Optimal inequalities and improvements via the use of linear and nonlinear flows

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Attainability and value of best constants in functional inequalities

Functional inequalities play an important role in obtaining a priori estimates for solutions of PDEs, in analyzing the long time behavior of solutions of evolution problems, in describing the blow-up profile for finite time blow-up phenomena, etc

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Many examples : Hardy, Hölder, Poincaré, Jensen, Nash, Sobolev, Gagliardo-Nirenberg, log Sobolev, Caffarelli-Kohn-Nirenberg,... : a very important toolbox in analysis and geometry.

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Important questions :

- What is the value of best constant ?
- Is it attained ? If yes, how do the optimal/extremal functions look like ?

Gagliardo-Nirenberg-Sobolev inequalities on the sphere

On the d-dimensional sphere, consider the interpolation inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(S^d)} \geq \frac{d}{p-2} \left(\|u\|^2_{\mathrm{L}^p(S^d)} - \|u\|^2_{\mathrm{L}^2(S^d)} \right) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\nu) \quad (1)$$

where the measure $d\nu$ is the uniform probability measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ and the exposant $p \geq 1$, $p \neq 2$, is such that $p \leq 2^* := \frac{2\,d}{d-2}$ if $d \geq 3$.

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The case $p = \frac{2\,d}{d-2}$ corresponds to the Sobolev inequality

$$\int_{\mathbb{R}^d} |\nabla v|^2\, dx \ge S \left(\int_{\mathbb{R}^d} |v|^{\frac{2\,d}{d-2}}\, dx\right)^{\frac{d-2}{d}} \quad \forall\, u \in \mathrm{H}^1(\mathbb{R}^d)\,,$$

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Proofs of (1) + minimizers are constants by : **Bidaut-Véron – Véron** (PDE, rigidity methods, 1991) ; **Beckner** (harmonic analysis methods, 1993) ; **Bakry-Emery (1985)**) ("carré du champ" method, linked to a flow method) only for $2 < p \leq 2^\# := \frac{2\,d^2+1}{(d-1)^2} < 2^*$).

Linear flow method - the Bakry-Emery point of view

Let us define $\rho = |u|^p$. The two inequalities below are equivalent

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$$\int_{S^d} |\nabla \rho^{\frac{1}{p}}|^2 \, d\omega \geq \frac{d}{p-2} \left[\left(\int_{S^d} \rho \, d\omega\right)^{\frac{2}{p}} - \int_{S^d} \rho^{\frac{2}{p}} \, d\omega\right] .$$

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So, we need to prove $\mathcal{I}_p[\rho] \geq d\,\mathcal{E}_p[\rho]$, with

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To establish such inequalities, one can use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho \ , \quad \text{with } \Delta = \text{ the Laplace-Beltrami operator on } \mathbb{S}^d$$

We have $\quad \frac{d}{dt}\left(\int_{S^d} \rho \, d\omega\right) = 0$

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and if $p \leq 2^{\#}$, $\displaystyle \frac{d}{dt} \mathcal{E}_p[\rho] = -\,2\,\mathcal{I}_p[\rho]$ and $\displaystyle \frac{d}{dt} \mathcal{I}_p[\rho] \leq -\,2\,d\,\mathcal{I}_p[\rho]$.

Nonlinear versus linear flow I

The goal is to prove $\mathcal{I}_p[\rho_0] - d\,\mathcal{E}_p[\rho_0] \geq 0\,,\ \forall\,\rho_0$. For $p \leq 2^\#$, $\rho(0,\cdot) = \rho_0$,

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \leq 2\,(-d+d)\,\mathcal{I}_p[\rho] = 0\,.$$

Not difficult to prove that ρ converges to a constant as $t \to +\infty$ and

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Actually,

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \leq -G[\ldots,|\nabla\rho|]\,, \quad G \geq 0\,,\ \ G[\ldots,s] = 0 \ \text{ iff } s = 0$$

proving that the only extremals for the inequality (the only minimizers for $I_p - \,d\,\mathcal{E}_p$, are the constant functions.

