Régularité des équations de Hamilton-Jacobi du premier ordre et applications aux jeux à champ moyen

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Séminaire Jacques-Luis Lions, 18 novembre 2016

The goal of the talk is to show Sobolev estimates for solutions of first order Hamilton-Jacobi (HJ) equations of the form

$$\partial_t u + H(t,x,Du) = f(t,x)$$

Here we assume that :

- H has a $p-$growth in the gradient variable ($H = H(t,x,\xi) \approx |\xi|^p$ at infinity, with $p > 1$)
- f is a continuous map that belongs to L^r

By Sobolev estimates, we mean estimates of u in Sobolev spaces which are independent of the regularity of H and f and depend only on the growth of H, the L^r norm of f and the L^∞ norm of u

The setting

For $\rho > 0$ set the cube $Q_\rho := (-\rho/2, \rho/2)^d$

Let $f : [0,1] \times \overline{Q_1} \to \mathbb{R}$ be continuous and nonnegative

u be continuous on $[0,1] \times \overline{Q_1}$ and satisfy in the viscosity solutions sense

$$\partial_t u + \frac{1}{\bar{C}} |Du|^p \leq f(t,x) \qquad \text{in } (0,1) \times Q_1$$

and

$$\partial_t u + \bar{C} |Du|^p \geq -\bar{C} \qquad \text{in } (0,1) \times Q_1$$

Definition

Given a continuous function $u : (0,1) \times Q_1 \to \mathbb{R}$ we say that u satisfies :

i)

$$\partial_t u + \frac{1}{\bar{C}}|Du|^p \leq f(t,x) \qquad \text{in } \ (0,1) \times Q_1$$

in the viscosity sense, or equivalently that u is a viscosity sub-solution of

$$\partial_t u + \frac{1}{\bar{C}}|Du|^p = f(t,x) \qquad \text{in } \ (0,1) \times Q_1,$$

if for each $v \in C^\infty((0,1) \times Q_1)$ such that $u - v$ has a maximum at $(t_0, x_0) \in (0,1) \times Q_1$,

$$\partial_t v(t_0,x_0) + \frac{1}{\bar{C}}|Dv(t_0,x_0)|^p \leq f(t_0,x_0);$$

Definition

Given a continuous function $u : (0,1) \times Q_1 \to \mathbb{R}$ we say that u satisfies :

ii)

$$\partial_t u + \bar{C}|Du|^p \geq -\bar{C} \qquad \text{in } \ (0,1) \times Q_1$$

in the viscosity sense, or equivalently that u is a viscosity super-solution of

$$\partial_t u + \bar{C}|Du|^p = -\bar{C} \qquad \text{in } \ (0,1) \times Q_1,$$

if for each $v \in C^\infty((0,1) \times Q_1)$ such that $u - v$ has a minimum at $(t_0, x_0) \in (0,1) \times Q_1$,

$$\partial_t v(t_0, x_0) + \bar{C}|Dv(t_0, x_0)|^p \geq -\bar{C}$$

Theorem (Cardaliaguet, Porretta, T.)

Assume $p > 1$ and $r > 1 + d/p$.

Then $u \in W^{1,1}_{loc}((0,1) \times Q_1)$ and, for any $\delta > 0$, there exists $\varepsilon > 0$ and M such that

$$\|\partial_t u\|_{L^{1+\varepsilon}((\delta,1-\delta)\times Q_{1-\delta})} + \|Du\|_{L^{p(1+\varepsilon)}((\delta,1-\delta)\times Q_{1-\delta})} \leq M,$$

where ε depends on d, p, r and $\bar{C}$ while M depends on d, p, r, $\bar{C}$, $\|f\|_r$, $\|u\|_\infty$ and δ.

Moreover u is differentiable at almost every point of $(0,1) \times Q_1$.

The result directly applies to viscosity solutions of (HJ) provided that

- f is non-negative
- the Hamiltonian satisfies the following growth condition: there exists $\bar{C} > 0$ and $p > 1$ such that

$$\frac{1}{\bar{C}}|\xi|^p - \bar{C} \leq H(t, x, \xi) \leq \bar{C}|\xi|^p + \bar{C}$$

Indeed a viscosity solution of

$$\partial_t u + H(t, x, Du) = f(t, x)$$

is a viscosity sub-solution and a viscosity super-solution, hence,

$$\partial_t u + \frac{1}{\bar{C}}|Du|^p \leq f(t, x) \qquad \text{in } (0,1) \times Q_1$$

and

$$\partial_t u + \bar{C}|Du|^p \geq -\bar{C} \qquad \text{in } (0,1) \times Q_1$$

