Lagrange-Projection scheme for two-phase flows.
1. Applications to realistic test cases.

F. Coquel¹, Q.L. Nguyen²*, M. Postel¹ et Q.H. Tran²

¹ Laboratoire Jacques Louis Lions, CNRS et Université Pierre et Marie Curie, B.C. 187, 75252 Paris Cedex 05, France
² Département Mathématiques Appliquées, Institut Français du Pétrole, 1 et 4 avenue de Bois-Préau, 92852 Rueil-Malmaison Cedex, France

We are concerned with the numerical simulation of two-phase flows representing oil transportation along a 1-D pipeline. More specifically, we wish to show that, for this kind of problems, it is highly recommended to make use of the Lagrange-Projection formalism. Indeed, this approach naturally splits the effects of fast (acoustic) and slow (kinematic) waves, thus enabling us to design numerical schemes with many desirable properties: explicit with respect to slow waves and implicit with respect to fast waves. The design of the overall explicit-implicit Lagrange-Projection method involves many interesting features, among which the positivity of the partial densities is of utmost interest to us. In this paper, the emphasis will be laid on the motivation and the general philosophy of the Lagrange-Projection method we propose. Extensive numerical results for realistic test cases are then presented in order to illustrate the capabilities of the method.

Key words Hyperbolic system, two-phase flows, finite volume schemes, relaxation, Lagrange-Projection.

1 Introduction

In the context of oil transportation, we are interested in simulating flows of mixture of different components such as oil, gas and water, over a long time range. In this work we are only interested in flows composed of two phases: liquid and gas. These two-phase flows can be seen in many configurations or regimes: stratified, emulsion or slug, depending strongly not only on the flow rates of each phase but also on the topography variation of the pipelines. Moreover, the pipeline is generally much longer than its diameter so we can consider this as a 1-D simulation problem. Such problems must be modeled by realistic models which respect the physical settings of the problem such as the regimes of the flows, the varying boundary conditions as well as the geometry of the pipeline. The model we are presenting here is based on a highly nonlinear system of PDEs of conservation laws. We can see that the genuine non-linearities come from two algebraic closures appearing in the system. There exist two phenomena in our problem: a slow kinematic wave and two fast acoustic waves. The first one is actually important for the engineers since the transportation is the only phenomenon of interest, a great accuracy is therefore expected for this wave. The remaining ones are less interesting since pressure propagation is usually a disregarded phenomenon, thus not so much accuracy is required. The main goal is to develop a fast numerical method for simulating realistic test cases while keeping good accuracy of the results. An explicit-implicit time integration strategy based on Lagrange-Projection decomposition (see [2] for mathematical foundation) is very efficient to achieve such a purpose.

2 Physical modeling

Before going into numerical details, let us first briefly describe the Drift-Flux model together with some notations that we use for modeling two-phase flows. The model is made of three PDEs (see [1])

$$\partial_t \begin{pmatrix} \rho \\ \rho Y \\ \rho v \end{pmatrix} + \partial_x \begin{pmatrix} \rho v \\ \rho Y v - \sigma \\ \rho v^2 + \pi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ S \end{pmatrix},$$  

where $\rho$ is the density of the mixture, $v$ is the velocity, $Y$ the gas mass fraction, and $S$ is the source term. Here, $\sigma$ and $\pi$ denote two nonlinear functions of the unknowns and are given as follows

$$\sigma(u) = \rho Y (1 - Y) \phi(u) \quad \text{and} \quad \pi(u) = p(u) + \rho Y (1 - Y) \phi^2(u) \quad \text{with} \quad u = (\rho, \rho Y, \rho v).$$  

In (2), $\phi$ is the slip law, i.e., the difference between the two phases’ velocities, and $p$ is the pressure of the mixture. The precise definition of these two laws depends on the regime of the flow as well as the nature of each phase (see [1]).

