigma_{2,1})
\end{align*}
\]

2. Update the surface unknown

\[
\begin{align*}
g_{12}^{n+1} & = -g_{21}^n + (S_1 + S_2)u_2^{n+1} \quad (\Sigma_{1,2}) \\
g_{21}^{n+1} & = -g_{12}^n + (S_1 + S_2)u_1^{n+1} \quad (\Sigma_{2,1})
\end{align*}
\]
Non-overlapping domain decomposition method

Gather the surface unknown in one vector

\[ g = (g_{i,j})_{i,j} \]

One iteration of the algorithm reads as:

\[ g^{n+1} = A g^n + b \]

- \( A \): iteration operator. Applying \( A \) is amount to solving \( N \) volume PDEs + \( N \) surface PDEs (with \( u^{\text{inc}} = 0 \))

- \( b \): right-hand side, containing physical information (\( u^{\text{inc}} \)).

At convergence, \( g \) is solution to:

\[ (I - A)g = b \]  \hspace{1cm} (1)

Krylov acceleration

System (1) can be solved using a Krylov subspace solver.
Non-overlapping domain decomposition method

2 subdomains and DtN

Let \( \Lambda_j : H^{1/2}(\Sigma) \to H^{-1/2}(\Sigma) \) be the DtN (Dirichlet-to-Neumann) map associated to \( \Omega_j \):

\[
\Lambda_j f = \partial_n w_j, \quad \text{on } \Sigma.
\]

with \( w_j \) solution of

\[
\left\{
\begin{array}{l}
(\Delta + k^2) w_j = 0 \quad \text{in } \Omega_j, \\
w_j = 0 \quad \text{on } \Gamma_j, \\
\partial_n w_j - ik w_j = 0 \quad \text{on } \Gamma_j^\infty, \\
w_j = f \quad \text{on } \Sigma.
\end{array}
\right.
\]

Then, if \( S_j = -\Lambda_j \), the algorithm converges in 2 iterations.

Remark

Entended to \( N \) subdomains: convergence in \( N \) iterations.
Non-overlapping domain decomposition method

One-dimensional case

\[
\begin{aligned}
  u'' + k^2 u &= 0, & \text{in } [0, 1], \\
  u(0) &= e^{ikx} = 1, \\
  u'(1) - iku(1) &= 0.
\end{aligned}
\]

Solution

\[ u(x) = e^{ikx} \]

Exact DtN

\[ \Lambda = ik \]
Non-overlapping domain decomposition method
Non-overlapping domain decomposition method

Residual

Solution

Error
Non-overlapping domain decomposition method
Non-overlapping domain decomposition method

Residual

Solution

Error
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Non-overlapping domain decomposition method
Non-overlapping domain decomposition method

Introduction

The Helmholtz case

The Maxwell case

ONELAB and GetDDM

Conclusion

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**Residual**

**Solution**

**Error**
Non-overlapping domain decomposition method
Non-overlapping domain decomposition method
Non-overlapping domain decomposition method
Non-overlapping domain decomposition method

Residual

Solution

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Non-overlapping domain decomposition method

Two major investigation fields

1. **Transmission condition**: find a suitable approximation of $-\Lambda_j$
2. **Coarse space**: decrease the linear convergence rate (in terms of $N$)
Non-overlapping domain decomposition method

Two major investigation fields

1. **Transmission condition**: find a suitable approximation of $-\Lambda_j$
2. **Coarse space**: decrease the linear convergence rate (in terms of $N$)

Problem

The DtN map is **non-local** and therefore is not suitable for FE framework.
Non-overlapping domain decomposition method

Two major investigation fields

1. Transmission condition: find a suitable approximation of $-\Lambda_j$
2. Coarse space: decrease the linear convergence rate (in terms of $N$)

Problem

The DtN map is non-local and therefore is not suitable for FE framework.

