Regularity for the planar optimal p-compliance problem

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Abstract

We prove a partial $C^{1,\alpha}$ regularity result in dimension $N = 2$ for the optimal p-compliance problem, extending for $p \neq 2$ some of the results obtained by A. Chambolle, J. Lamboley, A. Lemenant, E. Stepanov (2017). Because of the lack of good monotonicity estimates for the p-energy when $p \neq 2$, we employ an alternative technique based on a compactness argument leading to a p-energy decay at any flat point. We finally obtain that every optimal set has no loops, is Ahlfors regular, and $C^{1,\alpha}$ at $\mathcal{H}^1$-a.e. point for every $p \in (1, +\infty)$.

Introduction

Given an open bounded set $\Omega \subset \mathbb{R}^2$, an exponent $p \in (1, +\infty)$ and a function $f \in L^{p'}(\Omega)$ with $p' = p/(p-1)$. Let $\Sigma \subset \overline{\Omega}$ be a closed set and for $u \in W_0^{1,p}(\Omega \backslash \Sigma)$ define

$$E_p(u) := \frac{1}{p} \int_{\Omega \backslash \Sigma} |\nabla u|^p \, \mathrm{d}x - \int_{\Omega} f u \, \mathrm{d}x.$$

It is classical that the functional E_p admits a unique minimizer u_Σ, which is the solution of the Dirichlet problem

$$\begin{cases} -\Delta_p u &= f \text{ in } \Omega \backslash \Sigma \\ u &= 0 \text{ on } \Sigma \cup \partial\Omega \end{cases}$$

in the weak sense, which means that

$$\int_{\Omega \backslash \Sigma} |\nabla u_\Sigma|^{p-2} \nabla u_\Sigma \nabla \varphi \, \mathrm{d}x = \int_\Omega f \varphi \, \mathrm{d}x$$

for all $\varphi \in W_0^{1,p}(\Omega \backslash \Sigma)$.

We can interpret Ω as a membrane which is attached along $\Sigma \cup \partial\Omega$ to some fixed base (where Σ can be interpreted as the "glue line") and subjected to a given force f. Then u_Σ is the displacement of the membrane. The rigidity of the membrane is measured through the p-compliance functional, which is defined as

$$C_p(\Sigma) := -E_p(u_\Sigma) = \frac{1}{p'} \int_{\Omega \backslash \Sigma} |\nabla u_\Sigma|^p \, \mathrm{d}x = \frac{1}{p'} \int_\Omega f u_\Sigma \, \mathrm{d}x.$$

We study the following shape optimization problem.

Problem 1. *Given $\lambda > 0$, find a set $\Sigma \subset \overline{\Omega}$ minimizing the functional $\mathcal{F}_\lambda$ defined by*

$$\mathcal{F}_{\lambda,p}(\Sigma') := C_p(\Sigma') + \lambda \mathcal{H}^1(\Sigma')$$

among all sets $\Sigma' \in \mathcal{K}(\Omega)$, where $\mathcal{K}(\Omega)$ is the class of all closed connected subsets of $\overline{\Omega}$.

The physical interpretation of this problem may be the following: we are trying to find the best location Σ for the glue to put on the membrane Ω in order to maximize the rigidity of the latter, subject to the force f, while the penalization by $\lambda\mathcal{H}^1$ takes into account the quantity (or cost) of the glue.

The aim of the work

The aim of this work is to prove some $C^{1,\alpha}$ regularity properties about the minimizers of Problem 1. In particular, we prove that a minimizer has no loop and is Ahlfors regular. We establish some $C^{1,\alpha}$ regularity properties about the minimizers, as stated in the following main theorem.

Theorem. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded set, $p \in (1, +\infty)$, $f \in L^q(\Omega)$ with $q > 2$ if $2 < p < +\infty$ and $q > p'$ if $1 < p \leq 2$, and let $\Sigma \subset \overline{\Omega}$ be a minimizer for Problem 1. Then there exists a constant $\alpha \in (0,1)$ such that for $\mathcal{H}^1$-a.e. point $x \in \Sigma \cap \Omega$ one can find $r_0 > 0$, depending on x, such that $\Sigma \cap B_{r_0}(x)$ is a $C^{1,\alpha}$ regular curve.*

Problem 1 was studied earlier in the particular case $p = 2$ (see [2]) for which a full regularity result was established. We will explain later the differences between our context $p \neq 2$ and the Euclidean case $p = 2$. In the limit $p \to +\infty$, Problem 1 Γ-converges to the so-called *average distance problem* and for which it is known that minimizers may not be C^1 regular (see [3]). Our result can therefore be considered as making a link between $p = 2$ and $p = +\infty$, although it actually works for any $p > 1$.

Comments about the proof

In the proof of $C^{1,\alpha}$ regularity, as many other free boundary or free discontinuity problems, one of the main point is to prove a decay estimate on the local p-energy around a flat point. In other words we need to prove that the normalized energy

$$r \mapsto \frac{1}{r} \int_{B_r(x_0)} |\nabla u_\Sigma|^p \, \mathrm{d}x$$

converges to zero sufficiently fast at a point $x_0 \in \Sigma$, like a power of the radius, and this is where our proof differs from the case $p = 2$.

In the case $p = 2$, the decay on the normalized energy is obtained using a so-called *monotonicity formula* that was inspired by the one of A. Bonnet on the Mumford-Shah functional [1]. This monotonicity formula is also the key tool in the classification of blow-up limits.

