Optimisation d'une valeur propre asymétrique et fragmentation de l'habitat en écologie

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G. Nadin, The effect of the Schwarz rearrangement on the periodic principal eigenvalue of a non-symmetric operator, *to appear in SIAM J. Math. Anal.*

A reaction-diffusion model

$$\begin{cases} \partial_t u - \partial_{xx} u = \mu(x) u(1-u) \text{ in } (0,\infty)\times\mathbb{R}, \\ u(0,x) = u_0(x) \text{ compactly supported.} \end{cases} \quad (1)$$

u: population density

$\mu(x)$: growth rate per capita at small density, depends on the space variable x

Goal: Introduction of a population in a bounded area at $t = 0$
$\rightarrow$ Asymptotic behavior ? Speed of the invasion? **Influence of the "fragmentation of the habitat" on this behavior ?**

$$\mu = \text{positive constant}$$

$$\begin{cases} \partial_t u - \partial_{xx} u = \mu u(1-u) \text{ in } (0,\infty)\times\mathbb{R}, \\ u(0,x) = u_0(x) \text{ compactly supported.} \end{cases} \tag{2}$$

Theorem

(Aronson-Weinberger 78)

$$u(t, wt) \to \begin{cases} 1 \text{ if } 0 \leq w < w^*, \\ 0 \text{ if } w > w^*, \end{cases} \quad \text{as } t \to +\infty$$

where $w^* = 2\sqrt{\mu}$. *We say that* w^* *is a* **spreading speed**.

Interpretation: The population “spreads” with speed w^*.

Asymptotic spreading in periodic media

$$\mu(x+L) = \mu(x)$$

$$\begin{cases} \partial_t u - \partial_{xx} u = \mu(x) u(1-u) \text{ in } (0,\infty)\times\mathbb{R}, \\ u(0,x) = u_0(x) \text{ compactly supported.} \end{cases} \tag{3}$$

Theorem

Assume that μ is positive, there exists a speed $w^* = w^*(\mu)$ such that

$$u(t, wt) \to \begin{cases} 1 \text{ if } 0 \leq w < w^*, \\ 0 \text{ if } w > w^*. \end{cases} \quad \text{as } t \to +\infty$$

We say that w^* is a **spreading speed**.

Several proofs: Gartner-Freidlin 79, Nolen-Xin 08 (probabilistic tools), Weinberger 02 (discrete formalism), Berestycki-Hamel-N. 08 (periodic eigenvalues+PDE tools), Majda-Souganidis 94 (homogenization techniques).

Goal: What is the dependence of $w^*(\mu)$ with respect to μ?

Characterization of the spreading speed

$$\begin{cases} \partial_t u - \partial_{xx} u = \mu(x)u(1-u) \text{ in } (0,\infty)\times\mathbb{R}, \\ u(0,x) = u_0(x) \text{ compactly supported.} \end{cases} \tag{4}$$

$$\mathcal{L}\phi = \phi'' + \mu(x)\phi \text{ (linearized operator near 0)}$$

$$L_p\phi = e^{-px}\mathcal{L}(e^{px}\phi) = \phi'' + 2p\phi' + (p^2 + \mu(x))\phi$$

Set $k_p(\mu)$ the periodic principal eigenvalue of L_p (Krein-Rutman theory):

$$\begin{cases} L_p\phi = k_p(\mu)\phi, \\ \phi > 0, \\ \phi \text{ is periodic.} \end{cases} \tag{5}$$

Proposition

$$w^*(\mu) = \min_{p>0} \frac{k_{-p}(\mu)}{p}$$

$\Rightarrow$ Simplifies the investigation of $\mu \mapsto w^*(\mu)$.

General influence of the heterogeneity

Proposition

(Berestycki-Hamel-Roques 05)
Set $\overline{\mu}$ the average of μ, then

$$w^*(\mu) \geq w^*(\overline{\mu})(= 2\sqrt{\overline{\mu}}).$$

Example: $\mu(x) = 1 + \delta \sin x$, $w^*(\mu) \geq 2 = w^*(1)$.

Interpretation: The heterogeneity speeds-up the propagation.

