Convex representation for curvature dependent functionals

Antonin Chambolle

CMAP, Ecole Polytechnique, CNRS, Palaiseau, France

joint work with **T. Pock** (T.U. Graz)

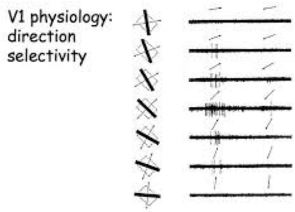
Laboratoire Jacques-Louis Lions, 25 nov. 2016

Introduction

- based on "roto-translation" group;
- a simple formula for curvature-dependent line energies;
- a general relaxation for functions;
- tightness result (C^2 sets);
- dual formulation and link with previous works [Bredies-Pock-Wirth'15];
- numerical results

Curvature information: a "natural" idea

Experiments and discovery of Hubel-Wiesel (62, 77)



Observation: the brain reacts to orientation. Corresponding cells are stacked and connected together to provide sensitivity to curvature. First mathematical theories: Koenderink-van Doorn (87), Hoffman (89), Zucker (2000), Petitot-Tondut (98/2003), Citti-Sarti (2003/2006).

Main idea: use the sub-Riemanian structure of the roto-translation group ($(a, R) \in SE(2) \simeq \mathbb{R}^2 \rtimes SO(2) \sim \mathbb{R}^2 \times \mathbb{S}^1$ in dimension 2) to describe the geometry of the visual cortex $\rightarrow$ sub-Riemanian diffusion and mean curvature motion (Citti-Sarti 3/6, Duits-Franken 10, Boscain et al 14, Citti et al, 2015) for inpainting.

Variational approaches

To complete contours, Mumford (94) suggested to use the "elastica" functional

$$\int_\gamma \kappa^2 d\mathcal{H}^1$$

for contour completion. (Idea suggested by psychological experiments, *cf* for instance Kanizsa 1980.) General theory by Masnou-Morel 98.
Issues: not lower semicontinuous. Studied by Bellettini-Mugnai 2004/2005, Nardi (PhD 2011), Dayrens-Masnou 16, Ambrosio-Masnou 2003. *[Examples]*
Minimisation is computationally challenging. A few approaches trying to exploit the roto-translation metric: Schoenemann with Cremers (2007), Kahl and Cremers (2009), Masnou and Cremers (2011): discrete approach on a graph (or LP) where vertices encode position and orientation (also, El Zehiry-Grady 2010, ...);
Length computation by JM Mirebeau (anisotropic Eikonal equations, 2014)

Variational approaches

Bredies-Pock-Wirth 2013, 2015: "vertex" penalization ("TVX"), then general energies $\int_\gamma f(x, \tau, \kappa)$, f convex, $f \geq 1$. Need to "lift" the image in $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}$ where last component $=$ curvature, with compatibility condition.
This work: a new (and simpler) representation for the latter approach (with $f(\kappa)$).

Example: a C^2 curve

$\gamma(t)$ planar curve, with $|\dot{\gamma}| = 1$ ($\dot{\gamma} = \tau_\gamma$), and $\ddot{\gamma} = \kappa_\gamma \tau_\gamma^\perp$.
Lifted as $\Gamma(t) = (\gamma(t), \theta(t))$ where $\tau_\gamma = (\cos\theta, \sin\theta)$.
Then: the length of $\Gamma(t)$ in $\Omega \times \mathbb{S}^1$ is

- Finite: sub-Riemanian structure, local metric is infinite in direction $\theta^\perp$ (we will also take into account orientation);
- Given by $\int_0^L \sqrt{\dot{\gamma}^2 + \dot{\theta}^2} dt = \int_0^L \sqrt{1 + \kappa^2} dt$: encoding curvature penalization information.

Example: a C^2 curve

Let now $f : \mathbb{R} \to \mathbb{R}$ be convex, assume $f \geq 1$, and consider the energy

$$\int_0^L f(\kappa) = \int_0^L f(\dot{\Gamma}^\theta(t))dt.$$

Observe that if one considers a reparametrization $\lambda(s)$, $s \in [0, a]$, of the curve Γ, then $\lambda^x(s)$ is a reparametrization of γ, $\dot{\lambda}^x = |\dot{\lambda}^x|\tau$, $\kappa = d\theta/dt = \dot{\lambda}^\theta ds/dt = \dot{\lambda}^\theta/|\dot{\lambda}^x|$ hence the energy becomes

$$\int_0^L f(\kappa)dt = \int_0^a f(\dot{\lambda}^\theta/|\dot{\lambda}^x|)|\dot{\lambda}^x|ds.$$

