
**Concentration (or not)
in the Vlasov-Navier-Stokes system**

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Sommaire

- 1 The Vlasov-Navier-Stokes system
- 2 Existence theory
- 3 Large time behavior : concentration
- 4 ... or not ?

Fluid/kinetic systems

Description of a dispersed phased moving within a continuous one

- $f(t, \mathbf{x}, \mathbf{v})$: probability density function (kinetic equation for the particles)
- $\mathbf{u}(t, \mathbf{x})$ and $p(t, \mathbf{x})$: velocity and pressure (fluid mechanics equation)

First models by O'Rourke and Williams ('80) and since :

Fluid	Spray	Interaction
Incompressible	Fragmentation	Drag force
Viscous	Coalescence	Retroaction
Homogeneous	Polydispersed	...

→ # Models \gg 1

→ « relevancy » vs. « mathematical challenge »

The Vlasov-Navier-Stokes system

- Monodispersed spray without internal interaction
- Viscous, homogeneous, incompressible fluid
- Interaction : **drag** force and **Brinkman** force

$$\left\{ \begin{array}{l} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla p = - \int_{\mathbf{v}} (\mathbf{u} - \mathbf{v}) f \, d\mathbf{v}, \\ \operatorname{div} \mathbf{u} = 0, \\ \partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}} \cdot (f(\mathbf{u} - \mathbf{v})) = 0. \end{array} \right.$$

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The energy-dissipation identity

Defining

$$E(t) = \frac{1}{2} \int_{\mathbf{x}} |\mathbf{u}(t)|^2 + \frac{1}{2} \int_{\mathbf{x}, \mathbf{v}} f(t) |\mathbf{v}|^2,$$

$$D(t) = \int_{\mathbf{x}} |\nabla \mathbf{u}(t)|^2 + \int_{\mathbf{x}, \mathbf{v}} f(t) |\mathbf{u}(t) - \mathbf{v}|^2,$$

we have formally

$$\frac{d}{dt} E(t) + D(t) = 0.$$

Setting

- $t \geq 0$, $\mathbf{x} \in \mathbb{T}^d$, $\mathbf{v} \in \mathbb{R}^d$, $d \in \{2, 3\}$;
- Admissible initial data :
 - $0 \leq f_0 \in L^\infty \cap L^1(\mathbb{T}^d \times \mathbb{R}^d)$;
 - $\int_{\mathbb{T}^d \times \mathbb{R}^d} |\mathbf{v}|^2 f_0 < +\infty$;
 - $\mathbf{u}_0 \in L^2_{\text{div}}(\mathbb{T}^d)$.

Global weak solutions

Theorem (Boudin, Desvillettes, Grandmont, M. – 2009)

For $d = 2, 3$ and admissible initial data, there exists a global weak solution (\mathbf{u}, f) to the Vlasov-Navier-Stokes system that satisfies the energy-dissipation inequality for almost all $0 \leq s \leq t$

$$E(t) + \int_s^t D(\tau) d\tau \leq E(s). \quad (\star)$$

- Leray for \mathbf{u} , renormalized for f ;
- More realistic framework [Boudin, Grandmont, M. – 2017] ;
- Here : large time behavior for solutions satisfying (\star) .

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What to expect ?

- Consider the **uncoupled** Vlasov equation on $\mathbb{T}^d \times \mathbb{R}^d$

$$\partial_t g + \mathbf{v} \cdot \nabla_{\mathbf{x}} g - \nabla_{\mathbf{v}} \cdot ((\mathbf{U} - \mathbf{v})g) = 0.$$

- If $\mathbf{U} = 0$ (linearization around $(\mathbf{u} = 0, f = 0)$), **concentration** occurs in velocity :

$$g(t, \mathbf{x}, \mathbf{v}) \xrightarrow{t \rightarrow +\infty} \left(\int_{\mathbb{R}^d} g_0(\mathbf{x} - \mathbf{v}, \mathbf{v}) d\mathbf{v} \right) \otimes \delta_0.$$

- Similar for $t \mapsto \mathbf{U}(t) \in \mathbb{R}^d$, concentration around $\mathbf{v} = \mathbf{U}(t)$
- $\nexists g \in L^1_{\text{loc}}(\mathbb{T}^d \times \mathbb{R}^d) \setminus \{0\}$ stationary solution

What to expect ?

