Convex representation for curvature dependent functionals

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Introduction

- based on “roto-translation” group;
- a simple formula for curvature-dependent line energies;
- a general relaxation for functions;
- tightness result ($C^2$ sets);
- dual formulation and link with previous works [Bredies-Pock-Wirth’15];
- numerical results
Curvature information: a “natural” idea

Experiments and discovery of Hubel-Wiesel (62, 77)

**Observation:** the brain reacts to orientation. Corresponding cells are stacked and connected together to provide sensitivity to curvature. First mathematical theories: Koenderink-van Doorn (87), Hoffman (89), Zucker (2000), Petitot-Tondut (98/2003), Citti-Sarti (2003/2006).

**Main idea:** use the sub-Riemannian structure of the roto-translation group \((a, R) \in SE(2) \simeq \mathbb{R}^2 \rtimes SO(2) \simeq \mathbb{R}^2 \times S^1\) in dimension 2) to describe the geometry of the visual cortex \(\rightarrow\) sub-Riemannian diffusion and mean curvature motion (Citti-Sarti 3/6, Duits-Franken 10, Boscain et al 14, Citti et al, 2015) for inpainting.
Variational approaches

To complete contours, Mumford (94) suggested to use the “elastica” functional
\[
\int_\gamma \kappa^2 d\mathcal{H}^1
\]
Variational approaches

Bredies-Pock-Wirth 2013, 2015: “vertex” penalization (“TVX”), then general energies $\int_{\gamma} f(x, \tau, \kappa)$, $f$ convex, $f \geq 1$. Need to “lift” the image in $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}$ where last component = curvature, with compatibility condition.
This work: a new (and simpler) representation for the latter approach (with $f(\kappa)$).
Example: a $C^2$ curve

$\gamma(t)$ planar curve, with $|\dot{\gamma}| = 1$ ($\dot{\gamma} = \tau_\gamma$), and $\ddot{\gamma} = \kappa_\gamma \tau_\gamma^\perp$.

Lifted as $\Gamma(t) = (\gamma(t), \theta(t))$ where $\tau_\gamma = (\cos \theta, \sin \theta)$.

Then: the length of $\Gamma(t)$ in $\Omega \times S^1$ is

- Finite: sub-Riemanian structure, local metric is infinite in direction $\theta^\perp$ (we will also take into account orientation);

- Given by $\int_0^L \sqrt{\dot{\gamma}^2 + \dot{\theta}^2} \, dt = \int_0^L \sqrt{1 + \kappa^2} \, dt$: encoding curvature penalization information.
Example: a $C^2$ curve

Let now $f : \mathbb{R} \to \mathbb{R}$ be convex, assume $f \geq 1$, and consider the energy

$$\int_0^L f(\kappa) = \int_0^L f(\dot{\gamma}(t))dt.$$ 

Observe that if one considers a reparametrization $\lambda(s)$, $s \in [0, a]$, of the curve $\Gamma$, then $\lambda^x(s)$ is a reparametrization of $\gamma$, $\lambda^x = |\lambda^x| \tau$, $\kappa = d\theta/dt = \lambda^\theta ds/dt = \lambda^\theta / |\lambda^x|$ hence the energy becomes

$$\int_0^L f(\kappa)dt = \int_0^a f(\lambda^\theta / |\lambda^x|) |\lambda^x| ds.$$
Example: a $C^2$ curve

Denoting $\sigma$ the measure (charge) in $\mathcal{M}^1(\Omega \times S^1; \mathbb{R}^3)$ defined by the curve $\Gamma(t)$:

$$\int_{\Omega \times S^1} \psi \cdot \sigma = \int_0^L \psi(\Gamma(t)) \cdot \dot{\Gamma}(t) dt,$$

one obtains that

$$\int_0^L f(\kappa) = \int_{\Omega \times S^1} \bar{h}(\sigma^x \cdot \theta, \sigma^\theta)$$

where

$$\bar{h}(s, t) = \begin{cases} sf(t/s) & \text{if } s > 0, \\ f^\infty(t) & \text{if } s = 0, \\ +\infty & \text{else.} \end{cases}$$

where $f^\infty(t) = \lim_{s \to 0} sf(t/s)$ is the recession function of $f$. 
Example: a $C^2$ curve

It is standard that if $f$ is convex lsc, then also $h$ is, with

$$\bar{h}(s, t) = \sup \{as + bt : a + f^*(b) \leq 0\}.$$ 

In addition, as $\sigma^x = \lambda \theta$ where $\lambda$ is a positive measure in $\Omega \times S^1$, introducing for $p = (p^x, p^\theta) \in \mathbb{R}^3$

$$h(\theta, p) = \begin{cases} 
\bar{h}(p^x \cdot \theta, p^\theta) & \text{if } p^x \cdot \theta = |p^x| \iff p^x \parallel \theta, p^x \cdot \theta \geq 0 \\
+\infty & \text{else},
\end{cases}$$

which encodes the sub-Riemanian structure of $\Omega \times S^1$: we also have

$$\int_0^L f(\kappa) = \int_{\Omega \times S^1} \bar{h}(\sigma^x \cdot \theta, \sigma^\theta) = \int_{\Omega \times S^1} h(\theta, \sigma).$$
Example: a $C^2$ curve

Now, observe that $\text{div} \sigma = \delta_{\Gamma(L)} - \delta_{\Gamma(0)}$, in particular if $\gamma$ is a closed curve or has its endpoints on $\partial \Omega$, then $\text{div} \sigma = 0$.