Example: a C^2 curve

Denoting σ the measure (charge) in $\mathcal{M}^1(\Omega \times \mathbb{S}^1; \mathbb{R}^3)$ defined by the curve $\Gamma(t)$:

$$\int_{\Omega \times \mathbb{S}^1} \psi \cdot \sigma = \int_0^L \psi(\Gamma(t)) \cdot \dot{\Gamma}(t) dt,$$

one obtains that

$$\int_0^L f(\kappa) = \int_{\Omega \times \mathbb{S}^1} \bar{h}(\sigma^x \cdot \theta, \sigma^\theta)$$

where

$$\bar{h}(s,t) = \begin{cases} sf(t/s) & \text{if } s > 0, \\ f^\infty(t) & \text{if } s = 0, \\ +\infty & \text{else.} \end{cases}$$

where $f^\infty(t) = \lim_{s \to 0} sf(t/s)$ is the recession function of f.

Example: a C^2 curve

It is standard that if f is convex lsc, then also h is, with

$$\bar{h}(s,t) = \sup\left\{as + bt : a + f^*(b) \le 0\right\}.$$

In addition, as $\sigma^x = \lambda\theta$ where λ is a positive measure in $\Omega \times \mathbb{S}^1$, introducing for $p = (p^x, p^\theta) \in \mathbb{R}^3$

$$h(\theta, p) = \begin{cases} \bar{h}(p^x \cdot \theta, p^\theta) & \text{if } p^x \cdot \theta = |p^x| \Leftrightarrow p^x \parallel \theta, p^x \cdot \theta \ge 0 \\ +\infty & \text{else,} \end{cases}$$

which encodes the sub-Riemanian structure of $\Omega \times \mathbb{S}^1$: we also have

$$\int_0^L f(\kappa) = \int_{\Omega \times \mathbb{S}^1} \bar{h}(\sigma^x \cdot \theta, \sigma^\theta) = \int_{\Omega \times \mathbb{S}^1} h(\theta, \sigma).$$

Example: a C^2 curve

Now, observe that $\operatorname{div}\sigma = \delta_{\Gamma(L)} - \delta_{\Gamma(0)}$, in particular if γ is a closed curve or has its endpoints on $\partial\Omega$, then $\operatorname{div}\sigma = 0$.

Obviously, if one considers the marginal $\bar{\sigma} = \int_{\mathbb{S}^1} \sigma^x \in \mathcal{M}^1(\Omega;\mathbb{R}^2)$ defined by

$$\int_{\Omega\times\mathbb{S}^1} (\psi, 0)\cdot\sigma = \int_\Omega \psi\cdot\bar{\sigma}$$

for any $\psi \in C_c(\Omega;\mathbb{R}^2)$, then it also has zero divergence (as it vanishes if $\psi = \nabla\phi$ for some ϕ). In dimension 2, it follows that (assuming Ω is connected) there exists a BV function u such that $Du^\perp = \bar{\sigma}$. In our case, u is the characteristic function of a set E with $\partial E \cap \Omega = \gamma([0,T]) \cap \Omega$.

Generalization to BV functions

One can define for any $u \in BV(\Omega)$

$$F(u) = \inf \left\{ \int_{\Omega\times\mathbb{S}^1} h(\theta,\sigma) \;:\; \operatorname{div}\sigma = 0, \int_{\mathbb{S}^1} \sigma^x = Du^\perp \right\}.$$

If we assume that $f(t) \geq \sqrt{1+t^2}$, then one sees that $\bar{h}(s,t) \geq \sqrt{s^2+t^2}$ and $\int_{\Omega\times\mathbb{S}^1} h(\theta,\sigma) \geq \int_{\Omega\times\mathbb{S}^1} |\sigma|$. It easily follows that the "inf" is a min, and that F defines a convex, lower semicontinuous function on BV with $F(u) \geq |Du|(\Omega)$.
From the example above, we readily see that if E is a C^2 set, then

$$F(\chi_E) \leq \int_{\partial E} f(\kappa)\, d\mathcal{H}^1.$$

Tightness of the representation

We can show the following result:

Theorem if E is a C^2 set, then

$$F(\chi_E) = \int_{\partial E} f(\kappa) d\mathcal{H}^1.$$

Proof: we need to show $\geq$. In other words, we need to show the obvious fact that if σ is a measure with $\int_{\mathbb{S}^1} \sigma^x = D\chi_E^\perp$, then σ, above ∂E, consists at least in the measure defined by the lifted curve above ∂E (with its orientation as third component).
Maybe there is a simple way to do this (as it is obvious). We used S. Smirnov's theorem which shows that if σ is a measure with $\operatorname{div} \sigma = 0$, then it is a superposition of curves.