Comments:

- The fact that f is non-negative is irrelevant, bounded below is enough
- The result does not hold in general if H has linear growth in the gradient variable
- We do not expect the result to hold if H is coercive but has a different growth from below and from above
- Under the assumptions of the above Theorem, Sobolev regularity only holds for small ε. A quantification of such a constant is an open problem
- The result does not hold if, for instance, we assume that u satisfies the two inequalities a.e., the super-solution inequality has to hold in a viscosity sense

Our result confirm the fact that solutions of HJ equations which are coercive with respect to the gradient variable enjoy unexpected regularity

The idea goes back to Capuzzo Dolcetta, Leoni and Porretta (2010) who proved that subsolutions of stationary HJ equations of second order with super-quadratic growth in the gradient variable have Hölder bounds (see also Barles 2010)

The result was later extended to equations with unbounded right-hand side by Dall'Aglio and Porretta (2015).

In the evolutionary case, Hölder bounds were progressively obtained:

by Cardaliaguet (2009) for viscosity solution of

$$-\partial_t u + b(t,x)|Du(t,x)|^p + f(t,x)\cdot Du(t,x) = 0 \quad \text{in}(0,T)\times\mathbb{R}^d$$

$$u(T,x) = g(x)$$

with $p > 1$, b, f, g continuous and bounded by some constant M, $b(t,x) \geq \delta > 0$ for some $\delta > 0$

then u is Hölder continuous in time-space and the estimates do not depend on the smoothness of the coefficients

- representation of u as the value function of a problem of calculus of variation (the Hamiltonian is convex)
- reverse Hölder inequality that implies higher integrability of u

$$\text{for } \alpha \in L^p(0,1) \quad \frac{1}{h}\int_0^h |\alpha(s)|^p ds \leq C\left(\frac{1}{h}\int_0^h |\alpha(s)|ds\right)^p \quad \forall h \in [0,1]$$

by Cannarsa and Cardaliaguet (2010) for continuous bounded viscosity solution of the equation

$$\partial_t u + H(t, x, Du(t, x)) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d$$

with

$$\frac{1}{\bar{C}}|\xi|^p - \bar{C} \leq H(t, x, \xi) \leq \bar{C}|\xi|^p + \bar{C}$$

for $\bar{C} > 0$ and $p > 1$

then u is Hölder continuous in time-space and the estimates do not depend on the smoothness of the coefficients

- construction of generalized characteristics along which super-solutions exhibit a sort of monotone behavior
- reverse Hölder inequality that implies higher integrability of u

IDEA :

Let u be a subsolution, then using Hopf's formula

$$u(\tau,x) \le u(s,y) + C(\tau - s)^{1-p}|y-x|^p + \bar{C}(\tau - s) \quad \forall \tau > s, y \in \mathbb{R}^d$$

Let u be a supersolution, then $\forall (t,x)\ \exists\ \gamma \in W^{1,p}$ s.t. $\gamma(t) = x$

$$u(t,x) \ge u(s,\gamma(s)) + C\int_s^t |\dot{\gamma}(\tau)|^p d\tau - \bar{C}(t-s) \quad \forall s \in [0,t]$$

then chosing $y = \gamma(s)$, we have Hölder inverse

$$\frac{1}{t-s}\int_s^t |\dot\gamma(\tau)|^p d\tau \le C\left(\frac{1}{t-s}\int_s^t |\dot\gamma(\tau)| d\tau\right)^p \quad \forall s \in [0,t]$$

and using Gehring result one can prove

$$\int_s^t |\dot\gamma(\tau)| d\tau \le C(t-s)^{1-\frac{1}{\theta}} \|\dot\gamma\|_{L^p} \quad \forall s \in [0,t]$$

for some $\theta > p$ depending only on structural constants Finally one can show Hölder estimates with the same exponent θ

$$|u(t,x) - u(s,y)| \le C\left[|x-y|^{\frac{\theta-p}{\theta-1}} + |t-s|^{\frac{\theta-p}{p}}\right]$$

by Cardaliaguet and Rainer (2011) for fully nonlinear, nonlocal equations

and by Cardaliaguet and Silvestre (2012) for an unbounded right-hand side

their proof use an improvement of the oscillation Lemma:

the oscillation of a solution to a given equation in a parabolic cylinder decreases by a fixed factor (less than one) when the size of the cylinder is reduced by another fixed factor

In all these results the regularity holds for solutions but non for sub-solutions