- Navier-Stokes on \mathbb{T}^d : $\|\mathbf{u}(t) - \langle \mathbf{u}(t) \rangle\|_2 \rightarrow 0$
- **Concentration** is expected for the full VNS system
- Attempts
 - [Jabin – 2000] (Vlasov-Stokes)
 - [Choi, Kwon – 2015] (if-theorem)

How to quantify the concentration ...

... without knowing the limiting profile?

Conservative equation :

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}} \cdot (\mathbf{A}f) = 0$$

Define

$$\mathbf{j}_f(t, \mathbf{x}) := \int_{\mathbb{R}^d} f(t, \mathbf{x}, \mathbf{v}) \mathbf{v} \, d\mathbf{v}, \quad \rho_f(t, \mathbf{x}) := \int_{\mathbb{R}^d} f(t, \mathbf{x}, \mathbf{v}) \, d\mathbf{v},$$

$$\text{If } f \approx \rho_f \otimes \delta_{\mathbf{U}}, \text{ then } \mathbf{j}_f(t, \mathbf{x}) \approx \rho_f(t, \mathbf{x}) \mathbf{U}$$

→ Up to normalization, $\langle \mathbf{j}_f(t) \rangle \approx \mathbf{U}$.

The modulated energy of Choi and Kwon

Recall the energy

$$E(t) = \frac{1}{2} \int_{\mathbb{T}^d} |\mathbf{u}(t)|^2 + \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{R}^d} f(t) |\mathbf{v}|^2.$$

Choi and Kwon introduced

$$\begin{aligned} \mathcal{E}(t) = \frac{1}{2} \int_{\mathbb{T}^d} |\mathbf{u}(t) - \langle \mathbf{u}(t) \rangle|^2 + \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{R}^d} f(t) |\mathbf{v} - \langle \mathbf{j}_f(t) \rangle|^2 \\ + |\langle \mathbf{j}_f(t) \rangle - \langle \mathbf{u}(t) \rangle|^2. \end{aligned}$$

The modulated energy of Choi and Kwon

Recall the Wasserstein distance

$$W_1(\mu, \nu) = \inf_{\gamma \in \mu \bullet \nu} \int_{\Omega \times \Omega} |\omega_1 - \omega_2| d\gamma(\omega_1, \omega_2).$$

W_1 metrizes narrow convergence of measures.

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W_1 metrizes narrow convergence of measures.

\mathcal{E} captures concentration because

$$W_1\left(f(t), \rho_f(t) \otimes \delta_{\mathbf{v}=\langle \mathbf{j}_f(t) \rangle}\right) \lesssim \mathcal{E}(t)^{1/2}.$$

$\longrightarrow \mathcal{E}(t) \xrightarrow{t \rightarrow +\infty} 0 \implies \text{concentration.}$

If-theorem ?

The modulated energy satisfies

$$\frac{d}{dt} \mathcal{E}(t) + D(t) = 0.$$

Lemma (Choi, Kwon (2015))

If (\mathbf{u}, f) is a solution of VNS such that

$$\sup_{t \geq 0} \|\rho_f(t)\|_{3/2} < \infty,$$

then $\mathcal{E}(t) \lesssim D(t)$ and therefore (exponentially)

$$\mathcal{E}(t) \xrightarrow[t \rightarrow +\infty]{} 0.$$

The uniform bound

Estimate $\rho_f \in L^\infty(\mathbb{R}_+; L^{3/2}(\mathbb{T}^d))$ is **structurally** hard to obtain.

With smooth initial data, energy estimate + interpolation leads to

$$\rho_f \in L_{\text{loc}}^\infty(\mathbb{R}_+; L^p(\mathbb{T}^d)),$$

for some $p < \infty$.

→ Need of bootstrap argument.

Main result

Theorem (Han-Kwan, M., Moyano – 2019)

Consider an admissible initial data (\mathbf{u}_0, f_0) such that

- « $f_0 \searrow 0$ in \mathbf{v} uniformly in \mathbf{x} » ;
- $\mathcal{E}(0) \ll 1$;
- $(d=3) \|\mathbf{u}_0\|_{\dot{H}^{1/2}(\mathbb{T}^3)} \ll 1$.

Any weak solution of the VNS system initiated with (\mathbf{u}_0, f_0) satisfies $\mathcal{E}(t) \rightarrow 0$, with exponential rate.

