Quadratic study of the small time local controllability

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Small time local controllability = local surjectivity

We consider an scalar-input affine system

$$\frac{dx}{dt} = f_0(x) + u(t) f_1(x)$$

where $f_0, f_1 : \mathbb{R}^n \to \mathbb{R}^n$, $u : [0, T] \to \mathbb{R}$, $x : [0, T] \to \mathbb{R}^n$, $f_0(0) = 0$

Definition

This system is *small-time locally controllable* (STLC) when

$$\forall T, \eta > 0, \quad \exists \delta > 0, \quad \forall x_f \in B_{\mathbb{R}^n}(0, \delta), \quad \exists u : [0, T] \to \mathbb{R}, \\ \text{such that} \quad \|u\|_\infty \leqslant \eta, \quad \text{and} \quad x\big(T\,;\,u\,,\,0\big) = x_f\,.$$

= **Local surjectivity at** 0 **of the end point map** $\mathcal{F}_T$ **for every** $T > 0$

$$\left| \begin{array}{rcl} \mathcal{F}_T : \quad L^\infty(0, T) & \to & \mathbb{R}^n \\ u & \mapsto & x\big(T\,;\,u\,,\,0\big) \end{array} \right.$$

Functional framework influence : The choice of L^∞ is historical but arbitrary. One can study the local surjectivity at 0 of the map $\mathcal{F}_T$ with other base spaces, e.g. $W^{m,\infty}$ with $m \geq -1$.

A priori, the answer may depend on m, even in finite dimension.

Linear theory : insensible to the input's regularity

$$\mathcal{F}_T(u) = \mathcal{F}_T(0) + d\mathcal{F}_T(0).u + O(\|u\|^2)$$

By the inverse mapping theorem, if $\mathcal{F}_T : L^\infty(0,T) \to \mathbb{R}^n$ is C^1 and $d\mathcal{F}_T(0) : L^\infty(0,T) \to \mathbb{R}^n$ is onto, then $\mathcal{F}_T$ is locally onto at 0.

$$\left| \begin{array}{llll} \mathcal{F}_T : & L^\infty(0,T) & \to & \mathbb{R}^n \\ & u & \mapsto & x(T) \end{array} \right. \qquad \left\{ \begin{array}{l} \dot{x} = f_0(x) + u(t) f_1(x) \\ x(0) = 0 \end{array} \right.$$

$$\left| \begin{array}{llll} d\mathcal{F}_T(0) : & L^\infty(0,T) & \to & \mathbb{R}^n \\ & u & \mapsto & y(T) \end{array} \right. \qquad \left\{ \begin{array}{l} \dot{y} = Ay + bu(t) \\ y(0) = 0 \end{array} \right.$$

$$A := f_0'(0)\,, \quad b = f_1(0)\,.$$

- OK under **Kalman condition :** $\mathrm{rank}\{b, Ab, \dots A^{n-1}b\} = n$
- Controls explicit (A, b, x_f), can be $C_c^\infty(0,T)$ and C^k-small $\forall k$
  STLC = smooth STLC
- Otherwise, one needs to go further in the power series expansion
  $\mathcal{F}_T(u) = \mathcal{F}_T(0) + d\mathcal{F}_T(0).u + \frac{1}{2} d^2\mathcal{F}_T(0).(u,u) + O(\|u\|^3)$

Structure of this talk

Goal : Illustrate possible behaviors for scalar input control systems stemming from the analysis of their **second-order expansions** :

1. Quadratic obstructions **in finite dimension** (ODE) : complete classification
2. Specific behaviors **in infinite dimension** (PDE) :
   - New quadratic obstructions
   - Recovering directions

Underline the influence of the **functionnal framework**.

Quadratic examples

Let $n = 3$ and $x = (x_1, x_2, x_3)$. We consider a system for which the order 1 Taylor expansion is given by

$$\begin{cases} \dot{x}_1 = u & + \dots \\ \dot{x}_2 = x_1 & + \dots \\ \dot{x}_3 = 0 & + \dots \end{cases}$$

$$x_1(T) = \int_0^T u(t)dt \qquad x_2(T) = \int_0^T (T-t)u(t)dt$$

The linearized system is not controllable since we cannot move in the directions $\pm e_3$. We must compute a second-order Taylor expansion of the dynamics and of $\mathcal{F}_T$.

