Prescribed curvature flow on the torus

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Prescribed curvature

Let (M, g_b) be a closed surface of genus 1 with "background metric" g_b.
A classical question of Kazdan-Warner (1974/75):
Given a smooth function f on M, is there a conformal metric $g = e^{2u} g_b$ with Gauss curvature $K_g = f$?
In view of the Gauss equation

$$K_g = e^{-2u}(-\Delta_{g_b} u + K_{g_b})$$

we are then led to study the equation

$$-\Delta_{g_b} u + K_{g_b} = f e^{2u} \text{ on } M.$$

By the uniformization theorem, with no loss of generality we may assume that the background metric g_b is flat with $K_{g_b} = 0$ and has volume $vol(M, g_b) = 1$.

Necessary and sufficient conditions

In 1974 already, Kazdan-Warner obtained the following conditions which are both necessary and sufficient for the existence of a solution u to the equation

$$-\Delta_{g_b} u = f e^{2u} \quad \text{on } M. \tag{1}$$

Theorem

There exists a solution u of (1) if and only if either $f \equiv 0$, or if the function f changes sign and satisfies

$$\int_M f d\mu_{g_b} < 0. \tag{2}$$

Proof: Necessity. Suppose u solves (1). Compute

$$\int_M f \, d\mu_{g_b} = -\int_M \Delta_{g_b} u \, e^{-2u} \, d\mu_{g_b} = -2 \int_M |\nabla u|^2 \, e^{-2u} \, d\mu_{g_b} \leq 0,$$

with equality iff $u \equiv const$, $f \equiv 0$.

Kazdan-Warner result (cont.)

Sufficiency. Let f satisfy (2). A minimizer of the Liouville energy

$$E(u) = \frac{1}{2}\int_M |\nabla u|^2_{g_b}\, d\mu_{g_b}$$

in the class of functions

$$\mathcal{C} = \{u \in H^1(M, g_b)\,;\ \int_M fe^{2u}\, d\mu_{g_b} = 0\} \neq \emptyset$$

with $\bar{u} = \int_M u\, d\mu_{g_b} = 0$ satisfies the equation

$$-\Delta_{g_b} u = \alpha + \beta fe^{2u} = fe^{2(u+\frac{1}{2}\log\beta)}$$

with Lagrange multipliers $\alpha = 0$, $\beta > 0$ if $f \not\equiv 0$. Observe that $E(u) = E(u+c)$ for $c \in \mathbb{R}$; moreover, the constraint

$$\int_M f\, d\mu_g = \int_M f\, e^{2u}\, d\mu_{g_b} = 0, \quad g = e^{2u} g_b,$$

is natural in view of (1) or the Gauss-Bonnet theorem.

Prescribed curvature flow

Motivated by recent work of Anh Ngô-Xingwang Xu on a flow approach to the results of Escobar-Schoen (1986) on the higher-dimensional Kazdan-Warner problem in the case of vanishing Yamabe invariant, for initial data $u_0 \in \mathcal{C}$ we consider the prescribed curvature flow

$$u_t = \alpha f - K \quad \text{on } M \times [0, \infty[, \; u\big|_{t=0} = u_0, \tag{3}$$

where $K = K_g$ with $g = g(t) = e^{2u(t)} g_b$ at any time $t \geq 0$ and with a function $\alpha = \alpha(t)$ so that $u(t) \in \mathcal{C}$ for all $t \geq 0$; that is, we require the condition

$$\frac{1}{2}\frac{d}{dt}\Big(\int_M f \, d\mu_g\Big) = \int_M u_t f \, d\mu_g = \int_M (\alpha f - K) f \, d\mu_g = 0.$$

Solving for α then we find

$$\alpha = \int_M fK \, d\mu_g \Big/ \int_M f^2 \, d\mu_g. \tag{4}$$

Energy inequality, apriori bounds

For a smooth solution $u = u(t)$ of (3)-(4) there holds

$$\frac{d}{dt}E(u(t)) = -\int_M u_t \Delta_{g_b} u \, d\mu_{g_b} = \int_M (\alpha f - K) K \, d\mu_g$$
$$= -\int_M |\alpha f - K|^2 \, d\mu_g = -\int_M |u_t|^2 \, d\mu_g \leq 0,$$

in view of $K = K_g = e^{-2u}(-\Delta_{g_b} u)$, $d\mu_g = e^{2u} d\mu_{g_b}$. Hence