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- maintain regularity of solutions

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- ~~maintain regularity of solutions~~ consider a smooth solution ;

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- ~~maintain regularity of solutions~~ consider a smooth solution ;
- produce a strong uniform (in time) bound :

$$\sup_{t \geq 0} \|\rho_f(t)\|_{\infty} < \infty.$$

Characteristics

For $(s, \mathbf{x}, \mathbf{v}) \in \mathbb{R}_+ \times \mathbb{T}^d \times \mathbb{R}^d$, define $t \mapsto (\mathbf{X}_s^t, \mathbf{V}_s^t)(\mathbf{x}, \mathbf{v})$, value at time t of

$$\begin{cases} \dot{\mathbf{X}} = \mathbf{V}, \\ \dot{\mathbf{V}} = \mathbf{u}(t, \mathbf{X}) - \mathbf{V}, \end{cases}$$

with $(\mathbf{X}_s^s, \mathbf{V}_s^s)(\mathbf{x}, \mathbf{v}) = (\mathbf{x}, \mathbf{v})$.

The solution of

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}} \cdot (f(\mathbf{u} - \mathbf{v})) = 0,$$

is then given by

$$f(t, \mathbf{x}, \mathbf{v}) = e^{dt} f_0(\mathbf{X}_t^0(\mathbf{x}, \mathbf{v}), \mathbf{V}_t^0(\mathbf{x}, \mathbf{v})).$$

Tracking the uniform bound

In particular

$$\rho_f(t, \mathbf{x}) = \int_{\mathbb{R}^d} e^{dt} f_0(\mathbf{X}_t^0(\mathbf{x}, \mathbf{v}), \mathbf{V}_t^0(\mathbf{x}, \mathbf{v})) d\mathbf{v}.$$

[Bardos, Degond – 1985] (for Vlasov-Poisson) :

- if $\Gamma_{t,\mathbf{x}} : \mathbf{v} \mapsto \mathbf{V}_t^0(\mathbf{x}, \mathbf{v})$ is \mathcal{C}^1 -diffeomorphism
- with $|\det D_{\mathbf{v}} \Gamma_{t,\mathbf{x}}| \gtrsim e^{dt}$, then $\mathbf{w} = \Gamma_{t,\mathbf{x}}(\mathbf{v})$

$$|\rho_f(t, \mathbf{x})| \lesssim \int_{\mathbb{R}^d} \|f_0(\cdot, \mathbf{w})\|_{\infty} d\mathbf{w} < +\infty.$$

The straightening change of variable

$\Gamma_{t,x}$ satisfies the implicit equation

$$\Gamma_{t,x}(\mathbf{v}) = \mathbf{V}_t^0(\mathbf{x}, \mathbf{v}) = e^t \mathbf{v} - \int_0^t e^s \mathbf{u}(s, \mathbf{X}_t^s(\mathbf{x}, \mathbf{v})) ds,$$

which relates $\mathbf{v} \mapsto e^{-t} \Gamma_{t,x}(\mathbf{v})$ to the identity

Lemma

If $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is such that $\|\nabla \Phi\|_\infty < 1$, then

- $\Theta := \text{Id}_{\mathbb{R}^d} + \Phi$ is a \mathcal{C}^1 -diffeomorphism of \mathbb{R}^d ;
- $\inf_{\mathbb{R}^d} |\det \nabla \Theta| > 0$.

Another estimate to track

Corollary

There exists $c_d > 0$ such that, if

$$\int_0^t \|\nabla \mathbf{u}(s)\|_\infty ds < c_d,$$

then $\Gamma_{t,x}$ is a \mathcal{C}^1 -diffeomorphism with $|\det D_{\mathbf{v}}\Gamma_{t,x}| \gtrsim e^{dt}$.

Again $\nabla \mathbf{u} \in L^1(\mathbb{R}_+; L^\infty(\mathbb{T}^d))$ not encoded in the energy estimate.

$$\text{Defining } t^* = \sup \left\{ t > 0, \int_0^t \|\nabla \mathbf{u}(s)\|_\infty ds < c_d \right\},$$

the goal is now to prove $t^* := +\infty$.