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THM [Dolbeault, E., Loss, 2017]. When $2^{\#} < p < 2^*$, we can find a function ρ_0 such that ρ solution of $\frac{\partial \rho}{\partial t} = \Delta \rho\,,\;\; \rho(t=0) = \rho_0\,$, and

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - \,d\,\mathcal{E}_p[\rho]\Big)\Big|_{t=0} > 0\,.$$

Then, we can get the same result by considering the flow $\frac{\partial \rho}{\partial t} = \Delta \rho^m\,$, for a well-chosen $m \neq 1$.

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REMARK. The proof above could actually be reduced to checking that for all ρ,

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big)\Big|_{t=0} = \left(-\Delta\rho_0 - \frac{d}{p-2}\left(\frac{C\rho_0^{p-1}}{p} - \rho_0\right)\right)\cdot \Delta\rho_0^m \leq 0\,,$$

as soon as $1 < p < 2^{\#}$, $p \neq 2$.

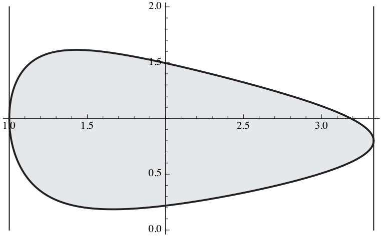
The evolution under the fast diffusion flow

To overcome the limitation $p \leq 2^{\#}$, we have considered a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m$$

[Demange], [Dolbeault, E., Kowalczyk, Loss] : for any $p \in [1, 2^*)$

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) < 0$$



(p, m) admissible region, $d = 5$

Use of linear/nonlinear flows may lead to improved inequalities (having in mind the Bianchi-Egnell result...)

Define the *entropy* and the *Fisher information* respectively by

$$\mathsf{e} := \frac{1}{p-2}\left(\|u\|^2_{\mathrm{L}^p(S^d)} - \|u\|^2_{\mathrm{L}^2(S^d)}\right) \quad \text{and} \quad \mathsf{i} := \|\nabla u\|^2_{\mathrm{L}^2(S^d)}\ .$$

Before we have proved that for all $p \in (1, 2^\sharp]$, $p \neq 2$, we have

$$\mathsf{i} \geq d\, \mathsf{e}$$

Towards an improved inequality I

For any two tensors, $\mathrm{A}_{i,j}$, $\mathrm{B}_{i,j}$, let us define,

$$\mathrm{A} : \mathrm{B} := g^{i,m}\, g^{j,n}\, \mathrm{A}_{i,j}\, \mathrm{B}_{m,n} \quad \text{and} \quad \|\mathrm{A}\|^2 := \mathrm{A} : \mathrm{A}\,,$$

where $g^{i,j}$ is the inverse of the metric tensor, *i.e.*, $g^{i,j}\, g_{j,k} = \delta^i_k$.

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Next we define the following trace free quantities :

$$\mathrm{L}u := \mathrm{H}u - \frac{1}{d}\,(\Delta u)\, g \text{ (trace free Hessian)}, \quad \mathrm{M}u := \frac{\nabla u \otimes \nabla u}{u} - \frac{1}{d}\,\frac{|\nabla u|^2}{u}\, g\,.$$

A not so straightforward computation shows that

$$\frac{1}{2}\,\frac{d}{dt}\,(\mathsf{i} - d\,\mathsf{e}) = -\,\frac{d}{d-1}\int_{S^d} \left\| \mathrm{L}u - (p-1)\,\tfrac{d-1}{d+2}\,\mathrm{M}u \right\|^2 d\mu - \gamma \int_{S^d} \frac{|\nabla u|^4}{u^2}\, d\mu \le 0$$

with $\gamma = \left(\frac{d-1}{d+2}\right)^2 (p-1)\,(2^\# - p)$ if $d \ge 2$, $\gamma = \frac{p-1}{3}$ if $d = 1$