Available methods

- Local approaches: Taylor, Padé, ...
- Integral Equation (Joly et. al)
- PML (Vion and Geuzaine)
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Transmission Operators for Helmholtz equation

Half-space case with straight interface $\Sigma$

$$\left\{ \begin{array}{ll}
\Delta u + k^2 u = 0 & \text{in } \mathbb{R}_+^3 = \{ \mathbf{x} \in \mathbb{R}^3; x_1 > 0 \}, \\
u = g & \text{on } \Sigma, \\
u \text{ is outgoing},
\end{array} \right.$$

Fourier transform (variable $\xi$ along $\Sigma$)

$$\partial_n u(0, \xi) = \mathcal{F}_\xi^{-1} (\sigma(\xi) \hat{u}(0, \xi))|_{\Sigma}.$$  

Symbol of the DtN

$$\sigma^{sq}(\xi) = ik \sqrt{1 - \frac{|\xi|^2}{k^2}}.$$  

DtN map

$$\Lambda^{sq} := \text{Op}(\sigma^{sq}) = ik \sqrt{1 + \frac{\Delta \Sigma}{k^2}}.$$
Model problem for convergence analysis

Model problem with two subdomains and a circular interface.

\[
\begin{cases}
(\Delta + k^2)u_0 = 0 & (\Omega_0) \\
\partial_{n_0} u_0 + Su_0 = g_0 & (\Sigma) \\
\lim_{|x| \to +\infty} |x|^{1/2}(\partial_{|x|} u_0 - iku_0) = 0
\end{cases}
\]

\[
\begin{cases}
(\Delta + k^2)u_1 = 0 & (\Omega_1) \\
\partial_{n_1} u_1 + Su_1 = g_1 & (\Sigma)
\end{cases}
\]
Model problem for convergence analysis

Rewrite $A$

$$A = \begin{pmatrix} 0 & T_0 \\ T_1 & 0 \end{pmatrix}, \quad T_j g_j^n = -g_j^n + 2S u_j^{n+1}.$$  

Modal decomposition

$$u_0 = \sum_m \alpha_m H^{(1)}_m(kr) e^{im\theta}, \quad u_1 = \sum_m \beta_m J_m(kr) e^{im\theta},$$

$$S = \sum_m S_m e^{im\theta}, \quad T_j = \sum_m T_{j,m} e^{im\theta}, \quad g_j = \sum_m g_{j,m}(r) e^{im\theta}.$$  

Recurrence relation

$$g_j^{n+1} = T_{0,m} T_{1,m} g_j^{n-1}.$$  

Convergence factor

$$\forall m, \quad \rho_m := T_{0,m} T_{1,m} = \begin{bmatrix} -kZ_{0,m} + S_m \\ kZ_{0,m} + S_m \end{bmatrix} \cdot \begin{bmatrix} -kZ_{1,m} + S_m \\ kZ_{1,m} + S_m \end{bmatrix},$$

$$Z_{0,m} = -\frac{H^{(1)'}_m(kR_0)}{H^{(1)}_m(kR_0)} \quad \text{and} \quad Z_{1,m} = \frac{J_m(kR_0)}{J_m(kR_0)}.$$  

Remark: $(S_m = 0) \Rightarrow (\rho_m = 1)$
Model problem for convergence analysis

Square root operator

\[ S^{sq} = -\Lambda^{sq} = -ik\sqrt{1 - \frac{\Delta \Sigma}{k^2}}. \]

Modal decomposition

\[ S_m^{sq} = -ik\sqrt{1 - \frac{m^2}{k^2 R_0^2}}. \]

Remark: if \( m^2 = k^2 R_0^2 \) then \( \rho_m^{sq} = 1 \).
Transmission Operators for Helmholtz equation

Impedance Boundary Condition (IBC) [Després, 1991]

Low frequency approximation ($\xi \to 0$):

$$\rho_{\text{sq}}(\xi) = ik \sqrt{1 - \frac{|\xi|^2}{k^2}} \approx ik.$$  

$$\mathcal{S}_{\text{IBC}} u = -iku.$$
Transmission Operators for Helmholtz equation