Obviously, if one considers the marginal $\bar{\sigma} = \int_{S^1} \sigma^x \in \mathcal{M}^1(\Omega; \mathbb{R}^2)$ defined by

$$
\int_{\Omega} (\psi, 0) \cdot \sigma = \int_{\Omega} \psi \cdot \bar{\sigma}
$$

for any $\psi \in C_c(\Omega; \mathbb{R}^2)$, then it also has zero divergence (as it vanishes if $\psi = \nabla \phi$ for some $\phi$). In dimension 2, it follows that (assuming $\Omega$ is connected) there exists a $BV$ function $u$ such that $Du^\perp = \bar{\sigma}$. In our case, $u$ is the characteristic function of a set $E$ with $\partial E \cap \Omega = \gamma([0, T]) \cap \Omega$. 
Generalization to $BV$ functions

One can define for any $u \in BV(\Omega)$

$$F(u) = \inf \left\{ \int_{\Omega \times S^1} h(\theta, \sigma) : \text{div} \sigma = 0, \int_{S^1} \sigma^x = Du \perp \right\}.$$

If we assume that $f(t) \geq \sqrt{1 + t^2}$, then one sees that $\overline{h}(s, t) \geq \sqrt{s^2 + t^2}$ and $\int_{\Omega \times S^1} h(\theta, \sigma) \geq \int_{\Omega \times S^1} |\sigma|$. It easily follows that the “inf” is a min, and that $F$ defines a convex, lower semicontinuous function on $BV$ with $F(u) \geq |Du|(\Omega)$.

From the example above, we readily see that if $E$ is a $C^2$ set, then

$$F(\chi_E) \leq \int_{\partial E} f(\kappa) d\mathcal{H}^1.$$
Tightness of the representation

We can show the following result:

**Theorem** if $E$ is a $C^2$ set, then

$$F(\chi_E) = \int_{\partial E} f(\kappa) d\mathcal{H}^1.$$ 

**Proof:** we need to show $\geq$. In other words, we need to show the obvious fact that if $\sigma$ is a measure with $\int_{S^1} \sigma^x = D\chi_E^1$, then $\sigma$, above $\partial E$, consists at least in the measure defined by the lifted curve above $\partial E$ (with its orientation as third component).

Maybe there is a simple way to do this (as it is obvious). We used S. Smirnov’s theorem which shows that if $\sigma$ is a measure with $\text{div} \sigma = 0$, then it is a superposition of curves.
Smirnov’s Theorem A (1994)

If $\text{div} \sigma = 0$ then it can be decomposed in the following way:

$$\sigma = \int_{\mathcal{C}_1} \lambda \, d\mu(\lambda), \quad |\sigma| = \int_{\mathcal{C}_1} |\lambda| \, d\mu(\lambda),$$

where $\lambda$ are of the form

$$\lambda_{\gamma} = \tau_{\gamma} \mathcal{H}^1 \mathcal{L} \gamma$$

for rectifiable (possibly closed) curves $\gamma \subset \Omega \times S^1$ of length at most one. ($\mathcal{C}_1$ is the corresponding set.)

[We do not need here the more precise “Theorem B”]
Thanks to the fact that the decomposition is convex (ie with $|\sigma| = \int_{\mathcal{C}_1} |\lambda| d\mu(\lambda)$) we can show that $|\sigma|$-a.e., for $\mu$-a.e. curve $\lambda$ one has $\sigma/|\sigma| = \lambda/|\lambda|$ $|\lambda|$-a.e., and in particular $\lambda^x$ is oriented along $\theta$, and

$$\int_{\Omega \times S^1} h(\theta, \sigma) = \int_{\mathcal{C}_1} \left( \int_{\Omega \times S^1} h(\theta, \lambda) \right) d\mu(\lambda) = \int_{\mathcal{C}_1} \left( \int_{\gamma} h(\theta, \tau_{\gamma}) d\mathcal{H}^1 \right) d\mu(\lambda_{\gamma}).$$

The horizontal projection $\lambda^x$ is a rectifiable curve, and one can deduce that its curvature is a bounded measure. For this we reparametrize $\lambda$ with the length of $\lambda^x$: that is we define $\tilde{\lambda}(t) = \lambda(s(t))$ in such a way that $\mathcal{H}^1(\tilde{\lambda}^x([0, t])) = t$ [if simple]. Then we show that $\tilde{\lambda}^\theta(t)$, which is the orientation of the tangent [because the energy is finite], has bounded variation.
Tightness