Maximal regularity

Maximal regularity for the heat flow :

$$\mathbf{w}(0, \cdot) = 0 \Rightarrow \|\Delta \mathbf{w}\|_{L^p(0, T; L^q(\mathbb{T}^d))} \lesssim \|\partial_t \mathbf{w} - \Delta \mathbf{w}\|_{L^p(0, T; L^q(\mathbb{T}^d))}.$$

Here \mathbf{u} solution of

$$\partial_t \mathbf{u} - \Delta \mathbf{u} = \mathbb{P}[\mathbf{j}_f - \rho_f \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{u}] := \mathbf{F},$$

decomposed as $\mathbf{u} = e^{t\Delta} \mathbf{u}_0 + \mathbf{w}$, where $\mathbf{w}(0, \cdot) = 0$ and

$$\partial_t \mathbf{w} - \Delta \mathbf{w} = \mathbf{F},$$

$$\Rightarrow \|\mathbf{D}^2 \mathbf{u}\|_{L^p(\varepsilon, T; L^q(\mathbb{T}^d))} \lesssim \|\mathbf{F}\|_{L^p(0, T; L^q(\mathbb{T}^d))} + \|\mathbf{u}_0\|_2.$$

The bootstrap argument

Combine the previous with

- estimates that are valid on $[0, t^*)$ + Choi-Kwon Lemma ;
- Gagliardo-Nirenberg-Sobolev estimate

$$\|\nabla \mathbf{u}(s)\|_{\infty} \lesssim \|\mathbf{D}^2 \mathbf{u}(s)\|_q^{\alpha} \|\mathbf{u}(s) - \langle \mathbf{u}(s) \rangle\|_2^{1-\alpha};$$

- $\|\mathbf{u}(s) - \langle \mathbf{u}(s) \rangle\|_2 \lesssim \mathcal{E}(s)^{1/2}$;

$$\begin{aligned} \int_0^{t^*} \|\nabla \mathbf{u}(s)\|_{\infty} ds &\lesssim \mathcal{E}(0)^{\gamma} \left(1 + \|\mathbf{F}\|_{L^p(0, t^*; L^q(\mathbb{T}^d))}\right) \\ &\lesssim \mathcal{E}(0)^{\gamma} \\ &\Rightarrow t^* = +\infty. \end{aligned}$$

The regularity issue

- Computations on NS and instantaneous regularization
 - $d = 2$: true for all positive times (! of Leray solution)
 - $d = 3$: true for $t \geq 0$ as long as

$$\|\mathbf{u}_0\|_{H^{1/2}(\mathbb{T}^3)}^2 + C \int_0^t \|(\mathbf{j}_f - \rho_f \mathbf{u})(s)\|_{H^{-1/2}(\mathbb{T}^3)}^2 ds \leq \frac{1}{C^2}.$$

- Computations on the system (energy identities)
 - $d = 2$: true for all positive times ([HKM³ – 2017])
 - $d = 3$: part of the assumption

Sommaire

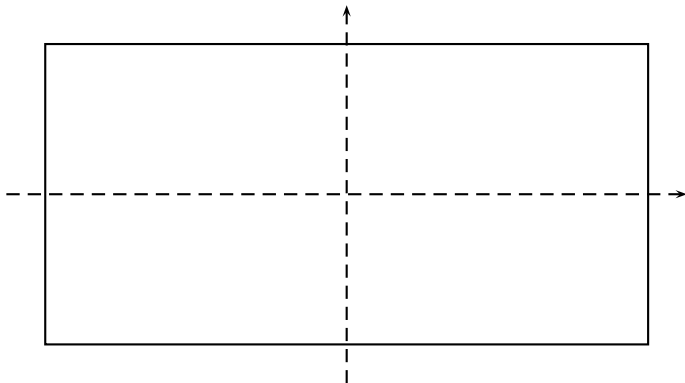
- ① The Vlasov-Navier-Stokes system
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Avoiding the concentration scenario ?

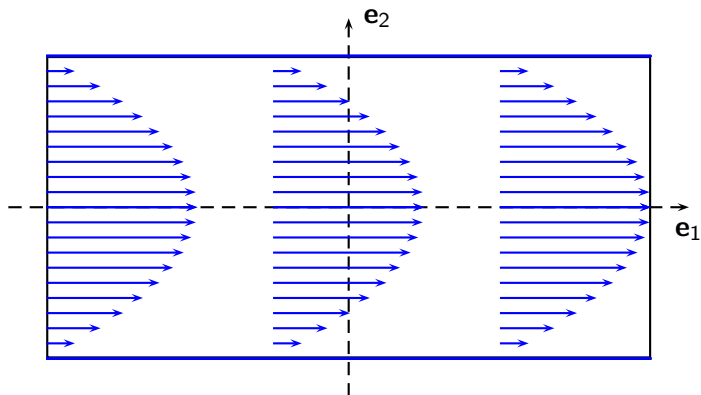
- \mathbb{R}^d ? Damping to 0 for \mathbf{u} (ongoing work)
- Common feature : absence of boundary
 - Eternity of influence of \mathbf{u} on f
 - Escape of particles ?
- \neq existence issues, geometry is important

→ An elementary example can reveal this phenomenon.