Optimized Order 2 [Gander, Magoulès and Nataf, 2002]

\[ \sigma^{sq}(\xi) = ik\sqrt{1 - \frac{|\xi|^2}{k^2}} \approx a(\delta\xi) - b(\delta\xi)\xi^2, \]

where \( a \) and \( b \) are solution of the min-max problem

\[ \min_{\alpha, \beta \in \mathbb{C}} \left( \max_{\xi_{\min} \in (0, k-\delta\xi) \cup (k+\delta\xi, \xi_{\max})} |\tilde{\rho}(\xi; \alpha, \beta)| \right), \]

where \( \tilde{\rho} \) is the convergence factor in the case \((-\infty, 0] \times \mathbb{R} \) and \([0, +\infty) \times \mathbb{R} \):

\[ \tilde{\rho}(\xi) = \left| \frac{\sigma^{sq}(\xi) - \sigma^{oo2}(\xi; a, b)}{\sigma^{sq}(\xi) + \sigma^{oo2}(\xi; a, b)} \right|^2. \]
Transmission Operators for Helmholtz equation

Optimized Order 2 [Gander, Magoulès and Nataf, 2002]

\[
\sigma^{sq}(\xi) = \nu k \sqrt{1 - \frac{|\xi|^2}{k^2}} \approx a(\delta \xi) - b(\delta \xi)\xi^2.
\]

\[
S^{002} u = a(\delta \xi) u + b(\delta \xi) \Delta_{\Sigma} u,
\]
Transmission Operators for Helmholtz equation

Modified DtN [Boubendir, Antoine and Geuzaine, 2012]

\[ S^{\text{sq},\varepsilon} u = -\kappa \sqrt{1 + \frac{\Delta \Sigma}{k^2}} u, \]

where \( k_\varepsilon = k + \nu \varepsilon \) and \( \varepsilon > 0 \).

Optimal \( \varepsilon \)

Searching for \( \varepsilon_{opt} \) such that:

\[ \min_{\varepsilon > 0} \max_m |\rho_m^{\text{sq},\varepsilon}|. \]

Assume that \( \max_m |\rho_m^{\text{sq},\varepsilon}| \) is reached on \( m = kR_0 \), then we can prove that

\[ \varepsilon_{opt} \approx 0.39k^{1/3}R_0^{-2/3}. \]

Formally extended to other curves by (\( \mathcal{H} \): local mean curvature):

\[ \varepsilon_{opt} \approx 0.39k^{1/3} \mathcal{H}^{2/3}. \]
Transmission Operators for Helmholtz equation

Modified DtN [Boubendir, Antoine and Geuzaine, 2012]

\[ S^{sq,\varepsilon} u = -ik\sqrt{1 + \frac{\Delta \Sigma}{k^{2\varepsilon}}} u, \]

where \( k_{\varepsilon} = k + i\varepsilon \) and \( \varepsilon = 0.39k^{1/3}\mathcal{H}^{2/3} \) (\( \mathcal{H} \): local mean curvature).

Modal decomposition

\[ S_{m}^{sq,\varepsilon} = -ik\sqrt{1 - \frac{m^2}{k^{2\varepsilon} R_0^2}} \quad \text{and} \quad |\rho_{m}^{sq,\varepsilon}| < 1, \quad \forall m. \]
Transmission Operators for Helmholtz equation

Classical Padé approximants on square root

\[
\sqrt{1 + X} \approx R_{N_p}(X) = c_0 + \sum_{\ell=1}^{N_p} \frac{a_\ell X}{1 + b_\ell X}, 
\]

\(N_p\) is the number of Padé approximants.

Localization of the nonlocal operator \(S^{sq, \varepsilon} u = -ik\sqrt{1 + \frac{\Delta \Sigma}{k_\varepsilon^2}} u\)

\[
S^{\text{GIBC}}(N_p, \varepsilon) u = -ikc_0 u - ik \sum_{\ell=1}^{N_p} a_\ell \text{div}_\Sigma \left( \frac{1}{k_\varepsilon^2} \nabla \Sigma \right) \left( I + b_\ell \text{div}_\Sigma \left( \frac{1}{k_\varepsilon^2} \nabla \Sigma \right) \right)^{-1} u. 
\]
Transmission Operators for Helmholtz equation

Modal decomposition for different number of $N_p$
Transmission Operators for Helmholtz equation