Quadratic examples : manifold case

$$\begin{cases} \dot{x}_1 = u & + 0 \\ \dot{x}_2 = x_1 & + 0 \\ \dot{x}_3 = 0 & + 2ux_1 \end{cases}$$

One sees that $\dot{x}_3$ is actually a total derivative. If we make a local change of coordinates for $(y_1 = x_1, y_2 = x_2, y_3 = x_3 - x_1^2)$, we obtain

$$\begin{cases} \dot{y}_1 = u & + 0 \\ \dot{y}_2 = y_1 & + 0 \\ \dot{y}_3 = 0 & + 0 \end{cases}$$

Rk : Idem with the crossed term "$x_1 x_2$"

Such quadratic terms bend the reachable space, without creating any new direction.

Quadratic examples : first drift case

$$\begin{cases} \dot{x}_1 = u & +0 \\ \dot{x}_2 = x_1 & +0 \\ \dot{x}_3 = 0 & +x_1^2 \end{cases}$$

There is a drift in the dynamics, along $[f_1, [f_1, f_0]](0)$ and one can never reach the half-space $x_3 < 0$ because

$$x_3(T) = \int_0^T x_1(t)^2 \mathrm{d}t = \int_0^T \left(\int_0^t u(s) \mathrm{d}s \right)^2 \mathrm{d}t \approx \|u\|_{H^{-1}}^2.$$

The system is $\dot{x} = f_0(x) + u f_1(x)$ with

$$f_0 = \begin{pmatrix} 0 \\ x_1 \\ x_1^2 \end{pmatrix} \quad f_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad [f_1, [f_1, f_0]](0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} \dot{x}_1 = u & + 0 \\ \dot{x}_2 = x_1 & + 0 \\ \dot{x}_3 = 0 & + x_2^2 \end{cases}$$

There is a drift in the dynamics, along $[\mathrm{Ad}_{f_0}(f_1), \mathrm{Ad}_{f_0}^2(f_1)](0)$ and one can never reach the half-space $x_3 < 0$ because

$$x_3(T) = \int_0^T x_2(t)^2 \mathrm{d}t = \int_0^T \left(\int_0^t (t-s) u(s) \mathrm{d}s \right)^2 \mathrm{d}t \approx \|u\|_{H^{-2}}^2.$$

The system is $\dot{x} = f_0(x) + u f_1(x)$ with

$$f_0 = \begin{pmatrix} 0 \\ x_1 \\ x_2^2 \end{pmatrix} \quad f_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad [\mathrm{Ad}_{f_0}(f_1), \mathrm{Ad}_{f_0}^2(f_1)](0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} \dot{x}_1 = u & + 0 \\ \dot{x}_2 = x_1 & + 0 \\ \dot{x}_3 = 0 & + x_1^2 - x_2^2 \end{cases}$$

There is a drift in the dynamics, along $[f_1, [f_1, f_0]](0)$ **for small enough times** and one can never reach the half-space $x_3 < 0$ because

$$x_3(T) = \int_0^T x_1^2(t)\mathrm{d}t - \int_0^T x_2^2(t)\mathrm{d}t$$
$$\geq (1 - T^2) \int_0^T x_1^2(t)\mathrm{d}t.$$

$[f_1, [f_1, f_0]]$ is a 'bad bracket'.

Quadratic examples : cubic-quad competition

$$\begin{cases} \dot{x}_1 = u & +0 & +0 \\ \dot{x}_2 = x_1 & +0 & +0 \\ \dot{x}_3 = 0 & +x_2^2 & +x_1^3 \end{cases}$$

Is it still true that $x_3(T) \geq 0$? When do we have

$$\|x_1\|_{L^3}^3 = o\left(\|x_2\|_{L^2}^2\right), \quad i.e. \quad \|\phi'\|_{L^3}^3 = o\left(\|\phi\|_{L^2}^2\right)?$$

This holds when $\|\phi'''\|_{L^\infty} \ll 1$ so **when the control u is small enough in $W^{1,\infty}$**. This is the good regularity to guarantee that the quadratic approximation is a good approximation of the nonlinear system.

$$\int_0^T (\phi')^3 = -2\int_0^T \phi\phi'\phi'' = \int_0^T \phi^2\phi''' \quad \leqslant \|\phi'''\|_\infty \int_0^T \phi^2$$

This system is not $W^{1,\infty}$-STLC, but it is L^∞-STLC.