Question: Quantification of the speed-up with the characteristics of the heterogeneity

Statement of the results (1)

$$0 < \mu^- \le \mu^+, \overline{\mu} \in \mathbb{R} \text{ given}$$

$$\mathcal{M} := \{\mu \in L^\infty_{per}, \mu^- \le \mu(x) \le \mu^+, \frac{1}{L}\int_0^L \mu = \overline{\mu}\}$$

Theorem

(N. 09) $\mu \in \mathcal{M} \mapsto w^*(\mu)$ *reaches its maximum when* $\mu^* = \mu^+ 1_{[0,a]} + \mu^- 1_{[a,L]}$, *where a is such that* $a\mu^+ + (L-a)\mu^- = L\overline{\mu}$.

Interpretation: The environment that maximizes the propagation speed is that where the favourable zones are gathered, i.e. fragmentation of the habitat slows down the propagation.

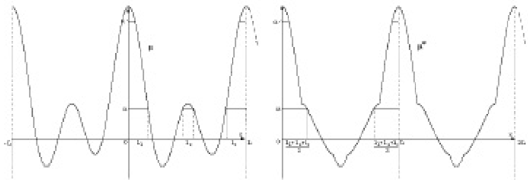
Possible to optimize w^* when a general distribution function is fixed?

Definition of the Schwarz rearrangement

Definition

The **Schwarz rearrangement** μ^* *of a measurable L−periodic function* $\mu = \mu(x), x \in \mathbb{R}$ *is the unique function of period L, with the same distribution function, even and nonincreasing on* $(0, L/2)$.

Example: if μ takes to values μ^+ and μ^-, then $\mu^* = \mu^+$ in the centered interval, μ^- otherwise.



A continuous function μ and its Schwarz rearrangement μ^*.

Observation (Berestycki-Hamel-Roques 05): *less fragmented* habitat is associated with the growth rate μ^*.

Statement of the results (2)

Definition

The **Schwarz rearrangement** μ^* *of a measurable* $L-$*periodic function* $\mu = \mu(x), x \in \mathbb{R}$ *is the unique function of period* L*, with the same distribution function, even and nonincreasing on* $(0, L/2)$*.*

Theorem

(N. 09) $w^*(\mu^*) \geq w^*(\mu)$*.*

Answer to an open problem stated by Berestycki-Hamel-Roques 05.

A related eigenvalue optimization problem

$$L_p\phi = \phi'' + 2p\phi' + (p^2 + \mu(x))\phi$$

$k_p(\mu)$ = periodic principal eigenvalue of L_p:

$$\begin{cases} L_p\phi = k_p(\mu)\phi, \\ \phi > 0, \\ \phi \text{ is periodic.} \end{cases} \tag{6}$$

Proposition

$$w^*(\mu) = \min_{p>0} \frac{k_{-p}(\mu)}{p}$$

$\Rightarrow$ If $k_p(\mu^*) \geq k_p(\mu)$ for all p, then $w^*(\mu^*) \geq w^*(\mu)$.

Reformulation of our problem Prove that for all $p \in \mathbb{R}$:

$$k_p(\mu^*) \geq k_p(\mu)$$

where μ^* is the Schwarz rearrangement of μ.

A Faber-Krahn inequality ($p = 0$)

Proposition

(Berestycki-Hamel-Roques 05)

$$k_0(\mu^*) \geq k_0(\mu).$$

Proof. If $p = 0$, $L_0\phi = \phi'' + \mu(x)\phi$ is self-adjoint and thus $k_0(\mu)$ is a *Rayleigh quotient*:

$$k_0(\mu) = \max_{\alpha \in \mathcal{C}^1_{per}} \frac{< L_0\alpha, \alpha >_{L^2}}{< \alpha, \alpha >_{L^2}} = \max_{\alpha \in \mathcal{C}^1_{per}} \frac{1}{\int_0^L \alpha^2} \int_0^L (-\alpha'^2 + \mu(x)\alpha^2). \quad (7)$$

The proposition follows from the two classical properties of the rearrangement:

$$\forall \mu, \alpha > 0, \int_0^L \mu^*(\alpha^*)^2 \geq \int_0^L \mu\alpha^2 \text{ and } \forall \alpha, \int_0^L (\alpha^*)'^2 \leq \int_0^L \alpha'^2. \square$$

A Faber-Krahn inequality ($p = 0$)

Proposition

(Berestycki-Hamel-Roques 05)

$$k_0(\mu^*) \geq k_0(\mu).$$

Issues when $p \neq 0$:

- No Rayleigh quotient since L_p is not symmetric.

$$L_p\phi = \phi'' + 2p\phi' + (p^2 + \mu(x))\phi.$$

- Rearrangement properties are integral ones.
- Very few litterature on the rearrangement of non-symmetric operators (Alvino-Trombetti-Lions 90-91, Hamel-Nadirashvili-Russ 05-07).