Vanishing modes are not well approximated

\[ S_{m}^{sq,\varepsilon} = -ik \sqrt{1 - \left( \frac{m^2}{k^2 \varepsilon R_0^2} \right)}. \]

Complex Padé approximants on square root (\( \alpha \): rotation of the branch cut)

\[ \sqrt{1 + X} = e^{i\alpha/2} \sqrt{(1 + X) e^{-i\alpha}} \approx R_{N_p}^{\alpha}(X) = C_0(\alpha) + \sum_{\ell=1}^{N_p} \frac{A_\ell(\alpha) X}{1 + B_\ell(\alpha) X}. \]

Localization of the nonlocal operator \( S^{sq,\varepsilon} u \)

\[ S^{GIBC}(N_p, \alpha, \varepsilon) u = -ik \sqrt{1 + \frac{\Delta \Sigma}{k^2 \varepsilon}} u \]

\[ S^{GIBC}(N_p, \alpha, \varepsilon) u = -ik C_0(\alpha) u - ik \sum_{\ell=1}^{N_p} A_\ell(\alpha) \text{div}_\Sigma \left( \frac{1}{k^2 \varepsilon} \nabla \Sigma \right) \left( I + B_\ell(\alpha) \text{div}_\Sigma \left( \frac{1}{k^2 \varepsilon} \nabla \Sigma \right) \right)^{-1} u. \]
Transmission Operators for Helmholtz equation

Modal decomposition for different number of $N_p$ and $\alpha = \pi/4$
Transmission Operators for Helmholtz equation

Eigenvalue distribution in the complex plane for the exact and Padé-localized square-root transmission operator of order 4 (left) and 8 (right).
Transmission Operators for Helmholtz equation

Eigenvalues distribution with respect to the number of Padé approximants
Comparison of the transmission operators

Convergence factor

![Graph of Convergence Factor](image-url)
Comparison of the transmission operators

Eigenvalue distribution in the complex plane for \((I - \mathcal{A})\)
Numerical Example

“concentric-” and “pie-” decomposition
Numerical Example

Convergence for the “circle-concentric” decomposition. Number of iterations vs. wavenumber.
Convergence for the "circle-pie" decomposition. Number of iterations vs. wavenumber.
Numerical Example

Convergence for the “circle-concentric” decomposition. Number of iterations vs. mesh density
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Non-Overlapping DDM for Maxwell

P-L. Lions algorithm

From iteration $n$ to $n+1$:

1. Solve, for $i = 1, \ldots, N$:

$$
\begin{align*}
\text{curl curl } E_i^{(n+1)} - k^2 E_i^{(n+1)} &= 0 & \text{in } \Omega_i, \\
\gamma_i^T(E_i^{(n+1)}) &= -\gamma_i^T(E_{\text{inc}}) & \text{in } \Gamma_i^D, \\
-\frac{i}{k} \gamma_i^t(\text{curl } E_i^{(n+1)}) + S(\gamma_i^T(E_i^{(n+1)})) &= g_{ij}^{(n)} & \text{in } \Sigma_{ij}.
\end{align*}
$$

2. Update surface quantities:

$$
\begin{align*}
g_{ji}^{(n+1)} &= \frac{i}{k} \gamma_i^t(\text{curl } E_i^{(n+1)}) + S(\gamma_i^T(E_i^{(n+1)})) \\
&= -g_{ij}^{(n)} + 2S(\gamma_i^T(E_j^{(n+1)})), \quad \text{on } \Sigma_{ij}.
\end{align*}
$$

where we have introduce the trace operators:

$$
\gamma_i^t : v_i \mapsto n_i \times v_i|_{\partial \Omega_i} \quad \text{and} \quad \gamma_i^T : v_i \mapsto n_i \times (v_i|_{\partial \Omega_i} \times n_i).
$$
Non-Overlapping DDM for Maxwell

Krylov acceleration
As for the Helmholtz case, the whole algorithm can be recast into a linear system:

\[(I - A)g = b.\]

Transmission operators
Again, the transmission operator \( S \) has a direct impact on the iteration operator \( A \).
Transmission Operators for Maxwell