Smirnov's Theorem A (1994)

If $\operatorname{div} \sigma = 0$ then it can be decomposed in the following way:

$$\sigma = \int_{\mathfrak{C}_1} \lambda d\mu(\lambda), \quad |\sigma| = \int_{\mathfrak{C}_1} |\lambda| d\mu(\lambda),$$

where λ are of the form

$$\lambda_\gamma = \tau_\gamma \mathcal{H}^1 \llcorner \gamma$$

for rectifiable (possibly closed) curves $\gamma \subset \Omega \times \mathbb{S}^1$ of length at most one. ($\mathfrak{C}_1$ is the corresponding set.)

[We do not need here the more precise "Theorem B"]

Smirnov's Theorem A (1994)

Thanks to the fact that the decomposition is convex (ie with $|\sigma| = \int_{\mathfrak{C}_1} |\lambda| d\mu(\lambda)$) we can show that $|\sigma|$-a.e., for μ-a.e. curve λ one has $\sigma/|\sigma| = \lambda/|\lambda|$ $|\lambda|$-a.e., and in particular λ^x is oriented along θ, and

$$\int_{\Omega \times \mathbb{S}^1} h(\theta, \sigma) = \int_{\mathfrak{C}_1} \left(\int_{\Omega \times \mathbb{S}^1} h(\theta, \lambda) \right) d\mu(\lambda) = \int_{\mathfrak{C}_1} \left(\int_\gamma h(\theta, \tau_\gamma) d\mathcal{H}^1 \right) d\mu(\lambda_\gamma).$$

The horizontal projection λ^x is a rectifiable curve, and one can deduce that its curvature is a bounded measure.
For this we reparametrize λ with the length of λ^x: that is we define $\tilde{\lambda}(t) = \lambda(s(t))$ in such a way that $\mathcal{H}^1(\tilde{\lambda}^x([0,t])) = t)$ [if simple]. Then we show that $\tilde{\lambda}^\theta(t)$, which is the orientation of the tangent [because the energy is finite], has bounded variation.

Tightness

Then one can show that if

$$\Gamma^+ = \{x \in \partial E \cap \lambda^x(0,L) : \text{ the curves have the same orientation } \}$$

then a.e. on Γ^+, the absolutely continuous part of the curvature $\kappa = \dot{\tilde{\lambda}}^\theta$ coincides with κ_E. Using that for any set I,

$$\int_{\lambda^x(I)} f(\kappa^a) \leq \int_{I \times \mathbb{S}^1} h(\theta, \lambda),$$

which more or less follows because this is precisely the way we have built h, we can deduce since $\kappa^a = \kappa_E$ a.e.:

$$\int_{\partial E} f(\kappa_E) d\mathcal{H}^1 = \int_{\mathfrak{C}_1} \int_{\partial E \cap \lambda^x} f(\kappa^a) d\mu(\lambda) \leq \int_{\mathfrak{C}_1} \int_{\partial E \times \mathbb{S}^1} h(\theta, \lambda) d\mu(\lambda)$$

which implies our inequality.

Tightness

- More cases?
- We know that F can be below the standard (L^1) relaxation of $\int_{\partial E} f(\kappa)$ (Bellettini-Mugnai 04/05, Dayrens-Masnou 16) (simple examples).

Dual representation

We can compute the dual problem of

$$F(u) = \inf \left\{ \int_{\Omega \times \mathbb{S}^1} h(\theta, \sigma) \,:\, \operatorname{div} \sigma = 0, \int_{\mathbb{S}^1} \sigma^x = Du^\perp \right\}.$$

by the standard perturbation technique, which consists in defining

$$G(p) = \inf \left\{ \int_{\Omega \times \mathbb{S}^1} h(\theta, \sigma + p) \,:\, \operatorname{div} \sigma = 0, \int_{\mathbb{S}^1} \sigma^x = Du^\perp \right\},$$

showing (exactly as for F) that $p \mapsto G(p)$ is (weakly-$*$) lsc and therefore that $G^{**} = G$, and in particular

$$F(u) = G(0) = \sup_{\eta \in C_0^0(\Omega \times \mathbb{S}^1; \mathbb{R}^3)} -G^*(\eta)$$

