Prescribed curvature flow on the torus

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Prescribed curvature

Let \((M, g_b)\) be a closed surface of genus 1 with “background metric” \(g_b\).
A classical question of Kazdan-Warner (1974/75):
Given a smooth function \(f\) on \(M\), is there a conformal metric 
\(g = e^{2u}g_b\) with Gauss curvature \(K_g = f\)?
In view of the Gauss equation

\[
K_g = e^{-2u}(-\Delta_{g_b} u + K_{g_b})
\]

we are then led to study the equation

\[
-\Delta_{g_b} u + K_{g_b} = fe^{2u} \text{ on } M.
\]

By the uniformization theorem, with no loss of generality we may assume that the background metric \(g_b\) is flat with \(K_{g_b} = 0\) and has volume \(\text{vol}(M, g_b) = 1\).
Necessary and sufficient conditions

In 1974 already, Kazdan-Warner obtained the following conditions which are both necessary and sufficient for the existence of a solution $u$ to the equation

$$ - \Delta_{g_b} u = fe^{2u} \text{ on } M. \quad (1) $$

**Theorem**

There exists a solution $u$ of (1) if and only if either $f \equiv 0$, or if the function $f$ changes sign and satisfies

$$ \int_M fd\mu_{g_b} < 0. \quad (2) $$

Proof: **Necessity.** Suppose $u$ solves (1). Compute

$$ \int_M f d\mu_{g_b} = -\int_M \Delta_{g_b} u e^{-2u} d\mu_{g_b} = -2 \int_M |\nabla u|^2 e^{-2u} d\mu_{g_b} \leq 0, $$

with equality iff $u \equiv \text{const}$, $f \equiv 0$. 
Kazdan-Warner result (cont.)

**Sufficiency.** Let $f$ satisfy (2). A minimizer of the Liouville energy

$$E(u) = \frac{1}{2} \int_M |\nabla u|^2_{g_b} \, d\mu_{g_b}$$

in the class of functions

$$C = \{u \in H^1(M, g_b); \int_M fe^{2u} \, d\mu_{g_b} = 0\} \neq \emptyset$$

with $\bar{u} = \int_M u \, d\mu_{g_b} = 0$ satisfies the equation

$$-\Delta_{g_b} u = \alpha + \beta fe^{2u} = fe^{2(u+\frac{1}{2}\log \beta)}$$

with Lagrange multipliers $\alpha = 0$, $\beta > 0$ if $f \not\equiv 0$. Observe that $E(u) = E(u + c)$ for $c \in \mathbb{R}$; moreover, the constraint

$$\int_M f \, d\mu_g = \int_M fe^{2u} \, d\mu_{g_b} = 0, \quad g = e^{2u} g_b,$$

is natural in view of (1) or the Gauss-Bonnet theorem.
Prescribed curvature flow

Motivated by recent work of Anh Ngô-Xingwang Xu on a flow approach to the results of Escobar-Schoen (1986) on the higher-dimensional Kazdan-Warner problem in the case of vanishing Yamabe invariant, for initial data \( u_0 \in C \) we consider the prescribed curvature flow

\[
  u_t = \alpha f - K \quad \text{on} \quad M \times [0, \infty], \quad u \big|_{t=0} = u_0, \tag{3}
\]

where \( K = K_g \) with \( g = g(t) = e^{2u(t)}g_b \) at any time \( t \geq 0 \) and with a function \( \alpha = \alpha(t) \) so that \( u(t) \in C \) for all \( t \geq 0 \); that is, we require the condition

\[
  \frac{1}{2} \frac{d}{dt} \left( \int_M f \, d\mu_g \right) = \int_M u_t f \, d\mu_g = \int_M (\alpha f - K)f \, d\mu_g = 0.
\]

Solving for \( \alpha \) then we find

\[
  \alpha = \frac{\int_M fK \, d\mu_g}{\int_M f^2 \, d\mu_g}. \tag{4}
\]
Energy inequality, apriori bounds