$[\mathrm{Ad}_{f_0}(f_1), \mathrm{Ad}_{f_0}^2(f_1)]$ is a bad bracket for $W^{1,\infty}$-STLC
But it is not necessarily a bad bracket for L^∞-STLC.

Classification of quadratic behaviors in finite dimension

$$\dot{x} = f_0(x) + u f_1(x)$$

$S_k = \text{Span}\{$ Lie brackets of f_0 and f_1 involving f_1 exactly k times$\}$

Theorem (KB Marbach 2017)

Let $f_0, f_1 \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $f_0(0) = 0$. Assume that the linearized system is not controllable. Then the following alternative holds :

- *either $S_2(0) \subset S_1(0)$ and then the quadratic terms bend the reachable space but without creating any new direction : the state lives within a manifold, up to cubic terms,*
- *or $S_2 \not\subset S_1(0)$ and then there exists a quadratic drift, quantified by the H^{-k} norm of the control, in the direction $[\mathrm{Ad}^k_{f_0}(f_1), \mathrm{Ad}^{k-1}_{f_0}(f_1)](0)$ which prevents small-time local controllability with regular controls $W^{2k-3,\infty}(0,T)$*

$$\mathbb{P}[x(T)] = \int_0^T u_k(t)^2 dt \quad [\mathrm{Ad}^k_{f_0}(f_1), \mathrm{Ad}^{k-1}_{f_0}(f_1)](0) \quad + O\left(\int_0^t |u_1|^3\right)$$

Conclusion :

- The condition $S_2(0) \subset S_1(0)$ is necessary for smooth-STLC.
- For $x' = f_0(x) + uf_1(x)$, with $x \in \mathbb{R}^n$, there are at most $(n-1)$ quadratic obstructions.
- The k-th one involves a H^{-k}-drift.
- It prevents $W^{2k-3,\infty}$-STLC.
- All functional spaces are optimal.

Key idea : Bring analysis into geometric arguments on Lie brakets. (Gagliardo Nirenberg inequality)

Previous ex of PDEs where the linear order misses a direction :

- [Coron, Crépeau 2003] : KdV, manifold-like behavior
- [KB, Morancey 2014] : bilinear Schrödinger, $[f_1, [f_1, f_0]](0)$
- [Marbach 2015] : Burgers, $H^{-5/4}$-quadratic drift (!!!)

Goal : Better understand the quadratic behaviors in **infinite dimension**.

We consider a simple nonlinear parabolic equation

$$\begin{cases} (\partial_t - \partial_{xx})z(t,x) = u(t)\Gamma[z(t)](x), \\ \partial_x z(t,0) = \partial_x z(t,\pi) = 0, \end{cases}$$

where $\Gamma \in C^2(H^1_N(0,\pi); H^{-1}_N(0,\pi))$ and $u \in L^\infty(0,T)$.
We will design Γ to illustrate new quadratic behaviors.

Linear theory : insensible to regularity, as for ODEs

Nonlinear system

$$\begin{cases} (\partial_t - \partial_{xx}) z(t,x) = u(t)\Gamma[z(t)](x), \\ \partial_x z(t,0) = \partial_x z(t,\pi) = 0, \\ z(0,x) = z_0(x), \end{cases}$$

Linearized system

$$\begin{cases} (\partial_t - \partial_{xx}) z_L = u(t)\Gamma[0], \\ \partial_x z_L(t,0) = \partial_x z_L(t,\pi) = 0, \\ z_L(0,x) = z_0(x). \end{cases}$$

$$z_L(T,x) = \sum_{k=0}^{\infty} b_k \int_0^T u(t) e^{-k^2(T-t)} dt \quad \varphi_k(x)$$

Its controllability is linked to the behavior of the sequence

$$b_k := \langle \Gamma[0], \varphi_k \rangle \in \mathbb{R},$$

where φ_k are the eigenfunctions of the Neuman-Laplacian on $(0,\pi)$. It holds if no term vanishes and if the sequence does not decay too fast. It yields regular controls (moment method on $u^{(p)}$).