Towards an improved inequality II

$$\frac{1}{2}\,\frac{d}{dt}\,(\mathsf{i}-d\,\mathsf{e})=-\,\frac{d}{d-1}\int_{S^d}\left\|\mathrm{L}u-(p-1)\,\tfrac{d-1}{d+2}\,\mathrm{M}u\right\|^2 d\mu-\gamma\int_{S^d}\frac{|\nabla u|^4}{u^2}\,d\mu\le 0$$

$$\mathsf{i}^2=\left(\int_{S^d}u\cdot\frac{|\nabla u|^2}{u}\,d\mu\right)^2\le\int_{S^d}u^2\,d\mu\int_{S^d}\frac{|\nabla u|^4}{u^2}\,d\mu=(1-(p-2)\,\mathsf{e})\int_{S^d}\frac{|\nabla u|^4}{u^2}\,d\mu$$

$\mathsf{e}'=-2\,\mathsf{i}$ implies that $\boxed{\mathsf{e}''+2\,d\,\mathsf{e}'-\dfrac{\gamma\,|\mathsf{e}'|^2}{1-(p-2)\,\mathsf{e}}\ge 0}$, so that

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$$\boxed{\mathsf{i} \ge d\,\|u\|^2_{\mathrm{L}^p(S^d)}\,\phi\left(\frac{\mathsf{e}}{\|u\|^2_{\mathrm{L}^p(S^d)}}\right)}$$

with

$$\phi(\mathsf{e}) = \frac{1-(p-2)\,\mathsf{e}-(1-(p-2)\,\mathsf{e})^{-\frac{\gamma}{p-2}}}{2-p-\gamma}\,.$$

And : $\phi(0)=0$, $\phi'(0)=1$ and $\phi''\ge 0$, so that $\phi(\mathsf{e})\ge\mathsf{e}$ with equality if and only if $\mathsf{e}=0$

Consequences of improved inequality

Among other things, if we take into account the generalized Csiszár-Kullback-Pinsker type inequality

$$\mathsf{e} = \frac{1}{p-2}\left[\|u\|^2_{\mathrm{L}^p(S^d)} - \|u\|^2_{\mathrm{L}^2(S^d)}\right] \geq \mathsf{C}\;\|u\|^{2\,(1-r)}_{\mathrm{L}^s(S^d)}\;\|u^r - \bar{u}^r\|^2_{\mathrm{L}^q(S^d)}$$

where $\bar{u} = \|u\|_{\mathrm{L}^r(S^d)}$ and r depends on p. We get

$$\mathsf{i} - d\,\mathsf{e} \geq d\;\|u\|^2_{\mathrm{L}^p(S^d)}\;\varphi\left(\mathsf{C}\,\frac{\|u\|^{2\,(1-r)}_{\mathrm{L}^s(S^d)}}{\|u\|^2_{\mathrm{L}^p(S^d)}}\;\|u^r - \bar{u}^r\|^2_{\mathrm{L}^q(S^d)}\right)\quad \forall\, u \in \mathrm{H}^1(S^d)\,.$$

where the function φ is nondecreasing, positive on $(0,\infty)$ and such that $\varphi(0) = \varphi'(0) = 0$.

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Bianchi-Egnell : for $p = 2^*$, $\mathsf{i} - d\,\mathsf{e} \ge C \min_{c>0} \|\nabla(\tilde{u} - \tilde{c})\|_{L^2(\mathbb{R}^d)}$, $\tilde{v} =$ ster-proj (v))

General link between interpolation inequalities and spectral estimates

For a second order elliptic operator L, consider the minimization problem

$$\mu_D(\alpha) := \min_{u\in H^1(D)} \frac{(Lu,u)+\alpha\,(u,u)}{\left(\int_D |u|^p\right)^{2/p}}\,, \quad 2<p<2^* := \frac{2d}{d-2}\,,$$

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Then, for $\mu = \|V_+\|_{L^q(D)}$, $\frac{1}{q} + \frac{2}{p} = 1$, $q \in (d/2, +\infty)$,

$$\frac{((L-V)u, u)}{(u, u)} \geq \frac{(Lu, u) - (V_+ u, u)}{(u, u)} \geq \frac{(Lu, u) - \mu\,\left(\int_D |u|^p\right)^{2/p}}{(u, u)} \geq -\alpha_D(\mu)$$