Motivation for having Hölder regularity: Homogeneization

Preliminaries

For simplicity, we work from now on with backward Hamilton-Jacobi equations, i.e. with a continuous map u which satisfies the following inequalities in the viscosity sense:

$$-\partial_t u + \frac{1}{\bar{C}}|Du|^p \leq f(t,x) \qquad \text{in } (0,1)\times Q_1 \tag{1}$$

and

$$-\partial_t u + \bar{C}|Du|^p \geq -\bar{C} \qquad \text{in } (0,1)\times Q_1 \tag{2}$$

We recall that $p > 1$ and $r > 1 + d/p$ are given

We denote by q the conjugate exponent of p: $1/p + 1/q = 1$

Let us start first with a consequence of inequality (1)

Lemma

Fix $r_1 \in (1 + d/p, r)$, $\bar{\alpha} > 0$ and $h > 0$ such that $2\bar{\alpha}h < 1$.

If u is continuous on $[0,1] \times \overline{Q_1}$ and satisfies (1) in the viscosity sense, then for any $(t,x), (s,y) \in (0,1) \times Q_{\bar{\alpha}h}$ with $s > t$,

$$u(t,x) \leq u(s,y) + C\frac{(\bar{\alpha}h)^q}{(s-t)^{q-1}} + C(s-t)\left(\fint_{(t,s)\times Q_{2\bar{\alpha}h}} f^{r_1}\right)^{1/r_1},$$

where $1/p + 1/q = 1$ and $C = C(p, \bar{C})$.

Then a more standard consequence of inequality (2)

Lemma

If u is continuous on $[0,1] \times \overline{Q_1}$ and satisfies (2) in the viscosity sense, then, for any $(t,x) \in (0,1) \times Q_1$, there exists an absolutely continuous curve γ with $\gamma(t) = x$ and, for any $s \in [t,1]$ such that $\gamma([t,s]) \subset Q_1$,

$$u(t,x) \geq u(s,\gamma(s)) + \frac{1}{C}\int_t^s |\dot{\gamma}(\sigma)|^q d\sigma - C(s-t),$$

where $C = C(p,\bar{C})$.

We say that γ is a generalized characteristic for $u(t,x)$

Indeed, if u is a solution of a Hamilton-Jacobi-Belmann equation, then any characteristic γ satisfies the above inequality

Lemma

If u is continuous on $[0,1] \times \overline{Q_1}$ and satisfies (1) in the viscosity sense, then

- *u is of bounded variation (BV) in $(0,1) \times Q_1$,*
- *$Du \in L^p((0,1) \times Q_1)$*
- *(1) holds in the sense of distributions*

A similar statement is not known for viscosity solution to (2), i.e. it is not known if this implies that (2) holds in the sense of distributions

Key estimate

Our aim is to show that, if Du and f are well estimated in some cube, then Du satisfies a reverse Hölder inequality

To this purpose, we will need to use cubes with an intrinsic scaling

Indeed as time and space play at different scales in (1) and (2), it is convenient to use ideas introduced by DiBenedetto (1993) for degenerate parabolic equations and refined by Kinnunen and Lewis (2000)

This consists in working on space-time cubes which size depends on the solution itself

Let us then introduce a family of parameters:

We fix $r_1 \in (1 + d/p, r)$

For constants $\lambda_0 \geq 1$, $\kappa \geq 1$ and $2 \leq c_1 \leq 5c_1 \leq c_2$ and variables $\lambda \geq \lambda_0$ and $h > 0$, we set

$$\sigma = \kappa \lambda^{1-p}$$

and

$$Q = Q_{\sigma h, h} \subset Q' = Q_{c_1 \sigma h, c_1 h} \subset Q'' = Q_{c_2 \sigma h, c_2 h} \subset Q_{1,1}$$

where for $\sigma, \rho > 0$

$$Q_{\sigma,\rho} := (-\sigma/2, \sigma/2) \times (-\rho/2, \rho/2)^d$$

Rough Idea: if $|Du| \sim \lambda$ then the equation looks like

$$-\partial_t u + |Du|^p \sim -\partial_t u + \lambda^{p-1}|Du|$$

and remains invariant through the scaling

$$u_h = u(h\lambda^{1-p}t, hx)$$

The inverse Hölder inequality

Proposition

There exists a suitable choice of the constants λ_0, κ, c_1, c_2, depending only on d, p, r_1, r and $\bar{C}$ such that, for any $\lambda \geq \lambda_0$ and $h > 0$, if the following estimate holds:

$$\lambda^p \leq \fint_Q (|Du|^p + f^{r_1}) \leq c_2^{d+1} \fint_{Q''} (|Du|^p + f^{r_1}) \leq c_2^{d+1} \lambda^p,$$

then we have

$$\fint_{Q''} |Du|^p \leq \hat{C} \left(\fint_{Q'} |Du| \right)^p + \hat{C} \fint_{Q'} (1 + f^{r_1}),$$

for some constant $\hat{C}$ independent of λ, h.