Half-space problem \((\Omega := (-\infty, 0) \times \mathbb{R}^2)\)

\[
\begin{aligned}
curl \mathbf{H} + ik \mathbf{E} &= 0 \quad (\Omega) \\
curl \mathbf{E} - ik \mathbf{H} &= 0 \quad (\Omega) \\
\gamma^T(\mathbf{E}) &= -\gamma^T(\mathbf{E}_{\text{inc}}) \quad (\Sigma) \\
\lim_{\|\mathbf{x}\| \to \infty} \|\mathbf{x}\| \left( \mathbf{E} + \frac{\mathbf{x}}{\|\mathbf{x}\|} \times \mathbf{H} \right) &= 0
\end{aligned}
\]

Surface electric and magnetic currents

\[
\mathbf{J} = \mathbf{n} \times \mathbf{E}, \quad \mathbf{M} = \mathbf{n} \times \mathbf{H}
\]

MtE

\[
\mathbf{M} + \Lambda^{sq}(\mathbf{n} \times \mathbf{J}) = 0,
\]

with

\[
\Lambda^{sq} = (\Lambda_1^{sq})^{-1} \Lambda_2^{sq},
\]

\[
\Lambda_1^{sq} = \left( \mathbf{I} + \nabla_{\Sigma} \frac{1}{k^2} \text{div}_{\Sigma} - \text{curl}_{\Sigma} \frac{1}{k^2} \text{curl}_{\Sigma} \right)^{1/2}, \quad \Lambda_2^{sq} = \left( \mathbf{I} - \frac{1}{k^2} \text{curl}_{\Sigma} \text{curl}_{\Sigma} \right).
Transmission Operators for Maxwell

\[
\Lambda^{sq} = (\Lambda^{sq}_1)^{-1} \Lambda^{sq}_2,
\]

\[
\Lambda^{sq}_1 = \left(I + \nabla_{\Sigma} \frac{1}{k^2} \text{div}_{\Sigma} - \text{curl}_{\Sigma} \frac{1}{k^2} \text{curl}_{\Sigma} \right)^{1/2}, \quad \Lambda^{sq}_2 = \left(I - \frac{1}{k^2} \text{curl}_{\Sigma} \text{curl}_{\Sigma} \right).
\]

0th-order transmission condition IBC(0) [Després, 1992]

\[
S_{\text{IBC}(0)}(\gamma^T(E)) = \gamma^T(E).
\]

Optimized impedance boundary condition GIBC(\(\alpha\)) [Alonso-Rodriguez and Gerardo-Giorda, 2006]

\[
S_{\text{GIBC}(\alpha)}(\gamma^T(E)) = \alpha \left(I - \frac{1}{k^2} \text{curl}_{\Sigma} \text{curl}_{\Sigma} \right) \gamma^T(E),
\]

where \(\alpha\) is chosen thanks to an optimization process.
Transmission Operators for Maxwell

\[ \Lambda^{sq} = (\Lambda_1^{sq})^{-1} \Lambda_2^{sq}, \]
\[ \Lambda_1^{sq} = \left( \mathbf{I} + \nabla_{\Sigma} \frac{1}{k^2} \text{div}_{\Sigma} - \text{curl}_{\Sigma} \frac{1}{k^2} \text{curl}_{\Sigma} \right)^{1/2}, \quad \Lambda_2^{sq} = \left( \mathbf{I} - \frac{1}{k^2} \text{curl}_{\Sigma} \text{curl}_{\Sigma} \right). \]

Optimized second-order GIBC(\(\alpha, \beta\)) [Rawat and Lee, 2010]

\[ S_{\text{GIBC}}(a, b)(\gamma^T(E)) = \left( \mathbf{I} + \frac{a}{k^2} \nabla_{\Sigma} \text{div}_{\Sigma} \right)^{-1} \left( \mathbf{I} - \frac{b}{k^2} \text{curl}_{\Sigma} \text{curl}_{\Sigma} \right) \gamma^T(E), \]

where \(a\) and \(b\) are chosen so that an optimal convergence rate is obtained for the (TE) and (TM) modes.

