Des solutions qui changent de signe de l'équation de la chaleur non linéaire avec données initiales positives

en collaboration avec
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The nonlinear heat equation

We discuss the Cauchy problem

$$\begin{cases} \partial_t u - \Delta u = |u|^\alpha u & \text{in } (0,T) \times \mathbb{R}^N, \\ u(0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{NLH}$$

where $\alpha > 0$.

Some known facts

- (NLH) is invariant under dilation, self-similar solutions are solutions invariant under this action.
- well posedness in $C_0(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ for $p > \frac{N\alpha}{2}$.
- ill posedness: let $\mu > 0$ and $u_0(x) = \mu|x|^{-\frac{2}{\alpha}}$.
  - $\frac{2}{N} < \alpha < \frac{4}{N-2} =: \alpha_S$ no local pos. sol. for $\mu > 0$ large.
  - $0 < \alpha \leq \frac{2}{N}$ no local pos. sol. for all $\mu > 0$.
  - $\alpha \geq \alpha_S$ no self-similar sol. for $\mu > 0$ large.

Question: are there solutions coming from positive initial data for which no positive solutions exist?

Self-similar solutions

One difficulty: usual fixed point methods
$u_{n+1} = S(t)u_0 + \int_0^t S(t-s)u_n(s)$ preserve positivity.

We turn to radial self-similar solutions
$u(t,x) = t^{-\frac{1}{\alpha}} f\left(\frac{x}{\sqrt{t}}\right) = \left(\frac{|x|}{\sqrt{t}}\right)^{\frac{2}{\alpha}} f\left(\frac{|x|}{\sqrt{t}}\right) |x|^{-\frac{2}{\alpha}}$
We would like to find $a \in \mathbb{R}$ such that

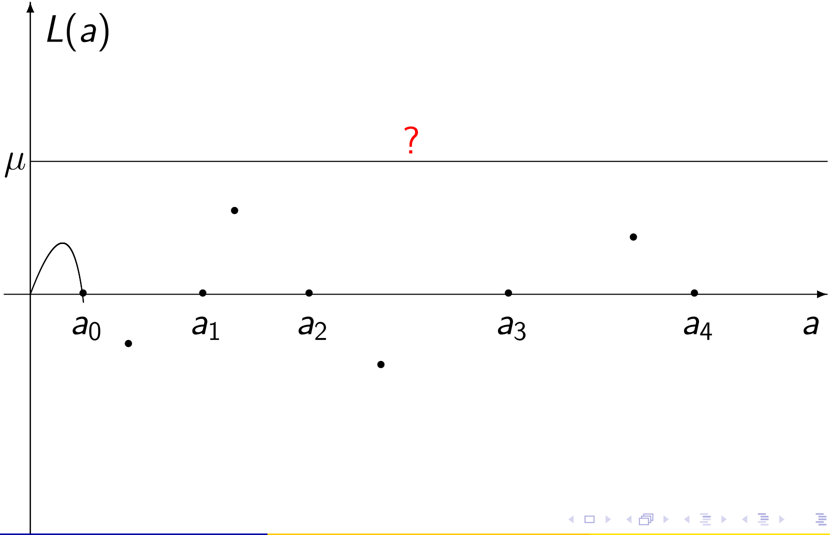
$$\begin{cases} f'' + \left(\frac{N-1}{r} + \frac{r}{2}\right) f' + \frac{1}{\alpha} f + |f|^\alpha f = 0, \\ f(0) = a,\ f'(0) = 0, \end{cases}$$

with $\lim_{r\to\infty} r^{\frac{2}{\alpha}} f(r) = \mu$ (so that $\lim_{t\to 0} u(t,x) = \mu|x|^{-\frac{2}{\alpha}}$.)