For a smooth solution $u = u(t)$ of (3)-(4) there holds

$$\frac{d}{dt} E(u(t)) = - \int_M u_t \Delta_{g_b} u \, d\mu_{g_b} = \int_M (\alpha f - K) K \, d\mu_g$$

$$= - \int_M |\alpha f - K|^2 \, d\mu_g = - \int_M |u_t|^2 \, d\mu_g \leq 0,$$

in view of $K = K_g = e^{-2u}(-\Delta_{g_b} u)$, $d\mu_g = e^{2u} d\mu_{g_b}$. Hence

$$E(u(T)) + \int_0^T \int_M |u_t|^2 \, d\mu_g \, dt \leq E(u_0) \quad (5)$$

for any $T > 0$. Also note that the volume is preserved with

$$\frac{1}{2} \frac{d}{dt} \text{vol}(M, g(t)) = \int_M u_t \, d\mu_g = \alpha \int_M f \, d\mu_g - \int_M K \, d\mu_g = 0$$

by (4) and Gauss-Bonnet. Normalizing $\text{vol}(M, g_0) = 1$, then

$$\text{vol}(M, g) = \int_M d\mu_g = \int_M e^{2u} \, d\mu_{g_b} = 1, \quad t > 0. \quad (6)$$
Apriori bounds (cont.)

By Jensen’s inequality, we then also find the uniform bound

$$2\bar{u} = 2 \int_M u \, d\mu_{g_b} \leq \log \left( \int_M e^{2u} \, d\mu_{g_b} \right) = 0.$$  

for the average of $u$. By the Moser-Trudinger inequality on any closed and orientable $(M, g_0)$ for any $\beta < 4\pi$ there holds

$$\sup \{ \int_M e^{u^2} \, d\mu_{g_0} : u \in H^1(M, g_0), \|\nabla u\|^2_{L^2} \leq \beta, \bar{u} = 0 \} < \infty.$$  

Estimating

$$2p(u - \bar{u}) \leq 2\pi (u - \bar{u})^2/\|\nabla u\|^2_{L^2(M, g_b)} + \frac{p^2}{2\pi}\|\nabla u\|^2_{L^2(M, g_b)}$$

via Young’s inequality, for any $p \in \mathbb{R}$ we then find

$$\int_M e^{2p(u - \bar{u})} \, d\mu_{g_0} \leq C_{TM}(2\pi)e^{\frac{p^2}{2\pi}\|\nabla u\|^2_{L^2(M, g_b)}} = C(p, E(u)).$$
Apriori bounds (cont.)

In particular, for a solution \( u \) to (3)-(4) we can bound

\[
1 = \int_M d\mu_g = \int_M e^{2u} d\mu_{g_0} = e^{2\bar{u}} \int_M e^{2(u - \bar{u})} d\mu_{g_0} \leq C e^{2\bar{u}}
\]

with a constant \( C = C(E(u_0)) \). We conclude the uniform bound

\[-m_0 \leq \bar{u} \leq 0.
\]

for the average of \( u = u(t) \). Likewise, by Hölder’s inequality

\[
0 < |\int_M f d\mu_{g_b}|^2 \leq \int_M e^{-2u} d\mu_{g_b} \int_M f^2 e^{2u} d\mu_{g_b} \leq C \int_M f^2 d\mu_g,
\]

\[
|\int_M fK d\mu_g| = |\int_M f(-\Delta_{g_b} u) d\mu_{g_b}| \leq \int_M |\nabla_{g_b} f||\nabla_{g_b} u| d\mu_{g_b}
\]

\[
\leq C\|f\|_{C^1} E^{1/2}(u) \leq C E^{1/2}(u_0),
\]

and we find the uniform bound

\[
|\alpha| = |\int_M fK d\mu_g|/\int_M f^2 d\mu_g \leq \alpha_0.
\]
Global existence

Thus, equation (3), that is,

$$u_t = \alpha f - K = e^{-2u} \Delta_g u + \alpha f \quad \text{on } M \times [0, \infty[,$$

is uniformly parabolic with uniformly bounded coefficients.

**Theorem**

Suppose that $f$ is smooth, changes sign, and satisfies (2). Then for any smooth $u_0 \in C$ there exists a unique, global smooth solution $u$ of (3)-(4) with initial data $u|_{t=0} = u_0$ and satisfying $u(t) \in C$ as well as the energy bound $E(u(t)) \leq E(u_0)$ for all $t$. Moreover, we have $u(t) \to u_\infty$ as $t \to \infty$ suitably, where $u_\infty + c_\infty$ is a smooth solution of (2) for some $c_\infty \in \mathbb{R}$.

Remark: Ngô-Xu (preprint 2016) obtained a similar result with almost the same methods.
Convergence

Our Theorem in general only gives convergence $u(t) \to u_\infty$ as $t \to \infty$ suitably.