$$E(u(T)) + \int_0^T \int_M |u_t|^2 \, d\mu_g \, dt \leq E(u_0) \qquad (5)$$

for any $T > 0$. Also note that the volume is preserved with

$$\frac{1}{2}\frac{d}{dt} vol(M, g(t)) = \int_M u_t \, d\mu_g = \alpha \int_M f \, d\mu_g - \int_M K \, d\mu_g = 0$$

by (4) and Gauss-Bonnet. Normalizing $vol(M, g_0) = 1$, then

$$vol(M, g) = \int_M d\mu_g = \int_M e^{2u} \, d\mu_{g_b} = 1, \quad t > 0. \qquad (6)$$

Apriori bounds (cont.)

By Jensen's inequality, we then also find the uniform bound

$$2\bar{u} := 2\int_M u\,d\mu_{g_b} \leq \log\Big(\int_M e^{2u}\,d\mu_{g_b}\Big) = 0.$$

for the average of u. By the Moser-Trudinger inequality on any closed and orientable (M, g_0) for any $\beta < 4\pi$ there holds

$$\sup\{\int_M e^{u^2}\,d\mu_{g_0};\ u \in H^1(M, g_0),\ \|\nabla u\|^2_{L^2} \leq \beta,\ \bar{u} = 0\} < \infty.$$

Estimating

$$2p(u - \bar{u}) \leq 2\pi(u - \bar{u})^2/\|\nabla u\|^2_{L^2(M,g_b)} + \frac{p^2}{2\pi}\|\nabla u\|^2_{L^2(M,g_b)}$$

via Young's inequality, for any $p \in \mathbb{R}$ we then find

$$\int_M e^{2p(u-\bar{u})}\,d\mu_{g_0} \leq C_{TM}(2\pi)e^{\frac{p^2}{2\pi}\|\nabla u\|^2_{L^2(M,g_b)}} = C(p, E(u)).$$

Apriori bounds (cont.)

In particular, for a solution u to (3)-(4) we can bound

$$1 = \int_M d\mu_g = \int_M e^{2u}\, d\mu_{g_0} = e^{2\bar{u}} \int_M e^{2(u-\bar{u})}\, d\mu_{g_0} \leq Ce^{2\bar{u}}$$

with a constant $C = C(E(u_0))$. We conclude the uniform bound

$$-m_0 \leq \bar{u} \leq 0.$$

for the average of $u = u(t)$. Likewise, by Hölder's inequality

$$0 < \Big| \int_M f\, d\mu_{g_b} \Big|^2 \leq \int_M e^{-2u}\, d\mu_{g_b} \int_M f^2 e^{2u}\, d\mu_{g_b} \leq C \int_M f^2\, d\mu_g,$$

$$\Big| \int_M fK\, d\mu_g \Big| = \Big| \int_M f(-\Delta_{g_b} u)\, d\mu_{g_b} \Big| \leq \int_M |\nabla_{g_b} f| |\nabla_{g_b} u|\, d\mu_{g_b}$$
$$\leq C \|f\|_{C^1} E^{1/2}(u) \leq CE^{1/2}(u_0),$$

and we find the uniform bound

$$|\alpha| = \Big| \int_M fK\, d\mu_g \Big| \Big/ \int_M f^2\, d\mu_g \leq \alpha_0.$$

Global existence

Thus, equation (3), that is,

$$u_t = \alpha f - K = e^{-2u}\Delta_{g_b} u + \alpha f \quad \text{on } M \times [0, \infty[,$$

is uniformly parabolic with uniformly bounded coefficients.

Theorem

Suppose that f is smooth, changes sign, and satisfies (2). Then for any smooth $u_0 \in \mathcal{C}$ there exists a unique, global smooth solution u of (3)-(4) with initial data $u|_{t=0} = u_0$ and satisfying $u(t) \in \mathcal{C}$ as well as the energy bound $E(u(t)) \leq E(u_0)$ for all t. Moreover, we have $u(t) \to u_\infty$ as $t \to \infty$ suitably, where $u_\infty + c_\infty$ is a smooth solution of (2) for some $c_\infty \in \mathbb{R}$.

Remark: Ngô-Xu (preprint 2016) obtained a similar result with almost the same methods.