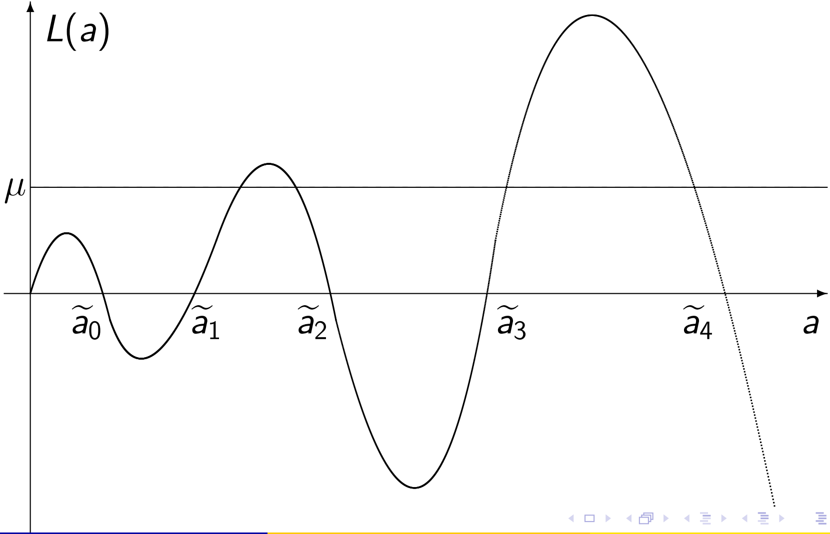
Known results about L

- for all $\alpha > 0$, $L(a) = \lim_{r\to\infty} r^{\frac{2}{\alpha}} f_a(r)$ is Lipschitz.
- f_a has a finite number $N(a)$ of zeros.
- if $\alpha \geq \alpha_S$ then $L(a) \neq 0$.
- if $\alpha < \alpha_S$, there exists $a_m \nearrow \infty$ s.t. $L(a_m) = 0$ and $N(a_m) = m$, $m \geq m_0$.
- $m_0 = 0$ iff $\frac{2}{N} < \alpha < \alpha_S$.
- if $L(a) \neq 0$ then $N(a)$ is stable under small perturbations of a.
- there exists $a_m < a < a_{m+1}$ s.t. $L(a) > 0$.

Summary



Main result


$L(a)$
μ
$\widetilde{a}_0$
$\widetilde{a}_1$
$\widetilde{a}_2$
$\widetilde{a}_3$
$\widetilde{a}_4$
a

Main result II

Let $0 < \alpha < \alpha_S$. Given $\mu > 0$ there exists $m_0 \geq 0$ s.t. for all $m \geq m_0$ there exist two radial self-similar solutions with initial value $\mu|x|^{-\frac{2}{\alpha}}$ and having exactly m zeros.

Remark:

- if $0 < \alpha \leq \frac{2}{N}$: no positive sol. with $u_0(x) = \mu|x|^{-\frac{2}{\alpha}}$, $\mu > 0$, but infinitely many sign-changing sol's.
- if $\frac{2}{N} < \alpha < \alpha_S$: no positive sol. with $u_0(x) = \mu|x|^{-\frac{2}{\alpha}}$, $\mu > 0$ large, but infinitely many sign-changing sol's.

Main ideas of the proof

The inverted profile $w(s) = s^{-\frac{1}{\alpha}} f\left(\frac{1}{\sqrt{s}}\right)$ satisfies

$$4s^2 w'' + 4\gamma s w' - w' - \beta w + |w|^\alpha w = 0,$$

where $\gamma > 1$ and β depend on α, N.

Note that $\lim_{s\to 0} w(s) = \lim_{r\to\infty} r^{\frac{2}{\alpha}} f(r)$.

We set $w(0) = \mu$. The solution is global. We wish $\lim_{r\to 0} f(r) = \lim_{s\to\infty} s^{\frac{1}{\alpha}} w(s) < \infty$.
This solves the problem (if $N \neq 1$).

Long time behavior

Write

$$Lw = 4s^2 w'' + 4\gamma s w' - \beta w = w' - |w|^\alpha w.$$

Then $Ls^{-\lambda_1} = Ls^{-\lambda_2} = 0$,

where $\lambda_1 = \frac{1}{\alpha}$, $\lambda_2 = \frac{1}{\alpha} - \frac{N-2}{2}$. ($\beta = -4\lambda_1\lambda_2$.)