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(2) $\qquad \lambda_1(L - V) \geq -\alpha_D\left(\|V_+\|_{L^q(D)}\right)$ (Keller-Lieb-Thirring inequality)

$$V_{opt} = \mu\,|u_{min}|^{p-2}$$

Caffarelli-Kohn-Nirenberg (CKN) inequalities (1984)

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$

with $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, $a \neq \frac{d-2}{2}$

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$$\frac{1}{C_{a,b}} = \inf_{\mathcal{D}_{a,b}} \quad \frac{\int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx}{\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}}$$

The symmetry issue

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$

$\mathsf{C}_{a,b}$ = best constant for general functions v
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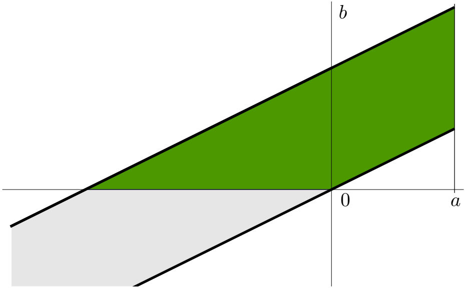
Questions : is optimality (equality) achieved ? do we have $v_{a,b} = v^*_{a,b}$?

Symmetry $(d \geq 3)$



Case $a > 0$: Th. Aubin, G. Talenti, E. Lieb, Chou-Chu, P.L. Lions, Horiuchi,...

Case $a < 0$: Lin, Wang ; Dolbeault, E., Tarantello (d=2) ; Betta-Brock-Mercaldo-Posteraro ($b > 0$)

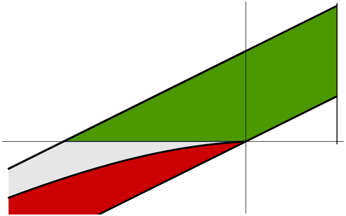
Linear instability of radial minimizers : the Felli-Schneider curve

Catrina, Wang (2001) looked for the set of pairs (a, b) such that the functional

$$C^\star_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla w|^2}{|x|^{2\,a}}\,dx - \left(\int_{\mathbb{R}^d} \frac{|w|^p}{|x|^{b\,p}}\,dx\right)^{2/p}$$

is linearly instable at $w = w^*_{a,b}$ (Catrina, Wang (2001) ; Felli, Schneider (2003)). This happens for

$$b < b^{FS}(a) := \frac{d(d-2-2a)}{2\sqrt{(d-2-2a)^2+4(d-1)}} - \frac{1}{2}\,(d-2-2a)$$

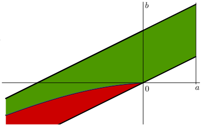


The conjecture

CONJECTURE : for the CKN problem, symmetry breaking can only arise from the instability of the symmetric extremal.

Solution of the conjecture : A Sobolev type inequality

THEOREM [2016].- In the stability region, if $d \geq 2$, the optimality is achieved by radial functions.



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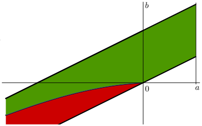

Idea of the proof : with the change of variables $r \mapsto r^\alpha$, $v(r,\omega) =: w(r^\alpha,\omega)$, and with

$$n = \frac{d - b\,p}{\alpha} = \frac{d - 2\,a - 2}{\alpha} + 2 \quad \left(\text{equivalent to } \; p = \frac{2n}{n-2}\right)$$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \leq \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \qquad \text{becomes}$$

$$\left(\int_{\mathbb{R}^d} |w|^{\frac{2n}{n-2}}\,r^{n-1}\,dr\,d\omega\right)^{\frac{n-2}{n}} \leq \mathsf{C}_{\alpha,n} \int_{\mathbb{R}^d} |\mathsf{D}_\alpha w|^2\,r^{n-1}\,dr\,d\omega\,,$$

$$\text{with} \quad \mathsf{D}_\alpha w = \left(\alpha\,\frac{\partial w}{\partial r}, \frac{1}{r}\,\nabla_\omega w\right)$$

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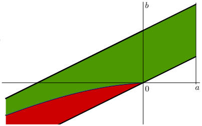

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In this setting, the stability region is characterized by $\alpha \leq \alpha_{\rm FS} := \sqrt{\frac{d-1}{n-1}}$

The flow

STRATEGY : Find a flow such that for all $\alpha \leq \alpha_{\mathrm{FS}}$,

1) Define $u := |w|^{\frac{2n}{n-2}}$ and $\mathcal{I}[u(t,\cdot)] := \int_{\mathbb{R}^d} |\mathrm{D}_\alpha w|^2 \, r^{n-1} \, dr \, d\omega$.