Convergence

Our Theorem in general only gives convergence $u(t) \to u_\infty$ as $t \to \infty$ suitably.
Ngô-Xu (2016), as an application of the Łojasiewicz-Simon inequality, obtain unconditional convergence of the flow (with a polynomial rate) as $t \to \infty$ for analytic functions f satisfying (2).
For the normalized Ricci flow (with $f \equiv 0$) Hamilton (1988) established global existence and exponentially fast convergence.
In fact, Hamilton shows global existence and exponentially fast convergence of the Ricci flow on any closed surface (M, g_0). For the sphere his work was completed by Chow (1991); in S. (2002) I gave a simpler proof of exponentially fast convergence in this case.
In fact, this method also gives exponentially fast convergence for general, sufficiently smooth functions f satisfying (2), whenever u_∞ is a strict relative minimizer of E.
Moreover, we obtain unconditional convergence of the flow (possibly not exponentially fast) whenever (1) is uniquely solvable.

A scenario for non-uniqueness of the limit

We non-rigorously describe a situation where the set of solutions of equation (1) is large, and where convergence of the flow (3), (4) cannot be expected.

There are two key ingredients for this:

• Non-uniqueness of subsequential limits of geometric flows was rigorously demonstrated by Topping (1997) in the case of the heat flow of harmonic maps from the standard 2-sphere S^2 to $T^2 \times S^2$, where the target is endowed with a suitable warped metric causing the "bubble" developing in the flow to follow a spiralling "groove".

• "Bubbling" of conformal metrics of prescribed mean curvature, as obtained by Galimberti (2015) in a certain limit regime, following work of Borer-Galimberti-Struwe (2015) for surfaces of genus > 1.

“Bubbling” of conformal metrics

Let f_0 be a smooth, non-constant function with $\max_{p \in M} f_0(p) = 0$. Then for any sufficiently small $\lambda > 0$ the function $f_\lambda = f_0 + \lambda$ changes sign and satisfies (2). Hence there exists a solution $\hat{u}_\lambda = u_\lambda + c_\lambda$ of (1), obtained from a minimizer u_λ of E in the set

$$\mathcal{C}_\lambda = \{u \in H^1(M, g_b)\,;\ \int_M f_\lambda e^{2u}\, d\mu_{g_b} = 0\},$$

subject to the constraint $\int_M e^{2u_\lambda}\, d\mu_{g_b} = 1$, and some $c_\lambda \in \mathbb{R}$. As shown by Galimberti (2015), for suitable $\lambda = \lambda_n \downarrow 0$ the corresponding metrics $\hat{g}_\lambda = e^{2\hat{u}_\lambda} g_0$ of Gauss curvature $K_{\hat{g}_\lambda} = f_\lambda$ exhibit “bubbles” concentrated near points $p \in M$ where $f_0(p) = 0$. If now f_0 vanishes on a set F_0 spiralling into a circle on (M, g_b), and if the function f_0 satisfies suitable growth conditions away from F_0, conceivably for sufficiently small $\lambda > 0$ and suitable initial data the flow (5), (6) might choose to produce “bubbly” metrics concentrated along points that move along the spiral F_0 (the groove of the record) as $t \to \infty$, similar to Topping's flow (1997).

“Bubbling” of conformal metrics (cont.)

The key result that permitted Galimberti to invoke the blow-up analysis of B-G-S for the metrics $\hat{g}_\lambda$ is the following observation.

Lemma

The function

$$\lambda \mapsto \beta_\lambda = E(u_\lambda) = \min\{E(u);\ u \in \mathcal{C}_\lambda\}$$

is non-increasing in λ for small $0 < \lambda < \lambda_0$, and we have the bound

$$\frac{d\beta_\lambda}{d\lambda} \leq -\alpha_\lambda/2,$$

where

$$\alpha_\lambda = vol(M, \hat{g}_\lambda) = \int_M e^{2(u_\lambda + c_\lambda)}\, d\mu_{g_b} = e^{2c_\lambda}.$$

Galimberti’s proof is quite complicated. Here is a simple proof.

“Bubbling” of conformal metrics (cont.)