This condition has been generalized in [Dolean, Gander, Lanteri, Lee and Peng, 2015].
Transmission Operators for Maxwell

Modified square root operator

\[
\Lambda^{sq,\varepsilon} = (\Lambda_{1}^{sq,\varepsilon})^{-1} \Lambda_{2}^{sq,\varepsilon},
\]
\[
\Lambda_{1}^{sq,\varepsilon} = \left( I + \nabla_{\Sigma} \frac{1}{k^{2}\varepsilon} \text{div}_{\Sigma} - \text{curl}_{\Sigma} \frac{1}{k^{2} \varepsilon} \text{curl}_{\Sigma} \right)^{1/2}, \quad \Lambda_{2}^{sq,\varepsilon} = \left( I - \frac{1}{k^{2} \varepsilon} \text{curl}_{\Sigma} \text{curl}_{\Sigma} \right).
\]

Padé-localized square-root transmission condition \( \text{GIBC}(N_{p}, \alpha, \varepsilon) \) [El Bouajaji, Antoine, Geuzaine, Thierry, 2014]

\[
\mathcal{S}_{\text{GIBC}}(N_{p}, \alpha, \varepsilon)(\gamma^{T}(E)) = \left( C_{0} + \sum_{\ell=1}^{N_{p}} A_{\ell} X (I + B_{\ell} X)^{-1} \right)^{-1} \left( I - \text{curl}_{\Sigma} \frac{1}{k^{2} \varepsilon} \text{curl}_{\Sigma} \right) \gamma^{T}(E),
\]

with \( X := \nabla_{\Sigma} \frac{1}{k^{2} \varepsilon} \text{div}_{\Sigma} - \text{curl}_{\Sigma} \frac{1}{k^{2} \varepsilon} \text{curl}_{\Sigma} \), and where \( k \varepsilon, C_{0}, A_{\ell} \) and \( B_{\ell} \) are defined as in the Helmholtz case.
Transmission Operators for Maxwell

Convergence Analysis for a Model Problem

Model problem with two subdomains and a spherical interface:

\[ \Omega_0 = \{ x \in \mathbb{R}^3, ||x|| > R_0 \}, \quad \Omega_1 = \{ x \in \mathbb{R}^3, ||x|| < R_0 \}. \]
Convergence Analysis for a Model Problem

Let

\[
\begin{align*}
A_{m,1} &= \iota \mu_m,\varepsilon \xi_m^{(1)'}(kR_0) - \xi_m^{(1)}(kR_0), \\
B_{m,1} &= \iota \mu_m,\varepsilon \psi_m'(kR_0) - \psi_m(kR_0), \\
A_{m,2} &= \iota \xi_m^{(1)'}(kR_0) - \mu_m,\varepsilon \xi_m^{(1)}(kR_0), \\
B_{m,2} &= \iota \psi_m'(kR_0) - \mu_m,\varepsilon \psi_m(kR_0), \\
A_{m,3} &= \iota \mu_m,\varepsilon \psi_m'(kR_0) + \psi_m(kR_0), \\
B_{m,3} &= \iota \mu_m,\varepsilon \xi_m^{(1)'}(kR_0) + \xi_m^{(1)}(kR_0), \\
A_{m,4} &= \iota \psi_m'(kR_0) + \mu_m,\varepsilon \xi_m^{(1)}(kR_0), \\
B_{m,4} &= \iota \xi_m^{(1)'}(kR_0) + \mu_m,\varepsilon \xi_m^{(1)}(kR_0),
\end{align*}
\]

where

- \( \mu_m,\varepsilon = 1 - \frac{m(m+1)}{k^2 \varepsilon R^2} \)
- \( \psi_m \) and \( \zeta_m \) are respectively the first- and second-kind Ricatti-Bessel functions of order \( m \)
- \( \xi_m^{(1)} = \psi_m + \iota \zeta_m \) is the first-kind spherical Hankel’s function of order \( m \)
We can show that:

\[
g^{(n+1),m} = \begin{pmatrix}
  (g_{12}^{(n+1),m})_1 \\
  (g_{12}^{(n+1),m})_2 \\
  (g_{21}^{(n+1),m})_1 \\
  (g_{21}^{(n+1),m})_2 \\
\end{pmatrix} = A_m g^{(n),m} := \begin{pmatrix}
  0 & 0 & \frac{B_{m,1}}{A_{m,3}} & 0 \\
  0 & 0 & 0 & \frac{B_{m,2}}{A_{m,4}} \\
  B_{m,3} & 0 & 0 & 0 \\
  \frac{B_{m,4}}{A_{m,2}} & 0 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix}
  g_{12}^{(n),m} \\
  g_{12}^{(n),m} \\
  g_{21}^{(n),m} \\
  g_{21}^{(n),m} \\
\end{pmatrix}
\]

with \( A_m \) the iteration matrix for a mode \( m \geq 1 \), with eigenvalues

\[
\lambda_{m,1} = \sqrt{\frac{B_{m,1}}{A_{m,1}} \frac{B_{m,3}}{A_{m,3}}} = -\lambda_{m,2}, \quad \lambda_{m,3} = \sqrt{\frac{B_{m,2}}{A_{m,2}} \frac{B_{m,4}}{A_{m,4}}} = -\lambda_{m,4}.
\]

One can prove that \( A = \text{diag}((A_m)_{m \geq 1}) \). Therefore, studying the global convergence of the DDM for \( A \) requires the spectral study of the modal iteration matrices \( A_m \), for \( m \geq 1 \).
Convergence Analysis for a Model Problem

Quasi-optimality of GIBC(post, ε)

One can prove that $\rho(\Lambda_m) < 1, \forall m \geq 1$, and that

$$\lim_{m \to \infty} \lambda_{m,(1,3)} = -\lim_{m \to \infty} \lambda_{m,(2,4)} = \frac{i\varepsilon}{2k + i\varepsilon},$$

i.e., we have two opposite accumulation points in the complex plane for the evanescent modes.

Optimal parameters for GIBC(α) and GIBC(α, β)

In what follows, the optimal parameters $\alpha$ and $\beta$ were computed numerically by solving the min-max problem

$$\min_{(\alpha, \beta) \in \mathbb{C}^2} \max_{m \geq 1} \rho(\Lambda_m)$$

with the Matlab function fminsearch.
Convergence Analysis for a Model Problem

Influence of the Padé approximation on the eigenvalue distribution.

\[ \text{GIBC(sq, } \varepsilon) \]
\[ \text{GIBC}(1, \pi/2, \varepsilon) \]
\[ \text{GIBC}(1, \pi/2, \varepsilon) \]

\[ \text{GIBC}(2, \pi/2, \varepsilon) \]
\[ \text{GIBC}(2, \pi/2, \varepsilon) \]

\[ \text{GIBC}(4, \pi/2, \varepsilon) \]
\[ \text{GIBC}(4, \pi/2, \varepsilon) \]

\[ \text{GIBC}(8, \pi/2, \varepsilon) \]
\[ \text{GIBC}(8, \pi/2, \varepsilon) \]
Convergence Analysis for a Model Problem

Eigenvalue distribution in the complex plane for $(I - A)$ and different transmission operators.
Convergence Analysis for a Model Problem

Spectral radius of $\Delta_m$ for different transmission operators.
Numerical example

Concentric cylinder decomposition: Number of GMRES iterations vs. wavenumber \((N_{\text{dom}} = 5, n_\lambda = 20)\).
Concentric cylinder decomposition (TE case): Number of GMRES iterations vs. wavenumber ($N_{\text{dom}} = 5$, $n_{\lambda} = 20$).
Numerical example

Concentric cylinder decomposition (TM case): Number of GMRES iterations vs. wavenumber \((N_{\text{dom}} = 5, \ n_\lambda = 20)\).
Numerical example

GMRES convergence history for different Padé orders.
Numerical example

Falcon jet \((N_{\text{dom}} = 4, \lambda = 10, n_\lambda = 10)\)
Numerical example

Falcon jet \((N_{\text{dom}} = 4, \lambda = 10, n_{\lambda} = 10)\)
Numerical example: scalability issue

Concentric cylinder decomposition (TM case): of iterations vs. number of subdomains ($k = 30$, $n_\lambda = 20$).
1 Introduction to domain decomposition method

2 The Helmholtz case

3 The Maxwell case

4 ONELAB and GetDDM

5 Conclusion
Open Numerical Engineering LABoratory

Provides ready-to-use finite element codes for different community.