Ngô-Xu (2016), as an application of the Łojasiewicz-Simon inequality, obtain unconditional convergence of the flow (with a polynomial rate) as $t \to \infty$ for analytic functions $f$ satisfying (2).

For the normalized Ricci flow (with $f \equiv 0$) Hamilton (1988) established global existence and exponentially fast convergence. In fact, Hamilton shows global existence and exponentially fast convergence of the Ricci flow on any closed surface $(M, g_0)$. For the sphere his work was completed by Chow (1991); in S. (2002) I gave a simpler proof of exponentially fast convergence in this case. In fact, this method also gives exponentially fast convergence for general, sufficiently smooth functions $f$ satisfying (2), whenever $u_\infty$ is a strict relative minimizer of $E$.

Moreover, we obtain unconditional convergence of the flow (possibly not exponentially fast) whenever (1) is uniquely solvable.
A scenario for non-uniqueness of the limit

We non-rigorously describe a situation where the set of solutions of equation (1) is large, and where convergence of the flow (3), (4) cannot be expected.

There are two key ingredients for this:

- **Non-uniqueness** of subsequential limits of geometric flows was rigorously demonstrated by Topping (1997) in the case of the heat flow of harmonic maps from the standard 2-sphere $S^2$ to $T^2 \times S^2$, where the target is endowed with a suitable warped metric causing the “bubble” developing in the flow to follow a spiralling “groove”.

“Bubbling” of conformal metrics

Let $f_0$ be a smooth, non-constant function with $\max_{p\in M} f_0(p) = 0$. Then for any sufficiently small $\lambda > 0$ the function $f_\lambda = f_0 + \lambda$ changes sign and satisfies (2). Hence there exists a solution $\hat{u}_\lambda = u_\lambda + c_\lambda$ of (1), obtained from a minimizer $u_\lambda$ of $E$ in the set

$$C_\lambda = \{ u \in H^1(M, g_b); \int_M f_\lambda e^{2u} d\mu_{g_b} = 0 \},$$

subject to the constraint $\int_M e^{2u_\lambda} d\mu_{g_b} = 1$, and some $c_\lambda \in \mathbb{R}$. As shown by Galimberti (2015), for suitable $\lambda = \lambda_n \downarrow 0$ the corresponding metrics $\hat{g}_\lambda = e^{2\hat{u}_\lambda} g_0$ of Gauss curvature $K_{\hat{g}_\lambda} = f_\lambda$ exhibit “bubbles” concentrated near points $p \in M$ where $f_0(p) = 0$. If now $f_0$ vanishes on a set $F_0$ spiralling into a circle on $(M, g_b)$, and if the function $f_0$ satisfies suitable growth conditions away from $F_0$, conceivably for sufficiently small $\lambda > 0$ and suitable initial data the flow (5), (6) might choose to produce “bubbly” metrics concentrated along points that move along the spiral $F_0$ (the groove of the record) as $t \to \infty$, similar to Topping’s flow (1997).
The key result that permitted Galimberti to invoke the blow-up analysis of B-G-S for the metrics $\hat{g}_\lambda$ is the following observation.

**Lemma**

The function

$$\lambda \mapsto \beta_\lambda = E(u_\lambda) = \min\{E(u); \ u \in C_\lambda\}$$

is non-increasing in $\lambda$ for small $0 < \lambda < \lambda_0$, and we have the bound

$$\frac{d\beta_\lambda}{d\lambda} \leq -\alpha_\lambda/2,$$

where

$$\alpha_\lambda = \text{vol}(M, \hat{g}_\lambda) = \int_M e^{2(u_\lambda+c_\lambda)} \, d\mu_{g_b} = e^{2c_\lambda}.$$

Galimberti’s proof is quite complicated. Here is a simple proof.
Fix some sufficiently small $\lambda > 0$ so that $f_\lambda$ changes sign and satisfies (3), and let $u_\lambda \in C_\lambda$ as above. For sufficiently small $\delta \in \mathbb{R}$ then

$$\int_M f_\lambda e^{2(u_\lambda + \delta f_\lambda)} \, d\mu_{g_b} = 2\delta \int_M f_\lambda^2 e^{2u_\lambda} \, d\mu_{g_b} + O(\delta^2),$$

while

$$\int_M e^{2(u_\lambda + \delta f_\lambda)} \, d\mu_{g_b} = 1 + 2\delta \int_M f_\lambda e^{2u_\lambda} \, d\mu_{g_b} + O(\delta^2) = 1 + O(\delta^2).$$

It follows that

$$u_\lambda + \delta f_\lambda \in C_\mu$$

with

$$\mu = \lambda - 2\delta \int_M f_\lambda^2 e^{2u_\lambda} \, d\mu_{g_b} + O(\delta^2).$$

In particular, $\mu > \lambda$ for $0 < -\delta << 1$, and $\delta = O(\mu - \lambda)$. 
"Bubbling" of conformal metrics (cont.)

