A NEW CLASS OF MODELS: MEAN FIELD GAMES

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SUMMARY

1. MFG?

2. MATHEMATICAL PERSPECTIVES

- ONGOING JOINT RESEARCH PROGRAM WITH J.-M. LASRY
- COLLABORATIONS: O. GUEANT, P.-N. GIRAUD ...
- SUPPORTED BY VARIOUS INDUSTRIAL INITIATIVES: FDD (EDF + CALYON + CDC), NATIXIS, CHEVREUX
- DISCUSSIONS WITH: R. LUCAS (—), P.-A. CHIAPPORI, J. SCHEINKMAN ...

1. MFG ?

- AIM: INTRODUCE, JUSTIFY, ANALYSE AND APPLY NEW MATHEMATICAL MODELS FOR THE "COLLECTIVE BEHAVIOR OF A VERY LARGE NUMBER OF AGENTS"

- ≈ SYNTHESIS OF i) MEAN FIELD / SELF-CONSISTENT MODELS IN STATISTICAL MECHANICS AND PHYSICS

  ii) OPTIMAL CONTROL / OPTIMIZATION

  iii) NASH EQUILIBRIA (GAMES THEORY)

- VERY EFFICIENT MODELS FOR SITUATIONS INVOLVING A LARGE POPULATION (OR SEVERAL ONES) OF "INDISTINGUISHABLE" PLAYERS, WHERE EACH PLAYER HAS VERY LITTLE INFLUENCE ON THE OVERALL OUTCOME.

- MANY (POTENTIAL) APPLICATIONS TO : ECONOMICS, FINANCE AND POPULATION DYNAMICS, BIOLOGY (?) AND SOCIAL NETWORKS

...

- FROM A MATHEMATICAL VIEW-POINT :

  - NEW SYSTEMS OF EQUATIONS (PDE's)

CONTAINING AS PARTICULAR CASES MANY CLASSICAL ONES

(VLASOV, BOLTZMANN, HEAT, POROUS MEDIA, HARTREE,

EULER, HAMILTON-JACOBI-BELLMAN ...)

  - FOR DISCRETE CASES, NEW ISSUES ON DYNAMICAL SYSTEMS

  - JUSTIFICATION REQUIRES A NEW MATHEMATICAL

FRAMEWORK WITH MANY OTHER APPLICATIONS

  - MANY OPEN PROBLEMS / DIRECTIONS TO INVESTIGATE

  - COMPUTATIONAL ISSUES (ALREADY 3 GROUPS ...)

- "EXAMPLE" : THE MEXICAN WAVE

  - ASSUMPTIONS : COMFORT, EFFORT, "DO LIKE YOUR NEIGHBORS"

  - MFG APPROACH $\longrightarrow$ MODEL

  - RESULT : IF THE ARENA IS NOT TOO SMALL, $\exists$! "PARTIALLY STABLE" PERIODIC PROPAGATING WAVE

2. MATHEMATICAL PERSPECTIVES

2.1 DERIVATION / JUSTIFICATION

- NASH EQUILIBRIA, $N \to \infty$
- FUNCTIONS, DIFFERENTIALS, EQUATIONS OF N VARIABLES WITH SYMMETRY $\longrightarrow$ FUNCTIONS, DIFFERENTIALS, EQUATIONS ON THE SPACE OF PROBABILITY MEASURES OR ON SPACES OF RANDOM VARIABLES (WITH "SYMMETRY" = DEPEND ONLY ON THE LAW)

2.2 ANALYSIS OF MFG MODELS (CONTINUOUS CASE...)

- EXAMPLE: POPULATION REPRESENTED BY A DENSITY m, WHOSE DYNAMICS CORRESPOND TO THE "OPTIMAL" CHOICE OF A GENERIC AGENT GIVEN m

$$\frac{\partial u}{\partial t} + F(D^2 u, Du, x, t; m) = 0$$

$$\frac{\partial m}{\partial t} - D^2 \cdot \left(F_1(-)\, m\right) - D \cdot \left(F_2(-)\, m\right) = 0 .$$

AND $\quad u|_{t=T} = U(x; m(T)) \quad , \quad m|_{t=0} = m_0$

- SUB) EXAMPLE :

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\Delta u + H(Du, x, t; m) = 0$$

$$\frac{\partial m}{\partial t} - \frac{\sigma^2}{2}\Delta m - \operatorname{div}\left(H_1(-)m\right) = 0$$

CONTAINS MANY CLASSICAL EQUATIONS (...)

- MANY QUESTIONS STILL OPEN BUT BASIC REGIMES FOR UNIQUENESS, EXISTENCE, STABILITY HAVE BEEN IDENTIFIED ...

- MANY DIRECTIONS (IN ADDITION TO ↑)
  - SEVERAL POPULATIONS
  - NUMERICAL COMPUTATIONS
  - RECURSIVE UTILITY
  - RANDOM VARIATIONS IN THE POPULATION
  - PARTIAL OBSERVATIONS

- BEYOND MFG (DREAMING !) :
  - $N \to \infty$ AND THE EFFECT OF A FINITE N
  - THE "ROAD(S!) TO EQUILIBRIUM"
  - "FLUCTUATIONS AND LARGE DEVIATIONS" AROUND MFG
  - PHASE TRANSITIONS ...