- Magnetostatic
- Acoustic time reversal
- 2D Acoustic scattering
- GetDDM
- ...

http://onelab.info. Available on Android and iOS markets
GetDDM

**A simple, flexible and ready-to-use environment**

- Direct link between discrete and continuous weak-formulations

\[
\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega \quad \forall v \longleftrightarrow 1 \text{ Galerkin } \{ \text{ Grad Dof} \{u\}, \{\text{Grad} u\} \}; \\
2 \text{ In Omega; Jacobian JVol; Integration I1; } \}
\]

- Parallelism made simple
- Click & run: GUI, full examples and scripts, numerous geometries, ...
GetDDM

a. circle_concentric  
b. circle_pie  
c. cylinder_concentric  
d. sphere_concentric  
e. waveguide2d  
f. waveguide3d  
g. cobra  
h. marmousi

**Figure:** Sample models available online at http://onelab.info/wiki/GetDDM.
The Marmousi model is a synthetic 2D acoustic model that reproduces the complex velocity profile of a slice of earth. It features a wide range of speeds, from 1500 m/s to 5500 m/s, with many layers and normal faults as depicted on Figure 3.17. It has become a classic test case for benchmarking seismic inversion codes. We will simulate the propagation of the time-harmonic acoustic waves produced by a point source located at coordinates (6200, 2300) as was done in [169], so both methods can be compared.

We solve the Helmholtz equation with Sommerfeld boundary conditions on all sides except the top side where we impose a homogeneous Neumann boundary condition; we will also consider the case of Sommerfeld conditions on all sides, since we assume that this situation was used in [169]. We will also test the methods in a similar domain with an homogeneous medium; Figure 3.18 shows typical solutions in all these configurations.

In the following, we report the results of our experiments in tables for each of the configurations, with decompositions into 16, 64 and 256 domains. We have tested two different transmission conditions: the GIBC(2) and the IBC(0), which is the most simple one; each case has been run twice, with decompositions into vertical and horizontal layers. The tables present the number of iterations for each run to converge with a relative residual decrease of $10^{-3}$, with a maximum number of iterations of 1000. Values in parentheses are an estimation of the normalized time required to reach convergence when using as many CPUs as there are subdomains. The time unit is the time to solve a single subproblem. Hence, these values cannot be directly compared for different decompositions or frequencies.

Velocity profile and pressure field. Dimensions: $9192m \times 2904m$. 700 Hz ($\sim 4000\lambda$ in the domain) with $N = 358$ subdomains on 4296 CPUs: $> 2.3$ billions unknowns.
Remark: also works on non academic cases
1 Introduction to domain decomposition method

2 The Helmholtz case

3 The Maxwell case

4 ONELAB and GetDDM

5 Conclusion
Conclusion

Efficient transmission condition for Helmholtz and Maxwell

- Quasi-optimal in wavenumber and mesh refinement
- Suitable for FE framework

Open source implementation readily available for testing:

- Preprint, code and examples on http://onelab.info/wiki/GetDDM
- Work from laptops to massively parallel computer clusters:
  - marmousi.pro test-case (Helmholtz) at $700\, Hz$ (approx. 4000 wavelengths in the domain) with $N = 358$ subdomains on 4296 CPUs: $> 2.3$ billions unknowns.
  - waveguide3d.pro test-case (Maxwell) with $N = 3,500$ subdomains on 3,500 CPUs (cores): $> 300$ million unknowns.
Perspectives

Mathematics side

- “Padé” operator:
  - Fine analysis on the number of Padé approximants
  - Stability at high frequency regime

- Optimization method on complexified square root operator

GetDDM

- Link with HPDDDM library (P. Jolivet, P-H. Tournier, F. Nataf)
- Automatic partitioning (Scotch, Metis, ...)
- “Production mode”: real physical cases