The domain : a 2D box

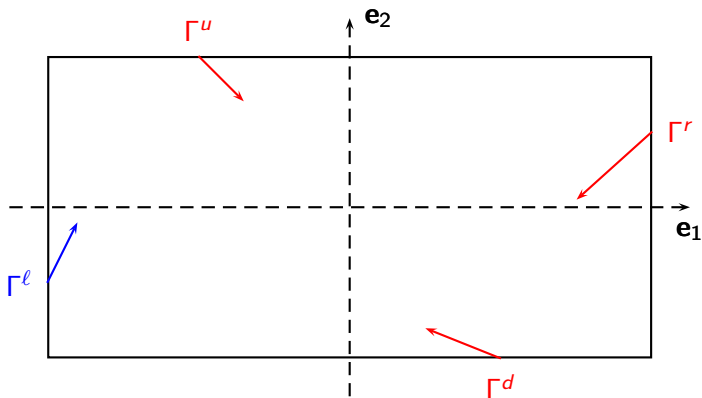


The Poiseuille flow



$\mathbf{u}_{\text{pois}}(t, \mathbf{x}) := (1 - \langle \mathbf{x}, \mathbf{e}_2 \rangle^2) \mathbf{e}_1$ is a stationary solution of NS

Entering phase space boundary



$$\Gamma^l \cup \Gamma^u \cup \Gamma^r \cup \Gamma^d = \{(\mathbf{x}, \mathbf{v}) \in \partial\Omega \times \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{n}(\mathbf{x}) < 0\}$$

The whole system

For $(t, \mathbf{x}, \mathbf{v}) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^2$ we consider

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla p = - \int_{\mathbf{v}} (\mathbf{u} - \mathbf{v}) f \, d\mathbf{v}, \\ \operatorname{div} \mathbf{u} = 0, \\ \partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}} \cdot (f(\mathbf{u} - \mathbf{v})) = 0, \end{cases}$$

with initial conditions (\mathbf{u}_0, f_0) and boundary conditions :

$$\mathbf{u} = \mathbf{u}_{\text{pois}} \text{ on } \mathbb{R}_+ \times \partial\Omega,$$

$$f = \psi \text{ on } \Gamma^\ell,$$

$$f = 0 \text{ on } \Gamma^u \cup \Gamma^d \cup \Gamma^r.$$

The exit geometric condition

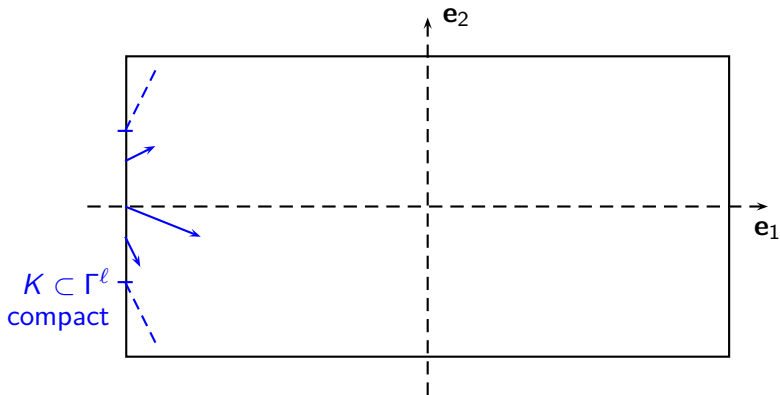
- $T > 0$;
- \mathbf{u} a vector field ;
- K a compact set of the phase space (\mathbf{x}, \mathbf{v}) ;