$\rightarrow$ Find an integral characterization of $k_p(\mu)$.

A integral characterization of $k_p(\mu)$

Proposition

(N. 09) $k_p(\mu) = \max_{\alpha \in \mathcal{C}^1_{per}} \frac{1}{\int_0^L \alpha^2} \left(\int_0^L -\alpha'^2 + \int_0^L \mu(x)\alpha^2 + p^2 \frac{L^2}{\int_0^L \frac{1}{\alpha^2}} \right)$

Corollary

$k_p(\mu^*) \geq k_p(\mu)$ *for all* p *and thus* $w^*(\mu^*) \geq w^*(\mu)$.

Proof. Follows from the two classical properties of the rearrangement:

$$\forall \mu, \alpha > 0, \int_0^L \mu^*(\alpha^*)^2 \geq \int_0^L \mu\alpha^2 \text{ and } \forall \alpha, \int_0^L (\alpha^*)'^2 \leq \int_0^L \alpha'^2,$$

and from $\int_0^L \frac{1}{\alpha^2} = \int_0^L \frac{1}{(\alpha^*)^2}$ since the rearrangement preserves the distribution function. □

Where does the integral characterization comes from?

A new characterization of the principal eigenvalue of a non-symmetric operator

Consider a general second order elliptic operator in multidimensional media:

$$L_0\phi := div(A(x)\nabla\phi) + q(x)\cdot\nabla\phi + \mu(x)\phi$$

Set $k_0(A, q, \mu)$ the periodic principal eigenvalue of L_0.

Theorem

(N. 09)
$k_0(A, q, \mu) = \min_{\beta \ periodic} k_0(A, 0, \mu + \nabla\beta A\nabla\beta + q\cdot\nabla\beta - divq/2)$

Proposition

(N. 09) $k_p(\mu) = \max_{\alpha\in\mathcal{C}^1_{per}} \frac{1}{\int_0^L \alpha^2}\Big(\int_0^L -\alpha'^2 + \int_0^L \mu(x)\alpha^2 + p^2\frac{L^2}{\int_0^L \frac{1}{\alpha^2}}\Big).$

Proof. Apply the Theorem to $L_p\phi = \phi'' + 2p\phi' + (\mu(x) + p^2)\phi$,

$$k_p(\mu) = \min_{\beta \ periodic} k_0(\mu + p^2(1+\beta')^2).$$

A new characterization of the principal eigenvalue of a non-symmetric operator

Proposition

(N. 09) $k_p(\mu) = \max_{\alpha \in \mathcal{C}^1_{per}} \frac{1}{\int_0^L \alpha^2} \Big(\int_0^L -\alpha'^2 + \int_0^L \mu(x)\alpha^2 + p^2 \frac{L^2}{\int_0^L \frac{1}{\alpha^2}} \Big).$

Proof. Apply the Theorem to $L_p\phi = \phi'' + 2p\phi' + (\mu(x) + p^2)\phi$,

$$k_p(\mu) = \min_{\beta \text{ periodic}} k_0(\mu + p^2(1+\beta')^2).$$

$k_0(\mu + p^2(1+\beta')^2)$ is associated with a self-adjoint operator $\Rightarrow$ Rayleigh quotient

$$k_p(\mu) = \min_{\beta \text{ periodic}} \max_{\alpha \in \mathcal{C}^1_{per}} \frac{1}{\int_0^L \alpha^2} \Big(\int_0^L -\alpha'^2 + \int_0^L (\mu(x) + p^2(1+\beta')^2)\alpha^2 \Big)$$

Reverse the max and the min, compute using Lax-Milgram lemma. □

Conclusion

Theorem

(N. 09)

$k_0(A, q, \mu) = \min_{\beta \text{ periodic}} k_0(A, 0, \mu + \nabla\beta A \nabla\beta + q \cdot \nabla\beta - divq/2)$

Remark: Similar formulas with different boundary conditions by Donsker-Varadhan (76), Holland (78).

Very useful to optimize principal eigenvalues of non-symmetric operators, like operators L_p.

$\Rightarrow$ Other applications to reaction-diffusion equations in periodic media.