2) for all $t \geq 0$, prove that $\frac{d}{dt} \int_{\mathbb{R}^d} u(t) \, d\mu = 0$ and $\frac{d}{dt} \mathcal{I}[u(t,\cdot)] \leq 0$,

3) prove that $\frac{d}{dt} \mathcal{I}[u(t,\cdot)] = 0$ means, in particular, that u is radially symmetric.

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$$\frac{\partial u}{\partial t} = \mathrm{L}_\alpha u^m\,, \quad m = 1 - \frac{1}{n}$$

where we define the self-adjoint operator L_α by

$$\mathrm{L}_\alpha w := -\,\mathrm{D}_\alpha^*\, \mathrm{D}_\alpha\, w = \alpha^2\, w'' + \alpha^2\, \frac{n-1}{r}\, w' + \frac{\Delta_\omega\, w}{r^2}$$

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The above strategy is carried out by performing lots of calculations, using some differential geometry tools and proving delicate regularity results. And again, and important here, we perform an elliptic proof !

Elliptic proof

– Consider the (elliptic) Euler-Lagrange equation associated to the inequality and multiply it by the right hand side of the evolution equation defining the flow.

– Then perform a lot of integration by parts and use Bochner-Lichnerovich-Weitzenböck's inequality on spheres.

– Get rid of the "boundary" terms in the integrations by parts (near the origin and at infinity) by proving quite delicate regularity properties for the positive solutions of the equation.

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So, the flow **guides us** as to which kind of multiplier we have to multiply the Euler-Lagrange equation by.

A little bit like what we do to obtain Pohozaev identities, where the multiplier is $x \cdot \nabla u$.

INEQUALITIES WITH EXTERNAL MAGNETIC FIELDS

Three magnetic interpolation inequalities and their dual forms : Magnetic Laplacian and spectral gap

(Dolbeault, E., Laptev, Loss)

In dimensions $d = 2$ and $d = 3$: the *magnetic Laplacian* is

$$-\Delta_{\mathbf{A}}\,\psi = -\Delta\,\psi - 2\,i\,\mathbf{A}\cdot\nabla\psi + |\mathbf{A}|^2\psi - i\,(\operatorname{div}\mathbf{A})\,\psi$$

where the magnetic potential (resp. field) is $\mathbf{A}$ (resp. $\mathbf{B} = \operatorname{curl}\mathbf{A}$) and

$$\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d) := \left\{\psi \in \mathrm{L}^2(\mathbb{R}^d)\,:\,\nabla_{\mathbf{A}}\psi \in \mathrm{L}^2(\mathbb{R}^d)\right\}\,,\quad \nabla_{\mathbf{A}} := \nabla + i\,\mathbf{A}$$

Spectral gap inequality :

$$\|\nabla_{\mathbf{A}}\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \Lambda[\mathbf{B}]\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d) \tag{1}$$

− Λ depends only on $\mathbf{B} = \operatorname{curl}\mathbf{A}$
− Assumption : *equality holds for some* $\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)$
− If $\mathbf{B}$ is a constant magnetic field, $\Lambda[\mathbf{B}] = |\mathbf{B}|$