On the other hand, we have

\[ E(u_{\lambda} + \delta f_{\lambda}) = E(u_{\lambda}) + \delta \int_{M} \nabla u_{\lambda} \cdot \nabla f_{\lambda} \, d\mu_{g_b} + O(\delta^2), \]

\[ \int_{M} \nabla u_{\lambda} \cdot \nabla f_{\lambda} \, d\mu_{g_b} = \int_{M} (-\Delta_{g_b} u_{\lambda} f_{\lambda} \, d\mu_{g_b} = \alpha_{\lambda} \int_{M} f_{\lambda}^2 e^{2u_{\lambda}} \, d\mu_{g_b}. \]

Hence for

\[ 0 < -2\delta \int_{M} f_{\lambda}^2 e^{2u_{\lambda}} \, d\mu_{g_b} = \mu - \lambda + O((\mu - \lambda)^2) \ll 1 \]

there holds

\[ \beta_{\mu} \leq E(u_{\lambda} + \delta f_{\lambda}) \leq E(u_{\lambda}) + \delta \alpha_{\lambda} \int_{M} f_{\lambda}^2 e^{2u_{\lambda}} \, d\mu_{g_b} + O(\delta^2) \]

\[ = \beta_{\lambda} - \frac{\alpha_{\lambda}}{2} (\mu - \lambda) + O((\mu - \lambda)^2) < \beta_{\lambda}, \]

and

\[ \limsup_{\mu \downarrow \lambda} \frac{\beta_{\mu} - \beta_{\lambda}}{\mu - \lambda} \leq -\frac{\alpha_{\lambda}}{2} < 0. \]
“Bubbling” of conformal metrics (cont.)

Moreover, suitable comparison functions give the bound

\[
\limsup_{\lambda \downarrow 0} \frac{\beta_\lambda}{\log(1/\lambda)} \leq 4\pi.
\]

Applying the monotonicity trick from S. (1987/88) we observe that the monotone function \( \beta \) is almost everywhere differentiable and can use the above bounds to obtain a sequence \( \lambda_k \downarrow 0 \) such that

\[
\limsup_{k \to \infty} \lambda_k \alpha_k \leq 2 \limsup_{k \to \infty} \lambda_k \left| \frac{d\beta_\lambda}{d\lambda}(\lambda_k) \right| \leq 8\pi,
\]

where \( \alpha_k = \alpha_{\lambda_k} \). Together with Gauss-Bonnet and the estimate

\[
|f_\lambda| \leq -f_0 + \lambda = -f_\lambda + 2\lambda
\]

this gives rise to the bound

\[
\limsup_{k \to \infty} \int_M |f_{\lambda_k}| e^{2(u_{\lambda_k} + c_{\lambda_k})} d\mu g_b \leq \limsup_{k \to \infty} 2\lambda_k \alpha_k \leq 16\pi
\]

for the total curvature of the metrics \( \hat{g}_\lambda = e^{2(u_{\lambda} + c_{\lambda})}, \lambda = \lambda_k \).
Conclusion

The analysis from B-G-S (2015) may now be invoked to see the desired bubbling.

It seems very likely that bubbling metrics with huge spherical bubbles of curvature $\lambda = \lambda_k \downarrow 0$ also may be constructed by matched asymptotic expansion, as was done by Del Pino-Román (2014) in the case of surfaces of higher genus, prompted by B-G-S.

Some open questions:

• What can be said when $f$ consists of a regular part and a sum of Dirac masses, where solutions of (2) would correspond to conical metrics of prescribed curvature with prescribed opening angles, as in the work of Troyanov (1991)?

• What can one say about the heat flow for Chern-Simons vortices, as in the work of Tarantello (2010), where a flow equation similar to (3), (4) may be expected to arise?