Definition

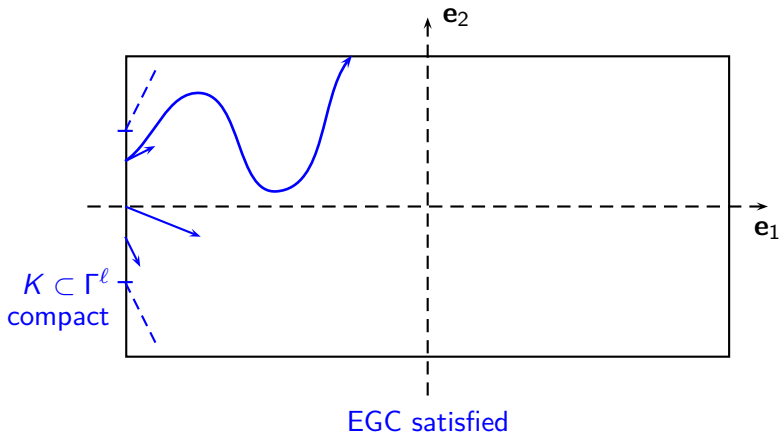
(\mathbf{u}, K) satisfies the EGC in time T if any trajectory starting from K has a lifetime $< T$ inside Ω (with transversal exit).

→ Reminiscent of [Bardos, Lebeau, Rauch – 1992]

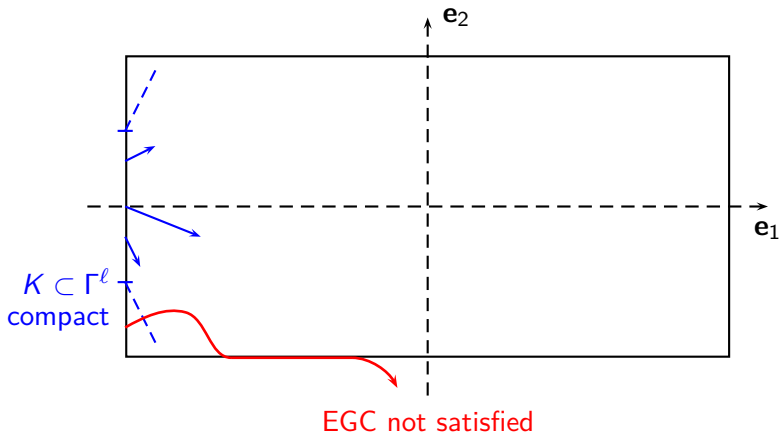
The exit geometric condition (EGC)



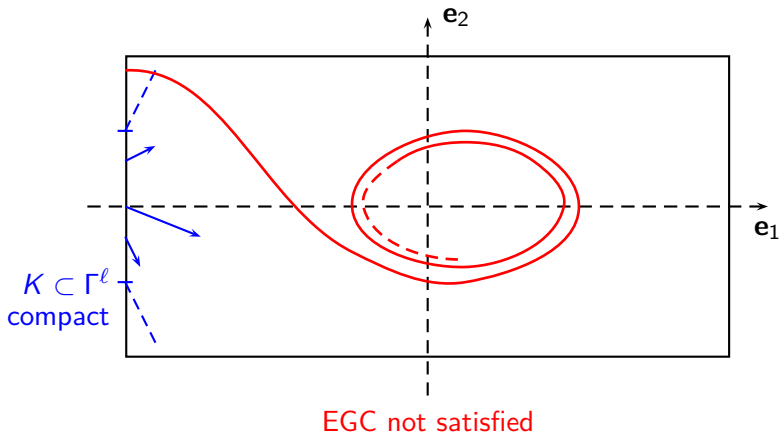
The exit geometric condition (EGC)



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The exit geometric condition (EGC)



Stability of the EGC

- given \mathbf{u} , not obvious to exhibit a compatible K
- \mathbf{u}_{pois} : EGC $\Leftrightarrow |x_2 + v_2| \neq 1$ for $(\mathbf{x}, \mathbf{v}) \in K$

Lemma

If (\mathbf{u}^*, K) satisfies the EGC in time T , then for $T^* > T$ there exists $\delta > 0$ such that

$$\sup_{t \geq 0} \int_t^{t+T^*} \|\mathbf{u}(\tau) - \mathbf{u}^*\|_{L^\infty(\Omega)} d\tau < \delta$$

$\Rightarrow (\mathbf{u}, K)$ satisfies EGC in time T^* .

Main result

Theorem (Glass, Han-Kwan, M. – 2018)

There exists non trivial smooth stationary solutions of VNS which are furthermore asymptotically stable (for adequate perturbations).

- example of non trivial non singular equilibrium ;
- stationary solutions built near $(\mathbf{u}_{\text{pois}}, 0)$;
- admissible perturbation : EGC remains satisfied.

Thank you !