The system is hyperbolic with three distinct eigenvalues among which the intermediate one corresponds to the slow transport wave, and the two others correspond to acoustic waves.

* Corresponding author  E-mail: q-long.nguyen@ifp.fr, Phone: +0033 147 526 042
3 Lagrange-Projection scheme and positivity of the partial densities

To solve the previous nonlinear hyperbolic system, we will now present an efficient numerical scheme ensuring the positivity of the density and the gas mass fraction. We will first make use of the relaxation method [1] to conveniently handle the non-linearities in the PDE model (1). More precisely, we replace the two nonlinear functions $\sigma$ and $\pi$ by two new state variables $\tau$ and $\Pi$ which coincide respectively with $\sigma$ and $\pi$ in the limit of an infinite relaxation parameter $\lambda$. The system is thus larger but it will help us treat the two non-linearities of the problem separately.

\[ \begin{pmatrix} \partial_t (\rho) \\ \partial_t (\rho Y) \\ \partial_x (\rho v) \\ \partial_x (\rho \Sigma) \\ \partial_x (\Pi) \end{pmatrix} + \begin{pmatrix} \partial_x (\rho v) \\ \partial_x (\rho Y v - \Sigma) \\ \partial_x (\rho v^2 + \Pi) \\ \partial_x (\rho \Sigma v - b^2 Y) \\ \partial_x (\rho \Pi v + a^2 v) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ S \\ \lambda \rho (\sigma - \Sigma) \end{pmatrix}. \]

(3)

The nonlinear stability of the relaxation procedure at the PDE level is achieved if the relaxation coefficients $a$ and $b$ satisfy: $a^2 > -\partial_x \pi(\mathbf{u})$ and $b > |\partial_x \sigma(\mathbf{u})|$ for all $\mathbf{u}$ under consideration (see [1]).

Equipped with this new system, we are able to introduce a positive explicit-implicit time integration Lagrange-Projection scheme. The main idea is to perform the resolution of the relaxation system in three steps. The first step is the transformation to Lagrangian coordinates. In this step, we can see that the Lagrangian system splits into two independent systems. The first one is related to fast acoustic (pressure) waves that we will solve implicitly. In the contrary, the second one will be solved explicitly as it corresponds to slow kinematic (transport) waves.

\[ (1^{st}) : \partial_t \begin{pmatrix} \tau \\ v \\ \Pi \\ \Pi^2 v \end{pmatrix} + \partial_y \begin{pmatrix} -u \\ S \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad (2^{nd}) : \partial_t \begin{pmatrix} Y \\ \Sigma \end{pmatrix} + \partial_y \begin{pmatrix} -\Sigma \\ -b^2 Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

(4)

The second step deals with the Eulerian remap which boils down to the transport system $\partial_t \mathbf{u} + \nu^* \partial_x \mathbf{u} = 0$. The transport Projection velocity $\nu^*$ can be chosen so as to ensure the conservativity of the scheme. The last step is the equilibrium process of the relaxation parameters following the relaxation method: $\Sigma = \sigma(\mathbf{u})$ and $\Pi = \pi(\mathbf{u})$.

This three-step resolution constitutes the Lagrange-Projection scheme which is stable under a CFL condition based on the transport velocity $\nu^*$. This CFL condition is devised so that positivity of the partial densities is preserved. Resulting time steps $\Delta t$ depend explicitly on the current solution. The mathematical foundation is given in [3] including the case of time dependent boundary conditions.

4 Numerical results

We now give numerical results of a realistic test case. It consists of simulating a horizontal pipeline of 4000 m length in which we impose time dependent boundary conditions: keep the gas mass flow rate constant 10 kg/m$^3$ and cut off the liquid mass flow rate from 1000 kg/m$^3$ during 200 s at the inlet while doubling the pressure from 1 bar to 2 bar at the outlet of the pipeline.

Fig. 1 Time dependent boundary conditions. The evolution of physical variables at different positions.

References