It gives controllability of the nonlinear system, with smooth controls, thanks to the source-term method of Liu-Takahashi-Tucsnak.

Quadratic expansion

Let us assume that $b_0 = \langle \Gamma[0], \varphi_0 \rangle = 0$, so that the first direction is lost at the linear order. We wonder if we can recover this direction thanks to the quadratic term $z_Q \quad (z = z_L + z_Q + O(|u|^3))$

$$\begin{cases} (\partial_t - \partial_{xx}) z_Q = u(t)\Gamma'[0] z_L(t), \\ \partial_x z_Q(t,0) = \partial_x z_Q(t,\pi) = 0, \\ z_Q(0,x) = \langle z_0, \varphi_0 \rangle \varphi_0(x). \end{cases}$$

$$\begin{aligned} \langle z_Q(T), \varphi_0 \rangle &= \langle z_0, \varphi_0 \rangle + \int_0^T u(t) \langle \Gamma'[0] z_L(t), \varphi_0 \rangle dt \\ &= \langle z_0, \varphi_0 \rangle + \int_0^T u(t) \int_0^t u(\tau) \underbrace{\sum_{j=1}^{\infty} c_j e^{-j^2(t-\tau)}}_{K(t-\tau)} d\tau dt \,. \end{aligned}$$

where $\quad c_j := \langle \Gamma'[0]\varphi_j, \varphi_0 \rangle \, \langle \Gamma[0], \varphi_j \rangle$.

We will design Γ to get a **weakly singular integral operator** and a drift quantified by a fractionnal Sobolev norm of the control.

Fractional drifts

Theorem (KB, Marbach 2018)

Let $n \in \mathbb{N}$, $s \in (0,1)$ and Γ with $\langle \Gamma[0], \varphi_0 \rangle = 0$. Assume that

$$c_j = \frac{1}{j^{4n-1+4s}} + l.o.t., \qquad \sum_{j=1}^{\infty} j^{2(2\ell-1)} c_j = 0, \quad \forall \ell < n.$$

For small times and small controls (in $H^{2n+2s+3/2}$), one has

$$\langle z(T), \varphi_0 \rangle \approx \langle z_0, \varphi_0 \rangle + (-1)^n \gamma(s) \| u_n \|^2_{H^{-s}}$$

which prevents small-time local controllability.

Fractional-order drifts a priori not linked with Lie brackets, but depend on the asymptotic behavior of the sequence c_j. Changing a finite number of coefficients does not remove the drift.

Proof strategy ($n = 0$)

We prove that $\langle z(T), \varphi_0 \rangle = Q_T(u) + O\left(\|u\|_{L^\infty}^3\right)$

$$Q_T(u) = \int_0^T \int_0^T u(t)u(\tau)K(t-\tau)d\tau dt \quad \text{where} \quad K(\sigma) = \sum_{j=1}^{\infty} c_j e^{-j^2|\sigma|}$$

$$Q_T(u) = \frac{1}{4\pi} \int_{\mathbb{R}} |\widehat{u}(\xi)|^2 \widehat{K}(\xi) \mathrm{d}\xi \quad \text{where} \quad \widehat{K}(\xi) = \sum_{j=1}^{\infty} \frac{2j^2 c_j}{j^4 + \xi^2}$$

Under our assumption on the asymptotic behavior of $c_j \sim j^{1-4s}$,

$$\widehat{K}(\xi) = 2\gamma(s)|\xi|^{-2s} + O_{|\xi|\to\infty}(|\xi|^{-2s-2\beta})$$

(for a finite sum, only polynomial asymptotics). Thus

$$Q_T(u) = \gamma(s)\|u\|_{H^{-s}}^2 + O\left(\|u\|_{H^{-s-\beta}}^2\right)$$

We conclude with :

- an **uncertainty principle**, to neglect the $H^{-s-\beta}$ norm for small enough times : $\|u\|_{H^{-s-\beta}} \leqslant CT^{\beta}\|u\|_{H^{-s}} +$ l.o.t.
- an **interpolation inequality**, to neglect the cubic term for small enough regular controls : $\|u\|_{\infty}^3 \leqslant \|u\|_{H^{-s}}^2 \|u\|_{H^{2s+3/2}}$