For $p \neq 2$, an analogous monotonicity formula can still be established for the p-energy. We indeed can prove that

$$r \mapsto \frac{1}{r^\alpha} \int_{B_r(x_0)} |\nabla u_\Sigma|^p \, \mathrm{d}x + Cr^{p'-\alpha}$$

is nondecreasing with a power α depending explicitly on the worst p-Wirtinger constant on arcs in $\partial B_r(x_0) \backslash (\Sigma \cup \partial\Omega)$, but unfortunately, if x_0 is a flat point, then the resulting power of r in that monotonicity formula is not high enough for our purposes and cannot be directly used to prove $C^{1,\alpha}$ estimates. We stress, for instance, that for $p = 2$ the monotonicity proof reaches the optimal power α, with equality for the harmonic homogeneous functions with smallest exponent, whereas the monotonicity for $p \neq 2$ produces a non-optimal exponent which is not the right power associated to the optimal homogeneous p-harmonic function. Consequently, we also miss a great tool which prevents us to establish the classification of blow-up limits.

As the p-monotonicity is not strong enough to get $C^{1,\alpha}$ regularity we therefore use another strategy, arguing by contradiction and compactness: as mentioned earlier, we know that $\int_{B_r(x_0)} |\nabla u_\Sigma|^p \, \mathrm{d}x$ behaves like Cr^2 at point x_0 lying on a line, for a p-harmonic function vanishing on that line (by reflection), thus by compactness it still has a similar behavior when Σ locally stays ε-close to a line.

Actually, as the compliance is a min-max type problem, the true quantity to control is not exactly $\int_{B_r(x_0)} |\nabla u_\Sigma|^p \, \mathrm{d}x$, but rather this other variant, as already defined and denoted by $\omega_\Sigma(x_0, r)$ in [2],

$$w_\Sigma(x_0, r) := \max_{\Sigma' \in \mathcal{K}(\Omega); \Sigma' \Delta \Sigma \subset \overline{B}_r(x_0)} \frac{1}{r} \int_{B_r(x_0)} |\nabla u_{\Sigma'}|^p \, \mathrm{d}x.$$

It can be shown that the quantity $w_\Sigma(x_0, r)$ controls, in many circumstances, the square of the flatness, leading to some $C^{1,\alpha}$ estimates when $w_\Sigma(x_0, r)$ decays fast enough.

In [2] the decay of the above quantity was still obtained by use of the monotonicity formula, applied to the function $u_{\Sigma'}$, where Σ' is the maximizer in the definition of $w_\Sigma(x_0, r)$.

As a consequence of our compactness argument, which provides a decay only for a set Σ' staying τ-close to a line, we need to introduce and work with the following slightly more complicated quantity

$$w_\Sigma^\tau(x_0, r) := \max_{\substack{\Sigma' \in \mathcal{K}(\Omega); \Sigma' \Delta \Sigma \subset \overline{B}_r(x_0) \\ \beta_{\Sigma'}(x_0, r) \leq \tau}} \frac{1}{r} \int_{B_r(x_0)} |\nabla u_{\Sigma'}|^p \, \mathrm{d}x,$$

where $\beta_\Sigma(x_0, r)$ is the flatness defined by

$$\beta_\Sigma(x_0, r) := \inf_{P \ni x_0} \frac{1}{r} d_H(\Sigma \cap \overline{B}_r(x_0), P \cap \overline{B}_r(x_0)),$$

(the infimum being taken over all affine lines P passing through x_0).

We indeed obtain a decay of $\omega_\Sigma^\tau(x_0, r)$ provided that $\beta_\Sigma(x_0, r)$ stays under control, which finally leads to the desired $C^{1,\alpha}$ result, and the same kind of estimate is also used to prove the absence of loops.

The strategy described above works, but curiously enough, only for the range of exponents $p > 4/3$. In order to get the regularity result for the full range $p > 1$, we need some little extra decay. This is obtained by using both the p-monotonicity formula, together with the compactness argument. Indeed, using a dyadic sequence of radii and inserting at each step the p-monotonicity estimate together with the compactness argument, we are able to reach the $C^{1,\alpha}$ regularity of Σ up to $p > 1$.

Results

Let $\Omega \subset \mathbb{R}^2$ be an open bounded set, $p \in (1, +\infty)$, $f \in L^q(\Omega)$ with $q > 2$ if $2 < p < +\infty$ and $q > p'$ if $1 < p \leq 2$, and let $\Sigma \subset \overline{\Omega}$ be a minimizer for Problem 1. Then the following assertions hold

- Σ contains no loops (homeomorphic images of S^1), hence $\mathbb{R}^2 \backslash \Sigma$ is connected.
- Σ is Ahlfors regular.
- There exists a constant $\alpha \in (0,1)$ such that for $\mathcal{H}^1$-a.e. point $x \in \Sigma \cap \Omega$ one can find $r_0 > 0$, depending on x, such that $\Sigma \cap B_{r_0}(x)$ is a $C^{1,\alpha}$ regular curve. Moreover, if Ω is a C^1 convex domain, then for $\mathcal{H}^1$-a.e. point $x \in \Sigma$ one can find $r_0 > 0$, depending on x, such that $\Sigma \cap B_{r_0}(x)$ is a $C^{1,\alpha}$ regular curve.

Forthcoming Research

The Problem 1 can be defined in higher dimension, provided that $p > N - 1$, still with a penalization with the one dimensional Hausdorff measure $\mathcal{H}^1(\Sigma)$. This instance of the problem in higher dimensions seems to be very original, leading to a free-boundary type problem with a high co-dimensional free boundary set. Due to the low dimension of the "free-boundary" in dimension $N > 2$, most of the usual competitors are no more valid and some new ideas and new tools have to be used. We believe that some techniques developed in our work for N=2 could be useful to prove a similar result in higher dimensions as well. This will be the purpose of a forthcoming work.

References

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Acknowledgments

This work was partially supported by the project ANR-18-CE40-0013 SHAPO financed by the French Agence Nationale de la Recherche (ANR)