Magnetic interpolation inequalities

THEOREM (E.-Lions (1989), Dolbeault-E.-Laptev-Loss (2017)).- Under some technical conditions on **A**, there exist constants $\mu_{\mathbf{B}}(\alpha)$, $\nu_{\mathbf{B}}(\beta)$ and $\xi_{\mathbf{B}}(\gamma)$ such that :

$$\|\nabla_{\mathbf{A}}\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \mu_{\mathbf{B}}(\alpha)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{R}^d)}\,, \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d) \tag{2}$$

for any $\alpha \in (-\Lambda[\mathbf{B}], +\infty)$ and any $p \in (2, 2^*)$,

$$\|\nabla_{\mathbf{A}}\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \beta\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{R}^d)} \ge \nu_{\mathbf{B}}(\beta)\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\,, \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d) \tag{3}$$

for any $\beta \in (0, +\infty)$ and any $p \in (1, 2)$

$$\|\nabla_{\mathbf{A}}\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \gamma \int_{\mathbb{R}^d} |\psi|^2 \log\left(\frac{|\psi|^2}{\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\right) dx + \xi_{\mathbf{B}}(\gamma)\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\,, \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R} \tag{4}$$

(limit case corresponding to $p = 2$) for any $\gamma \in (0, +\infty)$

Estimates for the best constants : general magnetic fields

(Dolbeault, E., Laptev, Loss, 2017)

– We know very little in the general case : some analytical lower estimates and some numerical upper estimates for the best constants $\mu_{\mathbf{B}}(\alpha)$ and $\nu_{\mathbf{B}}(\beta)$.

– In the case of constant magnetic fields, and $d = 2$, we can improve the lower estimates and we observe that those are very very close to the numerical upper estimates from radially symmetric test functions.

– **CONJECTURE :** for $d = 2$ and constant magnetic fields, the optimizers for the inequalities are radially symmetric and positive functions. This is true for α large enough.

Aharonov-Bohm magnetic fields

(Bonheure, Dolbeault, E., Laptev, Loss, 2018)

On the two-dimensional Euclidean space $\mathbb{R}^2$, let us introduce the Aharonov-Bohm magnetic potential

$$\mathbf{a}(\mathbf{x}) = a\left(\frac{x_2}{|\mathbf{x}|^2}, \frac{-x_1}{|\mathbf{x}|^2}\right)\,, \quad a \in \mathbb{R}\,;\;\; \mathbf{b} = \operatorname{curl}\mathbf{a}\,.$$

Magnetic Schrödinger energy (polar coordinates) :

$$\int_{\mathbb{R}^2} |(i\,\nabla + \mathbf{a})\,\Psi|^2\,d\mathbf{x} = \int_0^{+\infty}\!\!\int_{-\pi}^{\pi}\Big(|\partial_r\Psi|^2 + \frac{1}{|x|^2}\,|\,\partial_\theta\Psi - i\,a\,\Psi|^2\Big)\,r\,\mathrm{d}\theta\,\mathrm{d}r$$

Magnetic rings : magnetic interpolation in 1d, with Aharonov-Bohm potential

We want to characterize the *optimal constant* in the inequality

$$\|\psi' - i\, a\, \psi\|^2_{\mathrm{L}^2(S^1)} + \alpha\, \|\psi\|^2_{\mathrm{L}^2(S^1)} \ge \mu_{a,p}(\alpha)\, \|\psi\|^2_{\mathrm{L}^p(S^1)} \tag{5}$$

written for any $\psi \in X_a$, $p \in (2, +\infty)$,

$$\mu_{a,p}(\alpha) := \inf_{\psi \in X_a \setminus \{0\}} \frac{\int_{-\pi}^{\pi} \left(|\psi' - i\, a\, \psi|^2 + \alpha\, |\psi|^2\right) \mathrm{d}\sigma}{\|\psi\|^2_{\mathrm{L}^p(S^1)}}\,, \quad d\sigma = d\theta/2\pi\,.$$

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(ii) if $a \in (0, 1/2)$ and $a^2\,(p+2) + \alpha\,(p-2) > 1$, then $\mu_{a,p}(\alpha) < a^2 + \alpha$ and equality in (5) is not achieved by the constant functions

Case of dimension 2 (Bonheure, Dolbeault, E., Laptev, Loss)

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QUESTION : Is there a flow adapted to this case that could lead to optimal results ?

Très bon anniversaire au Labo Lions ! ! !