Fix some sufficiently small $\lambda > 0$ so that f_λ changes sign and satisfies (3), and let $u_\lambda \in \mathcal{C}_\lambda$ as above. For sufficiently small $\delta \in \mathbb{R}$ then

$$\int_M f_\lambda e^{2(u_\lambda + \delta f_\lambda)}\, d\mu_{g_b} = 2\delta \int_M f_\lambda^2 e^{2u_\lambda}\, d\mu_{g_b} + O(\delta^2),$$

while

$$\int_M e^{2(u_\lambda + \delta f_\lambda)}\, d\mu_{g_b} = 1 + 2\delta \int_M f_\lambda e^{2u_\lambda}\, d\mu_{g_b} + O(\delta^2) = 1 + O(\delta^2).$$

It follows that

$$u_\lambda + \delta f_\lambda \in \mathcal{C}_\mu$$

with

$$\mu = \lambda - 2\delta \int_M f_\lambda^2 e^{2u_\lambda}\, d\mu_{g_b} + O(\delta^2).$$

In particular, $\mu > \lambda$ for $0 < -\delta << 1$, and $\delta = O(\mu - \lambda)$.

"Bubbling" of conformal metrics (cont.)

On the other hand, we have

$$E(u_\lambda + \delta f_\lambda) = E(u_\lambda) + \delta \int_M \nabla u_\lambda \cdot \nabla f_\lambda \, d\mu_{g_b} + O(\delta^2),$$

$$\int_M \nabla u_\lambda \cdot \nabla f_\lambda \, d\mu_{g_b} = \int_M (-\Delta_{g_b} u_\lambda f_\lambda \, d\mu_{g_b} = \alpha_\lambda \int_M f_\lambda^2 e^{2u_\lambda} \, d\mu_{g_b}.$$

Hence for

$$0 < -2\delta \int_M f_\lambda^2 e^{2u_\lambda} \, d\mu_{g_b} = \mu - \lambda + O((\mu - \lambda)^2) << 1$$

there holds

$$\beta_\mu \leq E(u_\lambda + \delta f_\lambda) \leq E(u_\lambda) + \delta\alpha_\lambda \int_M f_\lambda^2 e^{2u_\lambda} \, d\mu_{g_b} + O(\delta^2)$$
$$= \beta_\lambda - \frac{\alpha_\lambda}{2}(\mu - \lambda) + O((\mu - \lambda)^2) < \beta_\lambda,$$

and

$$\limsup_{\mu \downarrow \lambda} \frac{\beta_\mu - \beta_\lambda}{\mu - \lambda} \leq -\frac{\alpha_\lambda}{2} < 0.$$

“Bubbling” of conformal metrics (cont.)

Moreover, suitable comparison functions give the bound

$$\limsup_{\lambda \downarrow 0} \frac{\beta_\lambda}{\log(1/\lambda)} \le 4\pi.$$

Applying the monotonicity trick from S. (1987/88) we observe that the monotone function β is almost everywhere differentiable and can use the above bounds to obtain a sequence $\lambda_k \downarrow 0$ such that

$$\limsup_{k \to \infty} \lambda_k \alpha_k \le 2 \limsup_{k \to \infty} \lambda_k \Big| \frac{d\beta_\lambda}{d\lambda}(\lambda_k) \Big| \le 8\pi,$$

where $\alpha_k = \alpha_{\lambda_k}$. Together with Gauss-Bonnet and the estimate

$$|f_\lambda| \le -f_0 + \lambda = -f_\lambda + 2\lambda$$

this gives rise to the bound

$$\limsup_{k \to \infty} \int_M |f_{\lambda_k}| e^{2(u_{\lambda_k} + c_{\lambda_k})} \, d\mu_{g_b} \le \limsup_{k \to \infty} 2\lambda_k \alpha_k \le 16\pi$$

for the total curvature of the metrics $\hat{g}_\lambda = e^{2(u_\lambda + c_\lambda)}$, $\lambda = \lambda_k$.

Conclusion

The analysis from B-G-S (2015) may now be invoked to see the desired bubbling.

It seems very likely that bubbling metrics with huge spherical bubbles of curvature $\lambda = \lambda_k \downarrow 0$ also may be constructed by matched asymptotic expansion, as was done by Del Pino-Román (2014) in the case of surfaces of higher genus, prompted by B-G-S.

Some open questions:

• What can be said when f consists of a regular part and a sum of Dirac masses, where solutions of (2) would correspond to conical metrics of prescribed curvature with prescribed opening angles, as in the work of Troyanov (1991)?

• What can one say about the heat flow for Chern-Simons vortices, as in the work of Tarantello (2010), where a flow equation similar to (3), (4) may be expected to arise?