Then one can show that if

$$\Gamma^+ = \{x \in \partial E \cap \lambda^x(0, L) : \text{the curves have the same orientation} \}$$

then a.e. on $\Gamma^+$, the absolutely continuous part of the curvature $\kappa = \ddot{\lambda}^\theta$ coincides with $\kappa_E$. Using that for any set $I$,

$$\int_{\lambda^x(I)} f(\kappa^a) \leq \int_{I \times S^1} h(\theta, \lambda),$$

which more or less follows because this is precisely the way we have built $h$, we can deduce since $\kappa^a = \kappa_E$ a.e.:

$$\int_{\partial E} f(\kappa_E) d\mathcal{H}^1 = \int_{c_1} \int_{\partial E \cap \lambda^x} f(\kappa^a) d\mu(\lambda) \leq \int_{c_1} \int_{\partial E \times S^1} h(\theta, \lambda) d\mu(\lambda)$$

which implies our inequality.
Tightness

- More cases?
- We know that $F$ can be below the standard ($L^1$) relaxation of $\int_{\partial E} f(\kappa)$ (Bellettini-Mugnai 04/05, Dayrens-Masnou 16) (simple examples).
Dual representation

We can compute the dual problem of

\[
F(u) = \inf \left\{ \int_{\Omega \times S^1} h(\theta, \sigma) : \text{div} \sigma = 0, \int_{S^1} \sigma^x = Du^\perp \right\}.
\]

by the standard perturbation technique, which consists in defining

\[
G(p) = \inf \left\{ \int_{\Omega \times S^1} h(\theta, \sigma + p) : \text{div} \sigma = 0, \int_{S^1} \sigma^x = Du^\perp \right\},
\]

showing (exactly as for \( F \)) that \( p \mapsto G(p) \) is (weakly-\( \ast \)) lsc and therefore that \( G^{**} = G \), and in particular

\[
F(u) = G(0) = \sup_{\eta \in C^0_0(\Omega \times S^1; \mathbb{R}^3)} -G^*(\eta).
\]
Dual representation

Then, it remains to compute $G^*(\eta)$:

$$G^*(\eta) = \sup_{\rho, \sigma : \text{div } \sigma = 0} \int_{\Omega \times S^1} \eta \cdot \rho - h(\theta, \sigma + \rho)$$

$$\int_{S^1} \sigma = Du^\perp$$

$$= \sup_{\sigma : \text{div } \sigma = 0} - \int_{\Omega \times S^1} \eta \cdot \sigma + \sup_{\rho} \eta \cdot (\sigma + \rho) - h(\theta, \sigma + \rho)$$

$$\int_{S^1} \sigma = Du^\perp$$

We find $\theta \cdot \eta^x + f^*(\eta^\theta) \leq 0$, and then $\eta = \psi(x) + \nabla \varphi(x, \theta)$ so that:

$$F(u) = \sup \left\{ \int_{\Omega} \psi \cdot Du^\perp : \psi \in C^0_c(\Omega; \mathbb{R}^2), \right. \left. \exists \varphi \in C^1_c(\Omega \times S^1), \theta \cdot (\nabla_x \varphi + \psi) + f^*(\partial_\theta \varphi) \leq 0 \right\}.$$
Numerical discretization

This is work in progress. We have a few approaches which work in theory but yield poorly concentrated measures $\sigma$. And better approaches which are not clearly justified.
We use both the primal and dual representation and solve the discretized problem using a saddle-point optimisation.
Examples: shape completion

Figure: Weickert’s cat: Shape completion using the function $f_2 = \sqrt{1 + k|\kappa|^2}$. 

(a) Original shape  
(b) Input  
(c) Inpainted shape
Examples: shape denoising
Examples: shape denoising

(a) AC, $\lambda = 8$

(b) AC, $\lambda = 4$

(c) AC, $\lambda = 2$

(d) EL, $\lambda = 8$

(e) EL, $\lambda = 4$

(f) EL, $\lambda = 2$

Figure: Shape denoising: First row: Using the function $f_1 = 1 + k|\kappa|$, second row: Using the function $f_3 = 1 + k|\kappa|^2$. 
Examples: Willmore flow
(cf for instance Dayrens-Masnou-Novaga 2016)

Figure: Motion by the gradient flow of different curvature depending energies. Energy $1 + |\kappa|$ gives the same as standard mean curvature flow for convex curves. Elastica/Willmore flow converges to a circle (shrinkage is still present due to the length term).
Conclusion, perspectives

- We have introduced a relatively simple systematic way to represent curvature-dependent energies in $2D$;
- It simplifies the (energetically equivalent) framework of [Bredies-Pock-Wirth 15];
- Open questions: characterize the functions for which the relaxation is tight (conjecture: functions with “continuous” curvature?);
- Discretization needs some improvement (issues: measure with orientation constraint).
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Thank you for your attention