Idea of the proof

Let us formally integrate inequality (1) over the cube Q to get:

$$\fint_Q |Du|^p \leq C \fint_{Q_h} \frac{u(\sigma h/2) - u(-\sigma h/2)}{\sigma h} + C \fint_Q f$$

In order to get a reverse Hölder inequality, one has to show that the right-hand side is bounded above by an expression of the form

$$\left(\fint_{Q'} |Du| \right)^p + C$$

Recall that

- u subsolution $\Longrightarrow$

$$u(t,x) \leq u(s,y) + C\frac{(\bar{\alpha}h)^q}{(s-t)^{q-1}} + C(s-t)\left(\fint_{(t,s)\times Q_{2\bar{\alpha}h}} f^{r_1}\right)^{1/r_1},$$

- u supersolution $\Longrightarrow \exists$ an absolutely continuous curve γ with $\gamma(t) = x$ and, for any $s \in [t,1]$ such that $\gamma([t,s]) \subset Q_1$,

$$u(t,x) \geq u(s,\gamma(s)) + \frac{1}{C}\int_t^s |\dot{\gamma}(\sigma)|^q d\sigma - C(s-t),$$

Remark: $u(t,x)$ is estimated from above with any $s > t$ and any γ. While $u(t,x)$ can be estimated from below with $s > t$ provided we follow a generalized characteristic

We estimate the RHS

$$C \fint_{Q_h} \frac{u(\sigma h/2) - u(-\sigma h/2)}{\sigma h} + C \fint_{Q} f$$

through the energy $\int_t^{t+\tau} |\dot\gamma|^q$ of a generalized characteristic starting form a suitable position (t, x) with $t < -\sigma h$

Here τ is the exit time of γ from a slightly larger ball Q_h

Actually we prove that $\tau \sim \sigma h$ and

$$C \fint_{Q_h} \frac{u(\sigma h/2) - u(-\sigma h/2)}{\sigma h} \lesssim \frac{1}{\tau} \int_t^{t+\tau} |\dot\gamma|^q \lesssim \left(\fint_{Q'} |Du| \right)^p + C$$

Higher integrability of Du

Proposition

There exists $\varepsilon_0 > 0$ depending only on d, p, r and $\bar{C}$ and a constant M, depending on d, p, r and $\bar{C}$, $\|u\|_\infty$ and $\|f\|_r$, such that

$$\int_{Q_{1/2,1/2}} |Du|^{p(1+\varepsilon_0)} \leq M.$$

The proof uses the inverse Hölder inequality and arguments developed by Kinnunen and Lewis (2000)

Higher integrability of $\partial_t u$

Corollary

The map u belongs to $W^{1,1}(Q_{1/2,1/2})$ and

$$\int_{Q_{1/2,1/2}} |\partial_t u|^{1+\varepsilon_0} \le M,$$

where ε_0 is the constant defined in the previous Proposition and M is a constant depending on d, p, r, $\bar{C}$, $\|u\|_\infty$ and $\|f\|_r$.

Almost everywhere differentiability of u

We no longer require the continuity of f but we only assume that $f \in L^r((0,1) \times Q_1)$

Proposition

Let $u \in W^{1,1}(Q_{1,1}) \cap C^0(\overline{Q_{1,1}})$ be such that $Du \in L^p(Q_{1,1})$. We assume that u satisfies (1) in the sense of distributions and (2) in the viscosity sense.
Then u is differentiable at almost every point of $Q_{1,1}$.

Remarks on the Result

- We can show that a map satisfying (1) and (2) does not necessarily belong to $W^{1,1+\varepsilon}$ for large values of ε
- It is important to note that the Sobolev estimates obtained are not true in general for a.e. solutions (2)
- Besides their intrinsic interest, our results are motivated by the theory of mean field games : our regularity result implies that "weak solutions" of the mean field game systems satisfy the equation in a more classical sense

Application to Mean Field Games

The Mean Field games system considered takes the form:

$$\begin{cases} (i) & -\partial_t u + H(x, Du) = f(x, m(t,x)) \quad \text{in } (0,T) \times \mathbb{T}^d \\ (ii) & \partial_t m - \operatorname{div}(mD_pH(x, Du)) = 0 \quad \text{in } (0,T) \times \mathbb{T}^d \\ (iii) & m(0,x) = m_0(x),\ u(T,x) = u_T(x) \quad \text{in } \mathbb{T}^d \end{cases} \tag{3}$$

where:

- $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is $d-$dimensional torus,
- $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is convex in the second variable,
- $f : \mathbb{T}^d \times [0, +\infty) \to [0, +\infty)$ is increasing with respect to the second variable,
- m_0 is a smooth probability density
- $u_T : \mathbb{T}^d \to \mathbb{R}$ is a smooth given function

- $f : \mathbb{T}^d \times [0, +\infty) \to \mathbb{R}$ is continuous in both variables, strictly increasing wrt the second variable m, and $\exists r > 1$ and C_1 s.t.

$$\frac{1}{C_1}|m|^{r'-1} - C_1 \leq f(x,m) \leq C_1|m|^{r'-1} + C_1 \qquad \forall m \geq 0\,,$$

  where r' is the conjugate exponent of r. Moreover we ask the following normalization condition:

$$f(x,0) = 0 \qquad \forall x \in \mathbb{T}^d\,.$$

- $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is continuous in both variables, convex and differentiable in the second variable, with D_pH continuous in both variable, and has a superlinear growth in the gradient variable: $\exists p > 1$ and $C_2 > 0$ such that $r > 1 + d/p$ and

$$\frac{1}{pC_2}|\xi|^p - C_2 \leq H(x,\xi) \leq \frac{C_2}{p}|\xi|^p + C_2 \qquad \forall (x,\xi) \in \mathbb{T}^d \times \mathbb{R}^d\,.$$

- $u_T : \mathbb{T}^d \to \mathbb{R}$ is of class $\mathcal{C}^1$, while $m_0 : \mathbb{T}^d \to \mathbb{R}$ is a continuous density, with $m_0 \geq 0$ and $\int_{\mathbb{T}^d} m_0\,dx = 1$.

Theorem (Cardaliaguet, Graber 2015)

There is a unique weak solution of (3), *i.e., a unique pair* $(m,u) \in L^{r'}((0,T)\times\mathbb{T}^d)\times BV((0,T)\times\mathbb{T}^d)$ *s.t.*

(i) *u is continuous in $[0,T]\times\mathbb{T}^d$, with*

$$Du \in L^p,\ mD_pH(x,Du) \in L^1$$

$$\text{and } \left(\partial_t u^{ac} - \langle Du, D_pH(x,Du)\rangle\right) m \in L^1 .$$

(ii) *Equation* (3)-*(i) holds in the following sense:*

$$-\partial_t u^{ac}(t,x) + H(x,Du(t,x)) = f(x,m(t,x)) \qquad a.e.\ in\ \{m>0\}$$

(where $\partial_t u^{ac}$ is the absolutely continuous part of the measure $\partial_t u$ wrt the Lebesgue measure) and inequality

$$-\partial_t u + H(x,Du) \le f(x,m) \quad \text{in } (0,T)\times\mathbb{T}^d$$

holds in the sense of distributions, with $u(T,\cdot) = u_T$ in the sense of trace,

(iii) Equation (3)-(ii) holds:

$$\partial_t m - \mathrm{div}(m D_p H(x, Du)) = 0 \quad \text{in } (0,T) \times \mathbb{T}^d, \qquad m(0) = m_0$$

in the sense of distributions,

(iv) The following equality holds:

$$\int_0^T \int_{\mathbb{T}^d} m \left(\partial_t u^{ac} - \langle Du, D_p H(x, Du)\rangle\right) = \int_{\mathbb{T}^d} m(T) u_T - m_0 u(0).$$

By uniqueness we mean that m is indeed unique and u is uniquely defined in $\{m > 0\}$.

The previous result is presented in Cardaliaguet, Graber 2015 under more general conditions

Under the assumptions stated above, they proved also that u is Hölder continuous.
u is also globally unique (not only in $\{m > 0\}$) if one requires that the additional condition

$$-\partial_t u + H(x, Du) \geq 0 \quad \text{in} \quad (0, T) \times \mathbb{T}^d$$

holds in the viscosity sense

As a consequence of our Theorem

Corollary

Let (u, m) be the unique weak solution of (3) which satisfies

$$-\partial_t u + H(x, Du) \geq 0 \qquad in\ (0, T) \times \mathbb{T}^d$$

in the viscosity sense.
Then u belongs to $W^{1,1}_{loc}((0, T) \times \mathbb{T}^d)$, u is differentiable a.e. and the following equality holds:

$$-\partial_t u(t, x) + H(x, Du(t, x)) = f(x, m(t, x)) \qquad a.e.\ in\ (0, T) \times \mathbb{T}^d.$$

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