Dual representation

Then, it remains to compute $G^*(\eta)$:

$$G^*(\eta) = \sup_{\substack{p,\sigma:\operatorname{div}\sigma=0\\ \int_{\mathbb{S}^1}\sigma=Du^\perp}} \int_{\Omega\times\mathbb{S}^1} \eta\cdot p - h(\theta,\sigma+p)$$

$$= \sup_{\substack{\sigma:\operatorname{div}\sigma=0\\ \int_{\mathbb{S}^1}\sigma=Du^\perp}} -\int_{\Omega\times\mathbb{S}^1} \eta\cdot\sigma + \sup_p \eta\cdot(\sigma+p) - h(\theta,\sigma+p)$$

We find $\underline{\theta}\cdot\eta^x + f^*(\eta^\theta) \le 0$, and then $\eta = \psi(x) + \nabla\varphi(x,\theta)$ so that:

$$F(u) = \sup\left\{ \int_\Omega \psi\cdot Du^\perp \;:\; \psi\in C_c^0(\Omega;\mathbb{R}^2),\right.$$
$$\left. \exists\,\varphi\in C_c^1(\Omega\times\mathbb{S}^1), \underline{\theta}\cdot(\nabla_x\varphi+\psi) + f^*(\partial_\theta\varphi)\le 0\right\}.$$

$\rightarrow$ SAME as Bredies-Pock-Wirth' 2015... This is how we find out that this representation is a simpler variant of theirs...

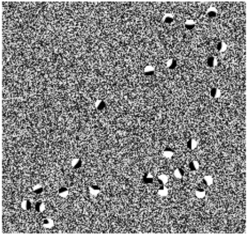
Numerical discretization

This is work in progress. We have a few approaches which work in theory but yield poorly concentrated measures σ. And better approaches which are not clearly justified.

We use both the primal and dual representation and solve the discretized problem using a saddle-point optimisation.

Examples: shape completion

(a) Original shape



(b) Input

(c) Inpainted shape

Figure : Weickert's cat: Shape completion using the function $f_2 = \sqrt{1 + k|\kappa|^2}$.

Examples: shape denoising

Examples: shape denoising

(a) AC, $\lambda = 8$

(b) AC, $\lambda = 4$

(c) AC, $\lambda = 2$

(d) EL, $\lambda = 8$

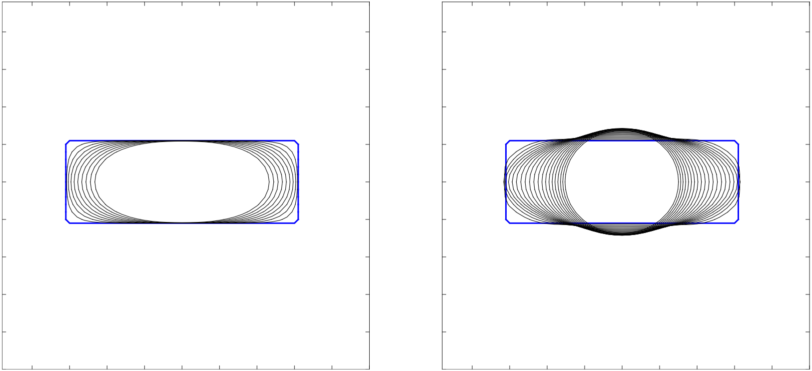
(e) EL, $\lambda = 4$

(f) EL, $\lambda = 2$

Figure : Shape denoising: First row: Using the function $f_1 = 1 + k|\kappa|$, second row: Using the function $f_3 = 1 + k|\kappa|^2$.

Examples: Willmore flow

(*cf* for instance Dayrens-Masnou-Novaga 2016)



(a) AC (b) EL

Figure : Motion by the gradient flow of different curvature depending energies. Energy $1 + |\kappa|$ gives the same as standard mean curvature flow for convex curves. Elastica/Willmore flow converges to a circle (shrinkage is still present due to the length term).

Conclusion, perspectives

- We have introduced a relatively simple systematic way to represent curvature-dependent energies in $2D$;
- It simplifies the (energetically equivalent) framework of [Bredies-Pock-Wirth 15];
- Open questions: characterize the functions for which the relaxation is tight (conjecture: functions with "continuous" curvature?);
- Discretization needs some improvement (issues: measure with orientation constraint).

Conclusion, perspectives

- We have introduced a relatively simple systematic way to represent curvature-dependent energies in $2D$;
- It simplifies the (energetically equivalent) framework of [Bredies-Pock-Wirth 15];
- Open questions: characterize the functions for which the relaxation is tight (conjecture: functions with "continuous" curvature?);
- Discretization needs some improvement (issues: measure with orientation constraint).

Thank you for your attention