Other drifts

One can construct systems (i.e. nonlinearities Γ) for which the drift is quantified by almost any weighted Sobolev-type norm (weaker than L^2) defined in the Fourier domain as

$$\|u\|_\Theta^2 := \int_{\mathbb{R}} |\hat{u}(\xi)|^2 |\xi|^{-2s} \Theta(|\xi|) \mathrm{d}\xi.$$

If Θ is sufficiently nice and the initial nonlinearity satisfies $c_j = j^{1-4s}\Theta(j)$, then a drift in $\|u\|_\Theta^2$ is observed.

Ex : $\Theta(x) = (\ln(1+x^2))^b (\ln\ln(1+x^2))^c$ with $b, c \geqslant 0$

Heuristically, as soon as c_j has an asymptotic sign, you can define an associated norm, prove the drift for small enough times and absorb the cubic residuals for regular enough controls.

Recovering directions in infinite dimension

We look for a system of which the linear order misses a direction

- but the quadratic order restores controllability,
- with only a single scalar control,
- **in small time**.

All previously known examples either :

- rely on cubic terms (e.g. KdV),
- use non-scalar controls
- require a large time (e.g. Schrodinger or Saint-Venant).

Sometimes one can recover one direction

$$\begin{cases} (\partial_t - \partial_{xx}) z(t,x) = u(t)\Gamma[z(t)](x), \\ \partial_x z(t,0) = \partial_x z(t,\pi) = 0 \\ z(0,x) = z_0(x) \end{cases} \quad (1)$$

Theorem (KB Marbach 2018)

There exists a nonlinearity Γ, *with* $\langle \Gamma[0], \varphi_0 \rangle = 0$, *such that (1) is small-time locally null controllable with* ***quadratic cost*** *:*

for every $T > 0$, $m \geq 1$, *there exists* $C > 0$ *such that, for each* $z_0 \in L^2(0,\pi)$, *there exists a control* $u \in H_0^m(0,T)$ *such that*

$$\|u\|_{H^m} \leq C \left(|\langle z_0, \varphi_0 \rangle|^{\frac{1}{2}} + \|z_0 - \langle z_0, \varphi_0 \rangle \varphi_0\|_{L^2} \right)$$

and the solution of (1) satisfies $z(T) = 0$.

Key idea : slow sign oscillations of $\widehat{K}$ up to ∞

Keep in mind $$\langle z_Q(T), \varphi_0 \rangle = \frac{1}{4\pi} \int_{\mathbb{R}} |\widehat{u}(\xi)|^2 \widehat{K}(\xi) \mathrm{d}\xi$$

- It is necessary that $\widehat{K}$ changes sign for large ξ because, due to the uncertainty principle, $\widehat{u}$ will be there. Maybe $\widehat{K}(\xi) = \cos|\xi|$?
- It is also convenient that $\widehat{K}$ changes sign more and more slowly, because the support of $\widehat{u}$ will become wide as $T \to 0$. Maybe $\widehat{K}(\xi) = \cos(\ln|\xi|)$?
- For $c_j = \cos(\ln j^2) j^{1-4s}$ then $\widehat{K}(\xi) = \sum_{j=1}^{\infty} \frac{2j^2 c_j}{j^4 + \xi^2} \approx |\xi|^{-2s} \cos(\ln|\xi|)$
- **Ex :** $\Gamma[z] := \sum_{k \geq 1} k^{\frac{1}{2} - 3s} \varphi_k + \left(\sum_{j \geq 1} \cos(\ln j^2) j^{\frac{1}{2} - s} \langle z, \varphi_j \rangle \right) \varphi_0$
- Which control ? $v_{\omega,\tau}(t) := \sin(\omega t) \mathbf{1}_{[0,\tau]}(t)$
  $\widehat{v_{\omega,\tau}}(\xi)$ is located near $\xi = \pm\omega$
  $v_{\omega,\tau}$ has an H^{-s} norm $\sim \omega^{-s}$ and can be corrected on the linear order with a correction $\sim \omega^{-1} << \omega^{-s}$

Picture of the base oscillations

Figure 1: Single periodic oscillating function.