If $v(t) = w(s)$, $t = \log s$ then

$$4v'' + 4(\gamma - 1)v' - e^{-t}v' + \beta v - |v|^\alpha v = 0,$$

$\gamma = 1 + \lambda_1 + \lambda_2$ $(> 1 \Leftrightarrow \alpha < \alpha_S)$

Suppose $N \geq 3$, $\alpha < \frac{2}{N-2}$. Then $0 < \lambda_2 < \lambda_1$. Set $w_\mu(0) = \mu$, $w'_\mu(k\mu^{-\frac{1}{\alpha}}) = 0$, $k > 0$.

We have

- the number of zeros $\mathcal{N}(\mu)$ of w_μ is finite.
- $\mathcal{N}(\mu) \to \infty$ as $\mu \to \infty$.
- $w_\mu(s) \to 0$ as $s \to \infty$.
- $\mathcal{L}(\mu) = \lim_{s\to\infty} s^{\lambda_2} w_\mu(s)$ is well defined and continuous.
- If $\mathcal{L}(\mu) = 0$ then $\lim_{s\to\infty} s^{\lambda_1} w_\mu(s) \neq 0$.

- if $w_\mu(s) = 0$ then $w'_\mu(s) \neq 0$.
- If $\mathcal{L}(\mu) \neq 0$ then $\mathcal{N}(\mu)$ est stable.
- if $\mathcal{L}(\mu) = 0$, then $\mathcal{N}(\nu) = \mathcal{N}(\mu)$ or $\mathcal{N}(\nu) = \mathcal{N}(\mu) + 1$ for $\nu \approx \mu$.

With these ingredients we close the proof.

The proof in the rest of the cases

- if $\frac{2}{N-2} < \alpha$ $(\lambda_2 < 0)$ then $w(s) \to \{0, \pm\beta^{1/\alpha}\}$.
- if $\alpha = \frac{2}{N-2}$ $(\lambda_2 = 0)$ slow decay means $(\log s)^{\frac{1}{\alpha}} w(s) \to \left(\frac{2}{\alpha}\right)^{\frac{2}{\alpha}}$.
- if $N = 2$ then $\lambda_1 = \lambda_2$ and slow decay means $s^{\lambda_1}(\log s)^{-1} w(s) \to a \neq 0$.
- if $N = 1$, then $\lambda_1 < \lambda_2$. We look for w s.t. $\lim_{s\to\infty} s^{\lambda_1} w(s) = 0$.

The case $N = 1$

- $\lambda_1 < \lambda_2$. Slow mode $s^{-\frac{1}{\alpha}}$ is stable.
- profile f is regular, $f(0)$ and $f'(0)$ are well-defined, but $f'(0) \neq 0$, in general.
- fast decay means $f(0) = 0$, $f'(0) \neq 0$ (odd solutions). They are associated to changes in the number of zeros of f.
- even solutions $f(0) \neq 0$, $f'(0) = 0$ can also be obtained.

Extensions by perturbation

Let U be a solution of (NLH) such that $|U(t,x)|^\alpha \leq C(t+|x|^2)^{-1}$, $\varphi \in C_0(\Omega)$, $\varphi(x) = 0$ if $|x| < \delta$.

- if $\Omega = \mathbb{R}^N$, there exists a local in time sign-changing solution $V = U + w$, $w(t) \in C_0(\mathbb{R}^N)$, $w(0) = \varphi$.
- if $\Omega \subset \mathbb{R}^N$ is bdd, there exists a local in time sign-changing solution $V = \eta U + w$, $w(t) \in C_0(\Omega)$

Question: can the red part be erased?

More questions

- the case $u_0(x) = \mu|x|^{-\eta}$, $\eta > \frac{2}{\alpha}$.
- can we find u_0 for which there is no solution?
- problem $u_t - \Delta u = |u|^\alpha u + |u|^\eta u$, $\eta > \alpha$
- study of the singular solutions corresponding to $w_\mu(s) \approx s^{-\lambda_2}$.
- for $N = 1$, study of general homogeneous initial data.