Here, we are in log scale, so the oscillations seem be periodic, but recall that we have $\hat{K}(\xi) \approx |\xi|^{-2s}\Theta(\ln|\xi|)$, so the oscillations are slowing down.

Recovering an infinity of directions

$$\begin{cases} (\partial_t - \partial_{xx}) z(t,x) = u(t)\Gamma[z(t)](x), \\ \partial_x z(t,0) = \partial_x z(t,\pi) = 0 \\ z(0,x) = z_0(x) \end{cases} \quad (2)$$

Theorem (KB Marbach 2018)

Let $s \in (0, \frac{1}{2})$. There exists a nonlinearity Γ, with $\langle \Gamma[0], \varphi_{2k+1} \rangle = 0$ for every $k \in \mathbb{N}$, but such that the nonlinear system is small-time locally null controllable with quadratic cost.

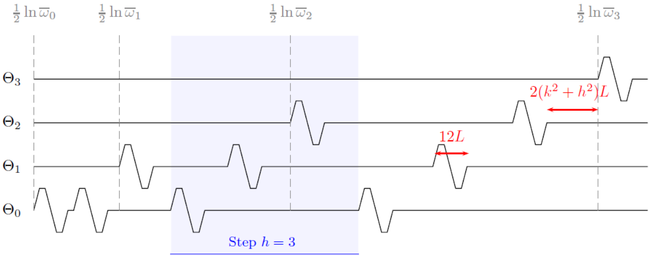
For every $T > 0$, there exists $C > 0$ such that, for each $z_0 \in L^2(0,\pi)$, there exists a control $u \in H^{-s}(0,T)$ and a solution z with $z(T) = 0$ and

$$\|u\|_{H^{-s}} \leq \|(\langle z_0, \varphi_{2k+1} \rangle)_{k \in \mathbb{N}}\|_{\ell^1}^{\frac{1}{2}} + C \|(\langle z_0, \varphi_{2k} \rangle)_{k \in \mathbb{N}}\|_{\ell^2}.$$

Key idea : slow asynchronous sign oscillations of $\widehat{K}_k$

$$\langle z_Q(T), \varphi_{2k+1}\rangle = \frac{1}{4\pi}\int_{\mathbb{R}} |\hat{u}(\xi)|^2 \widehat{K}_k(\xi)\mathrm{d}\xi.$$

- **Goal :** $\widehat{K}_k$ changes sign, increasingly slowly and at disjoint frequencies $\widehat{K}_k(\xi) \approx |\xi|^{-2s}\Theta_k(\ln|\xi|)$
- $\Gamma[z] := \sum_{k\geq 0} k^{\frac{1}{2}-3s}\varphi_{2k} + \sum_{k\geq 0}\left(\sum_{j\geq 1}\Theta_k(\ln j^2) j^{\frac{1}{2}-s}\langle z, \varphi_{2j}\rangle\right)\varphi_{2k+1}$
- We use a diagonal process to construct an infinity of base oscillations with disjoint supports, with values ± 1 arbitrary far, with constant length plateaus, with growing gaps.



Say we want to reach some virtual target $\sum y_k \varphi_{2k+1}$, we set

$$v(t) := \sum_{k\geq 0} |y_k|^{\frac{1}{2}} \omega_k^s \cos(\omega_k t) \mathbf{1}_{[0,T]}(t),$$

where the ω_k are chosen

- large enough,
- in the middle of a plateau of $\Theta_k(\ln \cdot)$,
- on a plateau of value ± 1 depending on the sign of the component y_k of the target state.

The sum converges in H^{-s} if the target sequence is in ℓ^1.
A priori, the system is ill-posed for H^{-s} controls. Nevertheless, with our semi-explicit controls, we can build a strong regular solution.

Conclusion, open problems, perspectives

- In finite dimension, the k-th obstruction can be saturated : the H^{-k}-drift prevents $W^{2k-3,\infty}$-STLC, but there exists a cubic perturbation which is $W^{2k-4,\infty}$-STLC.
- What about infinite dimension ?
  Control through a cubic term in spite of a quadratic term, bellow the threshold. [M. Bournissou, on going work]
- Can we treat examples from the real world ?
- For dispersive equations, are fractionnal drifts possible ?
- What can we say about higher order (e.g. quartic) drifts ?